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Research Article

Fredholm Weighted Composition Operators on Weighted Banach Spaces of Analytic Functions of Type H^{∞}

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We study Fredholm (weighted) composition operators between general weighted Banach spaces of analytic functions on the open unit disc with sup-norms.

1. Introduction

Let X, Y be Banach spaces (infinite dimensional spaces). We denote by $\mathscr{B}(X, Y)$ the space of bounded linear operators from X to Y. A bounded linear operator $T : X \rightarrow Y$ is said to be *Fredholm* if the spaces $\mathrm{Ker}T$ and $\mathrm{CoKer}T = Y/\mathrm{Im}T$ are finite dimensional. Every Fredholm operator has closed range. It is known that T is Fredholm if and only if T is invertible modulo the compact operators and if and only if its dual map is Fredholm. For background on Fredholm operators we refer to [1].

Let φ and ψ be analytic functions on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. These maps define, on the space $H(\mathbb{D})$ of analytic functions on $\mathbb{D}$, the so-called *weighted composition operator* $W_{\varphi,\psi}$ by $W_{\varphi,\psi}(f) = \psi(f \circ \varphi)$. It combines the classical composition operator $C_{\varphi} : f \mapsto f \circ \varphi$ with the pointwise multiplication operator $M_{\psi} : f \mapsto \psi \cdot f$. These operators have been studied by different authors on various types of function spaces. For more information about composition operators, we refer the reader to the excellent monographs of Cowen and MacCluer [2] and Shapiro [3]. In his thesis Schwartz [4] showed that a composition operator C_{φ} on the Hardy space $H^2(\mathbb{D})$ is invertible if and only if φ is a conformal automorphism of the unit disk. Cima et al. [5] proved that only the invertible operators on $H^2(\mathbb{D})$ are Fredholm. Various authors have considered the composition operator C_{φ} on several Banach spaces of analytic functions and have studied when C_{φ} is Fredholm. See, for example, [6] for the Hardy and Bergman spaces in $\mathbb{D}$, [7] for various spaces on domain in $\mathbb{C}^n$, [8] for a variety of Hilbert spaces of analytic functions on domains in $\mathbb{C}^n$, [9] for the space $H^{\infty}(B_E)$ of bounded analytic functions on the open unit ball of a complex Banach space E, and [10] for Banach spaces of analytic functions on $\mathbb{D}$ satisfying certain conditions. Recently, Zhao has given a characterization of bounded Fredholm weighted composition operator on Dirichlet spaces [11], on Hardy spaces [12], and on a class of weighted Hardy spaces [13].

In this paper we consider the weighted composition operator defined on the weighted Banach spaces of holomorphic functions H_v^{∞} and on the smaller spaces H_v^0 (see Section 2 for the definition). In this framework, Contreras and Hernández-Díaz [14] and Montes-Rodríguez [15], continuing work by [16], characterized the boundedness and compactness of $W_{\varphi,\psi}$ between weighted Banach spaces of analytic functions H_v^{∞} and H_v^0. We are interested in finding a characterization of Fredholm weighted composition operators in terms of properties of the symbol φ and of the multiplier function ψ.

Our paper is motivated mainly by the works [10, 17]. In the first one, Bonet et al. characterized when M_{ψ} is a Fredholm operator if it is considered between H_v^{∞} or H_v^0. Recently, Galindo and Lindström considered in [10] composition operators on a class of Banach spaces of analytic functions defined on $\mathbb{D}$ and E, satisfying certain conditions, and they proved that the composition operator $C_{\varphi} : E \rightarrow E$ associated with the analytic self-map $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is

Fredholm if and only if φ is a biholomorphic map and if and only if C_φ is invertible. We will present a characterization of Fredholm composition operators and Fredholm weighted composition operators when they are considered between H_v^∞ or H_v^0, for a general typical weight function v (see Theorem 10). Similar results to the ones presented here for more concrete weights are already established in Section 3 of [18]. Although some of the techniques are known and have already been used in the mentioned paper, we present here proofs obtained independently in a more general setting with some applications. Some particular cases of Banach spaces E considered in [10, 18] are the standard weighted Bergman spaces A_α^p with $\alpha > -1$ and $p \geq 1$, the little Bloch space, and the weighted Banach spaces $H_{v_p}^0$, $0 < p < \infty$, where $v_p(z) = (1-|z|^2)^p$. For any weight, H_v^0 does not satisfy in general condition (C1) of [10] or conditions (C1) and (C1)$'$ of [18]. Hence, we complement recent work by Contreras, Galindo, Hernández-Daz, Hyvärinen, Lindström, Nieminen, and Saukko, among others.

2. Notation and Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and let us denote by $H(\mathbb{D})$ the set of analytic functions from $\mathbb{D}$ into $\mathbb{C}$. A weight function $v : \mathbb{D} \to \mathbb{R}_+$ is a radial bounded continuous function that is nonincreasing with respect to $|z|$ such that $\lim_{|z|\to 1^-} v(z) = 0$. In the literature these types of weight functions are called *typical weight functions*. The weighted Banach spaces of analytic functions are defined as follows:

$$\begin{aligned} H_v^\infty &= H_v^\infty(\mathbb{D}) \\ &:= \left\{ f \in H(\mathbb{D}) : \|f\|_v = \sup_{z\in\mathbb{D}} v(z)\,|f(z)| < \infty \right\}, \\ H_v^0 &= H_v^0(\mathbb{D}) \\ &:= \left\{ f \in H_v^\infty(\mathbb{D}) : \lim_{|z|\to 1} v(z)\,|f(z)| = 0 \right\}, \end{aligned} \tag{1}$$

endowed with the norm $\|\cdot\|_v$. The function $v \equiv 1$ is not a weight function according to our assumptions. In this case $H_v^\infty = H^\infty$ and $H_v^0 = \{0\}$.

Many results on weighted spaces of analytic functions are formulated in terms of the so-called *associated weight function* which is defined by the formula

$$\begin{aligned} \tilde{v}(z) &:= \frac{1}{\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \leq 1\}} \\ &= \frac{1}{\|\delta_z\|_{(H_v^\infty)^*}}, \quad z \in \mathbb{D}, \end{aligned} \tag{2}$$

where δ_z denotes the point evaluation of z. By [19], we know that $\tilde{v}$ is also a weight function; we have that $\tilde{v} \geq v > 0$, and for each $z \in \mathbb{D}$ we can find $f_z \in H_v^\infty$, $\|f_z\|_v \leq 1$, such that $|f_z(z)| = 1/\tilde{v}(z)$. It is well known that $H_{\tilde{v}}^\infty$ is isometrically isomorphic to H_v^∞ and $(H_v^0)^{**}$ is isometric to H_v^∞, where the inclusion map is the canonical injection from a Banach space into its bidual. We refer to [19–21] for more information about these spaces.

For $\varphi, \psi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $E = H_v^\infty$ or H_v^0, we consider the composition operator

$$\begin{aligned} C_\varphi &: E \longrightarrow E \\ f &\longmapsto C_\varphi(f) = f \circ \varphi, \end{aligned} \tag{3}$$

and the weighted composition operator is defined as follows:

$$\begin{aligned} W_{\varphi,\psi} &: E \longrightarrow E \\ f &\longmapsto \psi(f \circ \varphi), \end{aligned} \tag{4}$$

provided it is well defined. We always assume that given a weighted composition operator $W_{\varphi,\psi}$ there exists a point $z \in \mathbb{D}$ such that $\psi(z) \neq 0$.

In case the composition operator C_φ is bounded and ψ is a bounded analytic function on the unit disk, $\psi \in H^\infty$, the multiplication operator $M_\psi : H_v^\infty \to H_v^\infty$, defined as $M_\psi(f) = \psi \cdot f$, is bounded ([17, Proposition 2.1]), and the weighted composition operator $W_{\varphi,\psi} : H_v^\infty \to H_v^\infty$ is also bounded and can be decomposed as $W_{\varphi,\psi} = M_\psi C_\varphi$, where $M_\psi : H_v^\infty \to H_v^\infty$ and $C_\varphi : H_v^\infty \to H_v^\infty$.

By Schwarz Lemma, if $\varphi(0) = 0$, then $|\varphi(z)| \leq |z|$ for every $z \in \mathbb{D}$ and therefore C_φ is bounded on every space H_v^∞. If v satisfies the Lusky condition

$$\inf_n \frac{\tilde{v}\left(1-2^{-n-1}\right)}{\tilde{v}(1-2^{-n})} > 0, \tag{L1}$$

then Theorem 2.3 in [16] ensures that C_φ is bounded on H_v^∞ and H_v^0. For instance, the standard weights $v_p(z) = (1-|z|)^p$, $p > 0$, are weight functions which have $(L1)$.

The point evaluations play an important role in our results. Our first two lemmas are known. We give a short proof for the reader's convenience.

Lemma 1. *For every $z \in \mathbb{D}$, the point evaluation $\delta_z : H_v^\infty \to \mathbb{C}$, $f \mapsto \delta_z(f) = f(z)$, is continuous and $\|\delta_z\|_{(H_v^0)^*} = \|\delta_z\|_{(H_v^\infty)^*}$.*

Proof. If $f \in H_v^\infty$, fix $z \in \mathbb{D}$:

$$\begin{aligned} |f(z)| &= \frac{1}{v(z)} v(z)\,|f(z)| \leq \frac{1}{v(z)} \sup_{\xi\in\mathbb{D}} v(\xi)\,|f(\xi)| \\ &\leq \frac{1}{v(z)} \|f\|_v. \end{aligned} \tag{5}$$

Then $\delta_z \in (H_v^\infty)^*$.

For all $u \in (H_v^\infty)^*$, it is clear that $\|u\|_{(H_v^0)^*} \leq \|u\|_{(H_v^\infty)^*}$. On the other hand,

$$\|\delta_z\|_{(H_v^\infty)^*} = \sup\{|f(z)| : \|f\|_v \leq 1, f \in H_v^\infty\}. \tag{6}$$

Under the hypothesis that v is a typical weight we know that the closed unit ball of H_v^0 is dense in the closed unit ball of H_v^∞

for the compact open topology (see [20]). Then if $f \in H_v^\infty$, $\|f\|_v \le 1$, there exists $(g_i)_{i\in\mathbb{N}}$, $g_i \in H_v^0$, $\|g_i\|_v \le 1$ such that $g_i \to f$ in the compact open topology c_0. Then $g_i(z) \to f(z)$. Hence $\delta_z(g_i) \to \delta_z(f)$. Since $|\delta_z(g_i)| \le \|\delta_z\|_{(H_v^0)^*}$, we obtain

$$\|\delta_z\|_{(H_v^\infty)^*} \le \|\delta_z\|_{(H_v^0)^*}. \tag{7}$$

□

Lemma 2. *If v is a weight function, then*

(i) $\lim_{|z|\to 1}\|\delta_z\|_{(H_v^\infty)^*} = \lim_{|z|\to 1}\|\delta_z\|_{(H_v^0)^*} = \infty$,

(ii) $\lim_{|z|\to 1}(\delta_z(f)/\|\delta_z\|_{(H_v^\infty)^*}) = 0$ *for all* $f \in H_v^0$.

Proof. (i) By [16, Proposition 1.1], we have $\lim_{r\to 1}\tilde{v}(r) = 0$. From formula (2) and Lemma 1 we obtain the conclusion.

(ii) Let P be a polynomial; then

$$\begin{aligned}&\lim_{|z|\to 1}\frac{|\delta_z(P)|}{\|\delta_z\|_{(H_v^\infty)^*}}\\ &\quad\le \sup\left\{|P(z)| : z\in\overline{\mathbb{D}}\right\}\lim_{|z|\to 1}\frac{1}{\|\delta_z\|_{(H_v^\infty)^*}} = 0.\end{aligned} \tag{8}$$

Since $\{\delta_z/\|\delta_z\|_{(H_v^\infty)^*} : z \in \mathbb{D}\}$ is equicontinuous in $(H_v^0)^*$ and the polynomials are dense in H_v^0, we have $\lim_{|z|\to 1}(\delta_z(f)/\|\delta_z\|_{(H_v^\infty)^*}) = 0$ for all $f \in H_v^0$. □

For X, Y Banach spaces (infinite dimensional spaces) a bounded linear operator $T : X \to Y$ is said to be *Fredholm* if the spaces $\mathrm{Ker}T$ and $\mathrm{CoKer}T = Y/\mathrm{Im}T$ are finite dimensional. Note that a consequence of the Fredholmness is that the condition $\mathrm{codim}(\mathrm{Im}T) = \dim(Y/\mathrm{Im}T) < \infty$ implies that the range of T is closed. It is known that $T : X \to Y$ is Fredholm if and only if $T^* : Y^* \to X^*$ is Fredholm and if and only if there are $S \in \mathscr{B}(Y, X)$, $K_1 \in \mathscr{K}(X)$, and $K_2 \in \mathscr{K}(Y)$ such that $ST = I_X + K_1$, $TS = I_Y + K_2$, where $\mathscr{K}(X)$ denote the set of compact operators on X. As a consequence, invertible operators are Fredholm and compact operators cannot be Fredholm. We refer to [1, Chapter III] for the proofs of these results.

3. Fredholm Weighted Composition Operators

Recall that the weighted composition operator $W_{\varphi,\psi} : H_v^\infty \to H_v^\infty$ is bounded if and only $\sup_{z\in\mathbb{D}}(|\psi(z)|v(z)/\tilde{v}(\varphi(z)))$ is finite ([14, Proposition 3.1]) and the operator $W_{\varphi,\psi} : H_v^0 \to H_v^0$ is bounded if and only if $\psi \in H_v^0$ and $\sup_{z\in\mathbb{D}}(|\psi(z)|v(z)/\tilde{v}(\varphi(z)))$ is finite ([14, Proposition 3.2]).

Lemma 3. *Let $\psi \in H_v^0$ and let $\varphi \in H(\mathbb{D})$ satisfy $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The operator $W_{\varphi,\psi} : H_v^\infty \to H_v^\infty$ is Fredholm if and only if $W_{\varphi,\psi} : H_v^0 \to H_v^0$ is Fredholm.*

Proof. It is a consequence of the fact that $(H_v^0)^{**} = H_v^\infty$ and that $W_{\varphi,\psi} : H_v^\infty \to H_v^\infty$ coincides with the biadjoint map of $W_{\varphi,\psi} : H_v^0 \to H_v^0$ whenever both operators are well defined. □

Lemma 4. *Let $\varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, and $\psi \in H_v^\infty$. If $W_{\varphi,\psi} : H_v^\infty \to H_v^\infty$ is Fredholm, then there can be at most finitely many points at which ψ vanishes.*

Proof. We first prove that $\delta_a \in (\mathrm{Im}(W_{\varphi,\psi}))^\perp$ for every zero a of ψ. To do this suppose that $\psi(a) = 0$ and $g \in \mathrm{Im}(W_{\varphi,\psi})$. Then there exists $f \in H_v^\infty$ such that $\langle \delta_a, g\rangle = \psi(a)(f\circ\varphi)(a) = 0$. Hence $\delta_a \in (\mathrm{Im}(W_{\varphi,\psi}))^\perp$.

Now, suppose that there exists an infinite sequence $\{z_i\}_{i=1}^\infty$ such that $\psi(z_i) = 0$. The sequence $\{\delta_{z_i}\}_{i=1}^\infty$ is linearly independent in $(H_v^\infty)^*$ (see [17] pp. 178) and $\{\delta_{z_i}\}_{i=1}^\infty \subseteq (\mathrm{Im}(W_{\varphi,\psi}))^\perp$. Then $\dim(\mathrm{Im}(W_{\varphi,\psi}))^\perp = \infty$.

On the other hand, since $W_{\varphi,\psi}$ is Fredholm, $W_{\varphi,\psi}^*$ is also Fredholm; hence $\dim(\mathrm{Im}(W_{\varphi,\psi})^\perp) = \dim(\mathrm{Ker}(W_{\varphi,\psi}^*)) < \infty$, which is a contradiction. □

Proposition 5. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic function and $\psi \in H_v^\infty$. If $W_{\varphi,\psi} : H_v^\infty \to H_v^\infty$ is Fredholm, then φ is injective.*

Proof. Our proof starts with the observation that for every point in $\mathbb{D}$ there is a neighborhood where φ is injective. Otherwise, there exists $x \in \mathbb{D}$ and there are disjoint infinite sequences $\{u_m\}_m$ and $\{v_m\}_m \subseteq \mathbb{D}$ such that $\lim_m u_m = \lim_m v_m = x$ and $\varphi(u_m) = \varphi(v_m)$. By Lemma 4, ψ has only a finite number of zeros in $\mathbb{D}$; then we can assume $\psi(u_m)\psi(v_m) \ne 0$. For every $m \in \mathbb{N}$, we may define $\vartheta_m := \delta_{u_m}/\psi(u_m) - \delta_{v_m}/\psi(v_m) \in (H_v^\infty)^*$. Then $\{\vartheta_m\}_m \subseteq \mathrm{Ker}W_{\varphi,\psi}^*$. Indeed,

$$\begin{aligned}W_{\varphi,\psi}^*(\vartheta_m)(f) &= \left\langle \frac{\delta_{u_m}}{\psi(u_m)} - \frac{\delta_{v_m}}{\psi(v_m)}, W_{\varphi,\psi}(f)\right\rangle\\ &= \frac{1}{\psi(u_m)}\psi(u_m)(f\circ\varphi)(u_m)\\ &\quad - \frac{1}{\psi(v_m)}\psi(v_m)(f\circ\varphi)(v_m)\\ &= f(\varphi(u_m)) - f(\varphi(v_m)) = 0.\end{aligned} \tag{9}$$

Moreover, $\{\vartheta_m\}_m$ is an infinite linearly independent sequence in $(H_v^\infty)^*$. Indeed, if we suppose

$$\alpha_{i1}\vartheta_{i1} + \cdots + \alpha_{ik}\vartheta_{ik} = 0 \quad \text{in } (H_v^\infty)^*, \tag{10}$$

then for all polynomial P we obtain

$$\begin{aligned}&\alpha_{i1}\left(\frac{P(u_{i1})}{\psi(u_{i1})} - \frac{P(v_{i1})}{\psi(v_{i1})}\right) + \cdots\\ &\quad + \alpha_{ik}\left(\frac{P(u_{ik})}{\psi(u_{ik})} - \frac{P(v_{ik})}{\psi(v_{ik})}\right) = 0.\end{aligned} \tag{11}$$

For every $j \in \{1, 2, \ldots, k\}$ it is sufficient to take a polynomial P_j such that

$$\begin{aligned}&P_j(u_{im}) = P_j(v_{im}) = 0 \quad \forall m \in \{1, 2, \ldots, k\}\setminus\{j\}\\ &P_j(u_{ij}) = 1,\\ &P_j(v_{ij}) = 0\end{aligned} \tag{12}$$

to conclude $\alpha_{ij} = 0$.

By assuming that $W_{\varphi,\psi}$ is Fredholm, then $W^*_{\varphi,\psi}$ is Fredholm and $\dim(\operatorname{Ker}W^*_{\varphi,\psi})$ is finite. This fact contradicts the fact that $\{\vartheta_m\}_m$ is an infinite linearly independent sequence in $\operatorname{Ker}W^*_{\varphi,\psi}$.

Note that φ cannot be constant. We are now in position to show that φ is injective. Assume φ is not injective. Then there are $x, y \in \mathbb{D}$, $x \neq y$, such that $\varphi(x) = \varphi(y) =: \omega$. By our observation above there are $B_x = B(x, r_1)$ and $B_y = B(y, r_2)$, $B_x \cap B_y = \emptyset$, such that $\varphi_{|B_x}$ and $\varphi_{|B_y}$ are injective. Then $\varphi(B_x)$ and $\varphi(B_y)$ are open in $\mathbb{D}$. Hence $\varphi(B_x) \cap \varphi(B_y)$ is open in $\mathbb{D}$ and it is an open neighborhood of ω. Therefore

$$\begin{gathered} \exists\{u_m\} \subseteq B_x, \\ \exists\{v_m\} \subseteq B_y \\ \text{such that } \varphi(u_m) = \varphi(v_m). \end{gathered} \tag{13}$$

As in the first part of the proof, we can assume that $\psi(u_m)\psi(v_m) \neq 0$. We define $\{\vartheta_m\}$ as above. It is an infinite linearly independent sequence in $\operatorname{Ker}W^*_{\varphi,\psi}$. A contradiction since $W^*_{\varphi,\psi}$ is Fredholm. □

Remark 6. For $\varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset r_0\mathbb{D}$ $(0 < r_0 < 1)$ the operator $W_{\varphi,\psi} : H^\infty_v \to H^\infty_v$ is compact for every $\psi \in H^\infty_v$. Then $W_{\varphi,\psi}$ is not Fredholm. In fact, the associated weight $\widetilde{v}$ is also radial and decreasing. Hence $\widetilde{v}(\varphi(z)) \geq \widetilde{v}(r_0)$ for all $z \in \mathbb{D}$ and

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)|\, v(z)}{\widetilde{v}(\varphi(z))} \leq \frac{1}{\widetilde{v}(r_0)} \sup_{z\in\mathbb{D}} \left(|\psi(z)|\, v(z)\right) < \infty. \tag{14}$$

From [14, Proposition 3.1], $W_{\varphi,\psi}$ is bounded. Since $\varphi(\mathbb{D}) \subset r_0\mathbb{D}$, with $0 < r_0 < 1$, by [16, Corollary 3.2] we obtain that the composition operator C_φ is compact and $C_\varphi(H^\infty_v) \subseteq H^\infty$. Then the operator $W_{\varphi,\psi}$ can be written as $W_{\varphi,\psi} = \widetilde{M_\psi}C_\varphi$, where $\widetilde{M_\psi} : H^\infty \to H^\infty_v$, $\widetilde{M_\psi}(f) = \psi f$. Since $C_\varphi \in \mathscr{K}(H^\infty_v, H^\infty)$ and $\widetilde{M_\psi} \in \mathscr{B}(H^\infty, H^\infty_v)$, we deduce that $W_{\varphi,\psi}$ is compact.

Proposition 7. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function and $\psi \in H^0_v$. If $W_{\varphi,\psi} : H^\infty_v \to H^\infty_v$ or $W_{\varphi,\psi} : H^0_v \to H^0_v$ is Fredholm, then φ is an automorphism.*

Proof. By Lemma 3, it is enough to consider only the case $W_{\varphi,\psi} : H^0_v \to H^0_v$. If $W_{\varphi,\psi}$ is Fredholm then $W^*_{\varphi,\psi}$ is also a Fredholm operator. From [1, Chapter III, Theorem 13] there are a bounded operator $S \in \mathscr{B}((H^0_v)^*, (H^0_v)^*)$ and a compact operator $K \in \mathscr{K}((H^0_v)^*)$, such that $SW^*_{\varphi,\psi} = I + K$. For $z \in \mathbb{D}$, we consider $\nu_z = \delta_z/\|\delta_z\|_{(H^\infty_v)^*}$. By Lemma 2, ν_z weakly converges to 0 in $(H^0_v)^*$ as $|z|$ goes to 1. Since K is a compact operator, $\|K\nu_z\| \to 0$ as $|z| \to 1$. Hence there is r, $0 < r < 1$, such that $\|K\nu_z\| < 1/2$ for all $z \in \mathbb{D}$ with $r < |z| < 1$. Since

$$W^*_{\varphi,\psi}\delta_z = \psi(z)\,\delta_{\varphi(z)}, \tag{15}$$

we obtain that

$$\begin{aligned} 1 - \|K\nu_z\| &= \|\nu_z\| - \|K\nu_z\| \leq \|\nu_z + K\nu_z\| = \left\|SW^*_{\varphi,\psi}\nu_z\right\| \\ &\leq \|S\|\, |\psi(z)|\, \frac{\left\|\delta_{\varphi(z)}\right\|}{\|\delta_z\|}. \end{aligned} \tag{16}$$

For $z \in \mathbb{D}$ with $r < |z| < 1$, we have

$$\|S\|\, |\psi(z)|\, \frac{\left\|\delta_{\varphi(z)}\right\|}{\|\delta_z\|} \geq 1 - \|K\nu_z\| \geq 1 - \frac{1}{2} = \frac{1}{2}. \tag{17}$$

Then for $z \in \mathbb{D}$ with $r < |z| < 1$,

$$\frac{1}{2\left\|\delta_{\varphi(z)}\right\|} \leq \frac{\|S\|\, |\psi(z)|}{\|\delta_z\|}. \tag{18}$$

From Lemma 2, for $f = \psi$, we know that $\lim_{|z|\to 1}(|\psi(z)|/\|\delta_z\|) = 0$. Hence, by (2),

$$\lim_{|z|\to 1} \widetilde{v}(\varphi(z)) = \lim_{|z|\to 1} \frac{1}{\left\|\delta_{\varphi(z)}\right\|} = 0. \tag{19}$$

It follows that $\lim_{|z|\to 1}|\varphi(z)| = 1$ and then φ is an inner function.

By [2, Corollary 3.28] it suffices to prove that φ is univalent to show that φ is an automorphism. Since $W_{\varphi,\psi} : H^0_v \to H^0_v$ is Fredholm, from Lemma 3 we obtain that $W_{\varphi,\psi} : H^\infty_v \to H^\infty_v$ is Fredholm. By Proposition 5, we get that φ is injective and the proof is complete. □

Lemma 8. *If φ is an automorphism on $\mathbb{D}$ and $C_\varphi : H^0_v \to H^0_v$ is bounded, then $C_{\varphi^{-1}} : H^0_v \to H^0_v$ is bounded and $C^{-1}_\varphi = C_{\varphi^{-1}}$.*

Proof. Since φ is an automorphism, it is known that there are $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$ such that $\varphi(z) = e^{i\theta}\varphi_a(z)$, where $\varphi_a(z) = (a-z)/(1-\bar{a}z)$. Then $\varphi^{-1}(z) = \varphi_a(e^{-i\theta}z)$. The Möbius transformation φ_a satisfies that $\varphi_a^{-1} = \varphi_a$. Now applying that $\widetilde{v}$ is radial, we obtain

$$\begin{aligned} \sup_{\omega\in\mathbb{D}} \frac{\widetilde{v}(\omega)}{\widetilde{v}(\varphi^{-1}(\omega))} &= \sup_{z\in\mathbb{D}} \frac{\widetilde{v}(\varphi(z))}{\widetilde{v}(z)} = \sup_{z\in\mathbb{D}} \frac{\widetilde{v}(\varphi_a(z))}{\widetilde{v}(z)} \\ &= \sup_{z\in\mathbb{D}} \frac{\widetilde{v}(z)}{\widetilde{v}(\varphi_a(z))} = \sup_{z\in\mathbb{D}} \frac{\widetilde{v}(z)}{\widetilde{v}(\varphi(z))}. \end{aligned} \tag{20}$$

As $C_\varphi : H^0_v \to H^0_v$ is bounded, by [16, Proposition 2.1] we obtain that $C_{\varphi^{-1}} : H^0_v \to H^0_v$ is bounded. An easy computation shows that $C^{-1}_\varphi = C_{\varphi^{-1}}$. □

We observe that Lemma 8 is also true if we replace H^0_v by H^∞_v.

Proposition 9. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function and $\psi \in H^0_v$. If C_φ and $W_{\varphi,\psi}$ are bounded on H^0_v or H^∞_v and $W_{\varphi,\psi}$ is also Fredholm, then C_φ and M_ψ are Fredholm.*

Proof. By Lemma 3 we only need to consider the case $E = H_v^0$. Proposition 7 implies that φ is an automorphism. Hence by the lemma above $C_\varphi^{-1} = C_{\varphi^{-1}} : H_v^0 \to H_v^0$ is bounded and is Fredholm. Thus the operator $M_\psi = C_\varphi^{-1} W_{\varphi,\psi} : H_v^0 \to H_v^0$ is well defined, bounded, and Fredholm. □

We obtain a characterization of Fredholm weighted composition operators by applying the results above.

Theorem 10. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function and $\psi \in H_v^0$. If C_φ is bounded on H_v^0 or H_v^∞, then the following assertions are equivalent:*

(1) $W_{\varphi,\psi} : H_v^0 \to H_v^0$ *is Fredholm.*

(2) $W_{\varphi,\psi} : H_v^\infty \to H_v^\infty$ *is Fredholm.*

(3) $M_\psi : H_v^0 \to H_v^0$ *and* $C_\varphi : H_v^0 \to H_v^0$ *are Fredholm.*

(4) $M_\psi : H_v^\infty \to H_v^\infty$ *and* $C_\varphi : H_v^\infty \to H_v^\infty$ *are Fredholm.*

(5) $\psi \in H^\infty$, $\exists \varepsilon > 0 : |\psi(z)| \geq \varepsilon \ \forall 1 > |z| \geq 1 - \varepsilon$, *and* φ *is an automorphism.*

Proof. We start with the observation that C_φ is bounded on H_v^0 if and only if it is bounded on H_v^∞ ([16, Proposition 2.1]). (1) ⇔ (2) It is Lemma 3. (1) ⇒ (3) and (2) ⇒ (4) are exactly Proposition 9. (3) ⇒ (1) and (4) ⇒ (2) follow by applying [1, Chapter III, Theorem 5]. (3) ⇒ (5) The condition on ψ follows by [17, Proposition 2.4]. That φ is an automorphism is deduced by Proposition 7 taking $\psi : \mathbb{D} \to \mathbb{C}$, $\psi(z) = 1$. (5) ⇒ (3) follows from [17, Proposition 2.4], Lemma 8, and [1, Chapter III, Theorem 13]. □

Our next result covers the case of composition operators on $H_{v_p}^0$, with $v_p(z) = (1 - |z|^2)^p$ and $0 < p < \infty$ considered in [10].

Corollary 11. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function. If C_φ is bounded on H_v^0 or H_v^∞, then the following assertions are equivalent:*

(1) $C_\varphi : H_v^0 \to H_v^0$ *is Fredholm.*

(2) $C_\varphi : H_v^\infty \to H_v^\infty$ *is Fredholm.*

(3) φ *is an automorphism.*

(4) $C_\varphi : H_v^0 \to H_v^0$ *is invertible.*

(5) $C_\varphi : H_v^\infty \to H_v^\infty$ *is invertible.*

Proof. (1) ⇔ (2) ⇔ (3) is a consequence of Theorem 10 taking $\psi : \mathbb{D} \to \mathbb{C}$, $\psi(z) = 1$. (3) ⇒ (4) and (3) ⇒ (5) follow by Lemma 8. (4) ⇒ (1) and (5) ⇒ (2) are a consequence of every invertible operator being Fredholm. □

4. Application to Composition Operators on Weighted Bloch Spaces

In this section, weighted composition operators on H_v^∞ are used to characterize Fredholm composition operators on weighted Bloch type spaces. These spaces are defined as follows:

$$\mathscr{B}_v^\infty = \left\{ f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} v(z) \left| f'(z) \right| < \infty \right\},$$
$$\mathscr{B}_v^0 = \left\{ f \in \mathscr{B}_v^\infty : \lim_{|z| \to 1} v(z) \left| f'(z) \right| = 0 \right\}. \tag{21}$$

$\mathscr{B}_v^\infty$ and $\mathscr{B}_v^0$ become Banach spaces under the norm

$$|||f|||_v = |f(0)| + \sup_{z \in \mathbb{D}} v(z) \left| f'(z) \right|. \tag{22}$$

For instance, if $v(z) = (1 - |z|^2)^\alpha$, where $\alpha > 0$, then we have the *Bloch type spaces*. In particular, for $\alpha = 1$ the spaces $\mathscr{B}_v^\infty$ and $\mathscr{B}_v^0$ are the usual *Bloch space* and the *little Bloch space*.

We also consider the Banach spaces defined by

$$\widetilde{\mathscr{B}}_v^\infty = \left\{ f \in H(\mathbb{D}) : f(0) = 0, \|f\|_v = \sup_{z \in \mathbb{D}} v(z) \left| f'(z) \right| < \infty \right\},$$
$$\widetilde{\mathscr{B}}_v^0 = \left\{ f \in \widetilde{\mathscr{B}}_v^\infty : \lim_{|z| \to 1} v(z) \left| f'(z) \right| = 0 \right\}. \tag{23}$$

These spaces are also called *weighted Bloch spaces*.

The map $S : \widetilde{\mathscr{B}}_v^\infty \to H_v^\infty$ given by $Sf = f'$ and $S^{-1} : H_v^\infty \to \widetilde{\mathscr{B}}_v^\infty$ defined by $S^{-1}f = F$, where $F' = f$, are isometric isomorphisms. The same is true if we replace the symbol ∞ by 0.

Our next result for $v(z) = 1 - |z|^2$ can be seen in [10].

Corollary 12. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function such that $C_\varphi : \mathscr{B}_v^0 \to \mathscr{B}_v^0$ is bounded. The following assertions are equivalent:*

(1) $C_\varphi : \mathscr{B}_v^0 \to \mathscr{B}_v^0$ *is Fredholm.*

(2) φ *is automorphism.*

(3) $C_\varphi : \mathscr{B}_v^0 \to \mathscr{B}_v^0$ *is invertible.*

Proof. (1) ⇔ (2) We take $\alpha = \varphi(0)$ and consider $\phi = \varphi_\alpha \circ \varphi$, where φ_α is the Möbius transformation $\varphi_\alpha(z) = (\alpha - z)/(1 - \overline{\alpha} z)$. Then $C_{\varphi_\alpha} : \mathscr{B}_v^0 \to \mathscr{B}_v^0$ is an isomorphism ([14, Lemma 6.2]). Hence $C_\varphi : \mathscr{B}_v^0 \to \mathscr{B}_v^0$ is Fredholm if and only if $C_\phi = C_\varphi \circ C_{\varphi_\alpha} : \mathscr{B}_v^0 \to \mathscr{B}_v^0$ is Fredholm. Since $\phi(0) = 0$, by Schwarz Lemma we have that $C_\phi : H_v^0 \to H_v^0$ is bounded, and $C_\phi : \mathscr{B}_v^0 \to \mathscr{B}_v^0$ is bounded if and only if $C_\phi : \widetilde{\mathscr{B}}_v^0 \to \widetilde{\mathscr{B}}_v^0$ is bounded. Moreover, since $\mathscr{B}_v^0 = \widetilde{\mathscr{B}}_v^0 \oplus \mathbb{C}$, $C_\phi : \mathscr{B}_v^0 \to \mathscr{B}_v^0$ is Fredholm if and only if $C_\phi : \widetilde{\mathscr{B}}_v^0 \to \widetilde{\mathscr{B}}_v^0$ is Fredholm. This is equivalent to $W_{\phi,\phi'} : H_v^0 \to H_v^0$ being Fredholm, because $C_\phi = S^{-1} \circ W_{\phi,\phi'} \circ S$, where S is the isomorphism given by $S : \widetilde{\mathscr{B}}_v^0 \to H_v^0$, $Sf = f'$. Finally, every automorphism on $\mathbb{D}$ is of the form $\mu(z) = e^{i\theta}((a - z)/(1 - \overline{a} z))$ with $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$, and applying Theorem 10, we obtain that $W_{\phi,\phi'}$

is Fredholm if and only if ϕ is an automorphism, which is equivalent to φ being an automorphism. (2) $\Rightarrow$ (3) follows by a similar argument to the proof of Lemma 8, applying in this case [22, Proposition 3]. (3) $\Rightarrow$ (1) is a consequence of [1, Chapter III, Theorem 13]. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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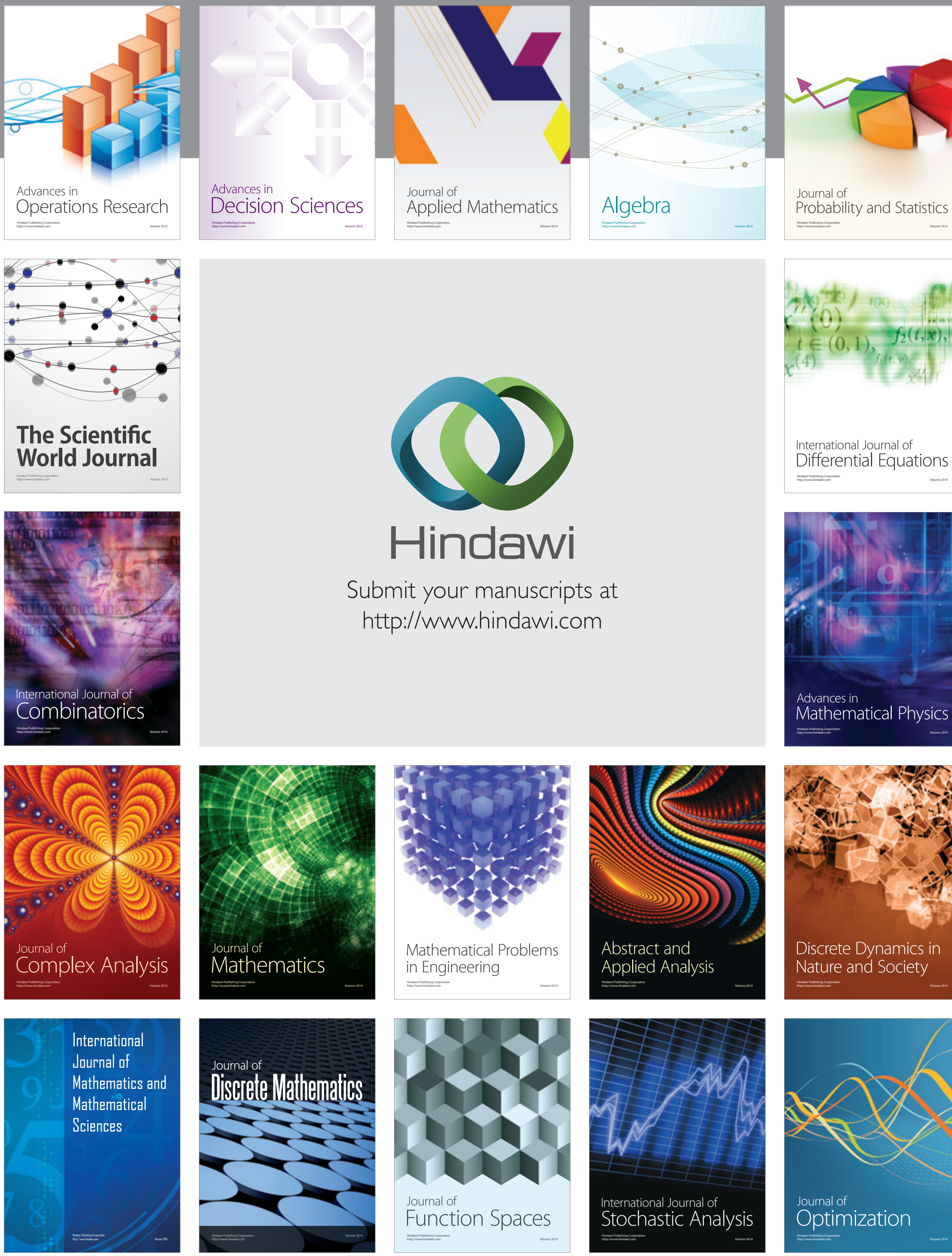
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