Mean ergodic operators on Banach spaces

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Introduction

The aim of this work is to review the main results about power bounded and (uniformly) mean ergodic operators on Banach spaces, including the theorems of Yosida and of Lin. These theorems are applied to the study of multiplication operators on weighted Banach spaces of analytic functions on the unit disc and on weighted Banach spaces of continuous functions. We conclude with an investigation of the mean ergodicity of the Cesàro operator on classical Banach sequence spaces.

In Chapter 1 relations between power boundedness, mean ergodicity and uniform mean ergodicity of operators defined in Banach spaces are shown. Some of them are presented as examples. Also, adding some conditions to the operators, one can ensure and characterize mean ergodicity (Yosida's Theorems 1.2.4 and 1.2.5) and uniform mean ergodicity (Lin's Theorem 1.3.2).

In Chapter 2 the space in which the operators are defined is not just Banach, but also a Grothendieck space with the Dunford-Pettis property. The objective of the Chapter is to prove Lotz's Theorem 2.2.2, which ultimately tells that mean ergodicity and uniform mean ergodicity for power bounded operators are equivalent in this kind of spaces.

Chapter 3 deals with the multiplication operator on weighted spaces of holomorphic functions defined on the unit disc of the complex plane. In the space of weighted vanishing functions it is shown that mean ergodicity of the multiplication operator is equivalent to power boundedness. In the space of weighted bounded functions it is shown that mean ergodicity of the operator implies power boundedness and that mean ergodicity is equivalent to uniform mean ergodicity.

In Chapter 4 the topic is also the multiplication operator but in this case it is defined on the space of continuous functions defined on a Hausdorff, locally compact, σ-compact, connected topological space. The main result is that mean ergodicity and uniform mean ergodicity are equivalent.

Lastly, in Chapter 5 the Cesàro operator is defined for the set of complex sequences.

Then it is shown that most of the usual sequence spaces are invariant by the Cesàro operator (so ergodicity can be studied for them). After some discussion about the spectrum of the operator in these spaces, it is shown that the operator is not mean ergodic for any of the spaces, but it is power bounded for some of them.

Preliminaries

If X is a Banach space, its topological dual is denoted by X'. If $x \in X$ and $x' \in X'$ the notations $x'(x)$ and $< x, x' >$ mean the same and are used indifferently.

The notation for the weak topology in X is $\sigma(X, X')$. A sequence $(x_n) \subset X$ converges to $x \in X$ for $\sigma(X, X')$ when $\lim_{n\to\infty} |x'(x_n) - x'(x)| = 0$ for each $x' \in X'$. Similarly the notations $\sigma(X', X)$ and $\sigma(X', X'')$ are defined.

If X, Y are both Banach spaces, $L(X, Y)$ denotes the linear and continuous operators from X onto Y. In particular, $L(X) = L(X, X)$. If $T \in L(X, Y)$, the adjoint operator is denoted T^t, i.e. $< x, T^t y' >=< Tx, y' >$. If $T \in L(X)$, then T^n denotes that the operator is iterated n times, i.e. $T^n = T \circ \overset{n}{\cdots} \circ T$.

We refer the reader to [9, Ch.7] for the spectral theory of operators on Banach spaces. $\sigma(T)$ denotes the spectrum of an operator T,

$$\sigma(T) = \{\lambda \in \mathbb{C} | T - \lambda I \text{ is not invertible}\},$$

and $r(T)$ is its spectral radius,

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|,$$

which, satisfies (see [9, Th. 7.5-5])

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.$$

Some spaces used in this work are the following:

$$l^\infty = \{(x_n) \in \mathbb{C}^{\mathbb{N}} \,:\, \|(x_n)\|_\infty = \sup_{n\in\mathbb{N}} |x_n| < \infty\},$$
$$c = \{(x_n) \in \mathbb{C}^{\mathbb{N}} \,:\, \lim_{n\to\infty} x_n \text{ exists }\},$$
$$c_0 = \{(x_n) \in \mathbb{C}^{\mathbb{N}} \,:\, \lim_{n\to\infty} x_n = 0\},$$
$$l^p = \{(x_n) \in \mathbb{C}^{\mathbb{N}} \,:\, \|(x_n)\|_p = \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{\frac{1}{p}} < \infty\}, \quad 1 \le p < \infty.$$

Recall that l^∞, c and c_0 are all Banach spaces when endowed with the norm $\|\cdot\|_\infty$ and l^p is a Banach space with the norm $\|\cdot\|_p$ for each $1 \leq p < \infty$.

All unexplained notation is as in [7], [9], [14], [16], [17].

Chapter 1

Ergodicity of operators in Banach spaces

1.1 Definitions and general results

Let $(X, \|\cdot\|)$ be a Banach space. Let $T \in L(X)$. We denote the n-th ergodic mean T_n as

$$T_n := \frac{1}{n}\sum_{m=1}^{n} T^m.$$

Definition 1.1.1. We say that an operator $T \in L(X)$ is **power bounded** if

$$\sup_{n\in\mathbb{N}} \|T^n\| < \infty.$$

We say it is **Cesàro bounded** if

$$\sup_{n\in\mathbb{N}} \|T_n\| < \infty.$$

Definition 1.1.2. We say that $T \in L(X)$ is **mean ergodic** if there exists $P \in L(X)$ such that

$$\lim_{n\to\infty} \|T_n x - Px\| = 0,$$

for each $x \in X$ (pointwise convergence).

We say it is **uniformly ergodic** if there exists $P \in L(X)$ such that

$$\lim_{n\to\infty} \|T_n - P\| = 0.$$

Proposition 1.1.3. *Let X be a Banach space and let $T \in L(X)$. Then:*

1. *If $r(T) < 1$, then T is power bounded.*
2. *If T is power bounded, $r(T) \leq 1$.*

PROOF. 1.: If $r(T) < 1$, then there exists $\alpha > 0$, with $r(T) < \alpha < 1$ therefore, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $\|T^n\| \leq \alpha^n < 1$. Thus $\{T^n\}_n$ is bounded and T is power bounded.

2.: If there exists $M > 0$ such that $\|T^n\| \leq M$ for each $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} M^{\frac{1}{n}} = 1,$$

and thus $r(T) \leq 1$. □

Example 1.1.4. We denote $\mathbb{R}^2$ as the Banach space $(\mathbb{R}^2, \|\cdot\|)$, in which we use the euclidean norm. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be linear with matrix

$$\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix},$$

then $r(T) < 1$. As seen before, T is power bounded, but $\|T\| > 1$, due to $\|T(0,1)\| = \|(1, 1/2)\| = \sqrt{5/4} > 1$. ■

Lemma 1.1.5. *Let $T \in L(X)$, then*

$$\frac{1}{n}T^n = T_n - \frac{n-1}{n}T_{n-1}.$$

Corollary 1.1.6. *Let $T \in L(X)$.*

1. *If T is mean ergodic, then*
$$\lim_{n\to\infty} \frac{1}{n}T^n x = 0,$$
for each $x \in X$.
2. *If T is uniformly ergodic, then*
$$\lim_{n\to\infty} \frac{1}{n}\|T^n\| = 0.$$

Proposition 1.1.7. *Every power bounded operator is Cesàro bounded.*

Proposition 1.1.8. *Every mean ergodic operator is Cesàro bounded.*

Proof. If $T \in L(X)$ is mean ergodic, then the sequence $(T_n x)$ is bounded for each $x \in X$. By the theorem of Banach-Steinhaus, the sequence of its norms $(\|T_n\|)$ is bounded, and thus T is Cesàro bounded. □

Example 1.1.9. In this example we show that $r(T) = 1$ does not necessarily imply power boundedness (see Proposition 1.1.3). Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be linear with matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Clearly $r(T) = 1$. We have that the matrix of T^n is:

$$\begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix},$$

and the matrix of T_n is:

$$\begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}.$$

Then, $T_n((0,1)) = (n+1, 1)$, whose norm is not bounded and hence T is not Cesàro bounded. Using Proposition 1.1.7, we have that T is not power bounded. ■

Example 1.1.10. The following example shows that the converse in Propositions 1.1.7 and 1.1.8 is not true.

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be linear with

$$T \rightsquigarrow \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix},$$

where the notation $T \rightsquigarrow A$ indicates that A is the matrix of the linear map T in the canonical basis. By induction we find

$$T^n \rightsquigarrow \begin{pmatrix} (-1)^n & (-1)^{n+1} 2n \\ 0 & (-1)^n \end{pmatrix},$$

for each $n \in \mathbb{N}$, and we see that T is *not power bounded*. Also, $\frac{1}{n} T^n(0,1) = \left((-1)^{n+1} 2, \frac{(-1)^n}{n} \right)$, which does not converge to 0 as $n \to \infty$. Thus T is *not mean ergodic*. However

$$nT_n = T + T^2 + \cdots + T^n \rightsquigarrow \begin{pmatrix} 0 & -n \\ 0 & 0 \end{pmatrix}, \quad \text{for } n \text{ even}$$

and

$$nT_n = T + T^2 + \cdots + T^n \rightsquigarrow \begin{pmatrix} -1 & n+1 \\ 0 & -1 \end{pmatrix}, \quad \text{for } n \text{ odd},$$

then

$$T_n \rightsquigarrow \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \text{for } n \text{ even}, \quad T_n \rightsquigarrow \begin{pmatrix} -1/n & 1+1/n \\ 0 & -1/n \end{pmatrix}, \quad \text{for } n \text{ odd},$$

and it is plain to see that T is *Cesàro bounded.* ■

Example 1.1.11. The following example shows that neither power boundedness, mean ergodicity nor both together imply uniform ergodicity.

Let $\lambda_i = 1 - \frac{1}{i+1} = \frac{i}{i+1}$, $i \in \mathbb{N}$. Let $1 < p < \infty$, define $T : l_p \longrightarrow l_p$ by $T((x_i)) = (\lambda_i x_i)$ for any $(x_i) \in l_p$. Then we have $T^n((x_i)) = (\lambda_i^n x_i)$ for every $n \in \mathbb{N}$. Now, using that $\lambda_i < 1$,

$$\|T^n((x_i))\|_p = \|(\lambda_i^n x_i)\|_p \leq \|(x_i)\|_p$$

and thus, $\|T^n\| \leq 1$ and T is *power bounded.*

We can see now that T is mean ergodic. Let $x \in l_p$,

$$\|T_n x\|_p^p = \frac{1}{n^p} \sum_{i=1}^{\infty} \left| \sum_{j=1}^{n} \lambda_i^j x_i \right|^p = \frac{1}{n^p} \sum_{i=1}^{\infty} \left| (\lambda_i + \lambda_i^2 + \cdots + \lambda_i^n) x_i \right|^p \leq \frac{i+1}{n^p} \|x\|_p^p,$$

since $\lambda_i + \lambda_i^2 + \cdots + \lambda_i^n \leq \frac{1}{1-\lambda_i} = i + 1$. This gives us that $\lim_n T_n x = 0$, for each $x \in l_p$ and T is *mean ergodic.* However, T is *not uniformly ergodic.* Indeed, take $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$ with 1 at the n-th position. Then,

$$\begin{aligned} \|T_n e_n\|_p &= \frac{1}{n}(\lambda_n + \lambda_n^2 + \cdots + \lambda_n^n) = \frac{1}{n} \frac{\lambda_n - \lambda_n^{n+1}}{1 - \lambda_n} = \frac{\lambda_n(1-\lambda_n^n)}{\frac{n}{n+1}} \\ &= 1 - \left(\frac{n}{n+1}\right)^n = 1 - (1 - \frac{1}{n+1})^n, \end{aligned}$$

which converges to $1 - \frac{1}{e}$. Since $\|e_n\|_p = 1$, T_n does not converge to 0 uniformly on the closed unit ball of l_p. ■

Lemma 1.1.12. *Let $T \in L(X)$ be mean ergodic (resp. uniformly mean ergodic) and let $Y \subset X$ be a closed subspace, which is T-invariant (i.e. $T(Y) \subset Y$). Then also $S := T_{|Y}$ is mean ergodic (resp. uniformly mean ergodic).*

Proof. Since Y is T-invariant, it is also T^n-invariant, and thus it is also invariant by $S_n = (T_n)_{|Y}$. Using that Y is closed, we get that both kind of limits (if they exist) are inside of Y, and thus S is mean ergodic (resp. uniformly mean ergodic). □

Lemma 1.1.13 (Sine). *Let $T \in L(X)$. Assume there exists a functional $u \in X'$, $u \neq 0$ such that $T^t u = u$ (i.e. it is invariant by the adjoint operator of T). If T is mean ergodic, then* $\ker(I - T) \neq \{0\}$.

Proof. Suppose $\ker(I-T)=\{0\}$. Take $x\in X$ such that $u(x)=1$, then, using that T is mean ergodic, there exists $x_0\in X$ such that

$$x_0=\lim_{n\to\infty}T_nx=\lim_{n\to\infty}\sum_{m=1}^{n}T^mx.$$

We have

$$Tx_0=\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}T^{m+1}x=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{m=1}^{n}T^mx-\frac{1}{n}Tx+\frac{1}{n}T^{n+1}x\right]=x_0.$$

Thus, $x_0\in\ker(I-T)$ and $x_0=0$. Then we have

$$0=\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}T^mx,$$

and, using that $u(T^mx)=[(T^m)^tu]\,x=u(x)$ for each $m\in\mathbb{N}$, we get

$$0=u(0)=\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}u(T^mx)=u(x)=1.$$

This contradiction implies $\ker(T-I)\neq\{0\}$. □

Example 1.1.14. Let $T:c_0\longrightarrow c_0$, defined by $T(x_1,x_2,\ldots)=(x_1,x_1,x_2,\ldots)$. We show that T is not mean ergodic but it is power bounded.

Firstly, we have that

$$T(a)=a\iff a=\text{const}\iff a=0,$$

then we get $\ker(I-T)=\{0\}$.

Now, we use $e_1=(1,0,0,\ldots)\in l^1=c_0'$. We have

$$(T^te_1)(x)=<e_1,Tx>=<e_1,(x_1,x_1,x_2,\ldots)>=x_1=<e_1,x>,$$

and thus, $T^te_1=e_1$. By the previous lemma, T is *not mean ergodic*. However, we have that

$$\|T^nx\|=\|x\|$$

and, thus, T is *power bounded*. ■

Example 1.1.15. Let $X=C(\overline{\mathbb{D}})$ the space of continuous functions in the closed unit disc in $\mathbb{C}$, endowed with the supremum norm. Let $T:X\longrightarrow X$ be a linear map defined by $(Tf)z=zf(z)$ for each $f\in X$.

We have

$$T^n f(z) = z^n f(z) \quad \forall f \in X \quad \forall z \in \overline{\mathbb{D}} \quad \forall n \in \mathbb{N},$$

therefore, $\|T^n f\| \le \|f\|$ for each $n \in \mathbb{N}$ and for each $f \in X$, and thus T is *power bounded.* Now, we prove that T is not mean ergodic.

Let $f \in X$ such that $Tf = f$. Then, we get

$$zf(z) = f(z)\,\forall z \in \overline{\mathbb{D}} \Rightarrow (z-1)f(z) = 0\,\forall z \in \overline{\mathbb{D}} \Rightarrow f(z) = 0\,\forall z \in \overline{\mathbb{D}} \setminus \{1\},$$

and, then $f \equiv 0$ by continuity and $\ker(T - I) = \{0\}$.

Now, let $\delta_1 : X \longrightarrow \mathbb{C}$ defined by $\delta_1(f) = f(1)$. Clearly, $\delta_1 \in (C(\overline{\mathbb{D}}))'$. We have

$$[T^t(\delta_1)](f) =< \delta_1, Tf >= \delta_1(Tf) = 1f(1) = f(1) = \delta_1(f)$$

and as $\delta_1 \neq 0$ and by the previous lemma, T is *not mean ergodic.* ■

1.2 Yosida's mean ergodic theorem

Lemma 1.2.1. *Let $T : X \longrightarrow Y$ a linear map between normed spaces with open unit balls B_X and B_Y respectively. Then the following conditions are equivalent:*

1. *T is open from X onto Y.*
2. *There exists $\delta > 0$ such that $\delta B_Y \subseteq T(B_X)$.*
3. *There exists $K > 0$ such that for any $y \in Y$, there exists $x \in X$ with $Tx = y$ and $\|x\| \le K\|y\|$.*

Proof. $\boxed{1. \Rightarrow 2.}$ Since $T(B_X)$ is open and $0 \in T(B_X)$, there exists $\delta > 0$ such that $\delta B_X = B(0, \delta) \subset T(B_X)$.

$\boxed{2. \Rightarrow 3.}$ Fix $y \in Y$, $y \neq 0$ and define $y' = \frac{\delta y}{2\|y\|}$. Then, $y' \in \delta B_Y \subset T(B_X)$, thus there exists $x' \in B_X$ such that $Tx' = y'$. Set $x = \frac{2\|y\|}{\delta} x'$. Clearly $Tx = y$ and $\|x\| \le \frac{2}{\delta}\|y\|$. 3. holds with $K = \frac{2}{\delta}$.

$\boxed{3. \Rightarrow 1.}$ Fix $U \subseteq X$ open and $x \in U$. There exists $\varepsilon > 0$ such that $x + B(0, \varepsilon) = B(x, \varepsilon) \subset U$. To show that $T(U)$ is open, we prove that $\big(Tx + B(0, \frac{\varepsilon}{K})\big) = B(Tx, \frac{\varepsilon}{K}) \subset T(U)$.

Take $y \in Y$ with $\|y\| < \frac{\varepsilon}{K}$, then by 3., there exists $z \in X$ with $Tz = y$ and $\|z\| < K\frac{\varepsilon}{K} = \varepsilon$. Thus, $x + z \in U$ and $Tx + y = Tx + Tz = T(x + z) \in T(U)$. Then $T(U)$ is open. □

Theorem 1.2.2. *Let $T \in L(X)$ with $\|T\| < 1$, then $(I - T)^{-1}$ exists.*

PROOF. Using that $\|T^j\| \le \|T\|^j$, we find

$$\sum_{j=0}^{\infty} \|T^j\| \le \sum_{j=0}^{\infty} \|T\|^j$$

and the second sum is convergent because $\|T\| < 1$, which gives us that

$$S = \sum_{j=0}^{\infty} T^j$$

is absolutely convergent. Since X is complete, also $L(X)$ is complete, thus the previous sum es convergent. Now we shall show that $S = (I - T)^{-1}$:

$$(I - T)(I + T + T^2 + \ldots + T^n) = (I + T + T^2 + \ldots + T^n)(I - T) = I - T^{n+1}$$

and taking the limit over n and using that $\|T\| < 1$, we get

$$(I - T)S = S(I - T) = I,$$

which finishes the proof. □

Lemma 1.2.3. *Let $T \in L(X)$ with $\lim_n \frac{1}{n} T^n x = 0$, for each $x \in X$. Then*

$$(I - T)^k X \cap \ker(I - T) = \{0\},$$

for each $k \in \mathbb{N}$.

PROOF. Recall that $T_n(I - T) = \frac{1}{n}(T - T^{n+1})$. Let $y \in (I - T)X$. Then, there exists $x \in X$ with $y = (I - T)x$ and we have

$$T_n y = T_n(I - T)x = \frac{1}{n}(Tx - T^{n+1}x) = \frac{1}{n}Tx - \frac{1}{n}T^{n+1}x,$$

which converges to 0 as $n \to \infty$ by assumption.

Now let $y \in (I - T)X \cap \ker(I - T)$, then $y = Ty$ and, clearly, $y = T^m y$, for each $m \in \mathbb{N}$. Thus,

$$T_n y = \frac{1}{n} \sum_{m=1}^{n} T^m y = y$$

and, as seen before, $T_n y \to \infty$ and then, $y = 0$. This way, we have seen that $(I - T)X \cap \ker(I - T) = \{0\}$. Also for each $k \in \mathbb{N}$, we have $0 \in (I - T)^k X \subseteq (I - T)X$ and this implies the assertion. □

Theorem 1.2.4 (Yosida). *Let $T \in L(X)$ and assume:*

a) $\lim_n \frac{1}{n} T^n x = 0$, *for each $x \in X$, and*

b) T is Cesàro bounded ($\sup_n \|T_n\| < \infty$).

Then,

$$\overline{(I-T)X} = \{x \in X \,:\, T_n x \to 0 \text{ as } n \to \infty\},$$

and

$$\overline{(I-T)X} \cap \ker(I-T) = \{0\}.$$

Proof. Let $Z = \{x \in X \,:\, T_n x \to 0 \text{ as } n \to \infty\}$. If $y \in (I-T)X$, using the first part of the proof of the Lemma 1.2.3, we conclude that $y \in Z$. Now let $z \in \overline{(I-T)X}$. By $b)$, there exists $C > 0$ such that $\|T_n y\| \le C\|y\|$ for all $y \in X$ and for all $n \in \mathbb{N}$. Fix $\varepsilon > 0$, then there exists $w \in (I-T)X \subset Z$ with $\|z - w\| < \frac{\varepsilon}{C}$. We get

$$\|T_n z\| \le \|T_n w\| + \|T_n(z-w)\| \le \|T_n w\| + C\|z-w\| < \|T_n w\| + \varepsilon.$$

As $w \in Z$, for a big enough n, we have $\|T_n z\| < 2\varepsilon$ and thus $\lim_n \|T_n z\| = 0$. This way we have proven that $\overline{(I-T)X} \subseteq Z$. To show the other inclusion, let $x \in Z$ and fix $n \in \mathbb{N}$. Then,

$$x - T_n x = x - \frac{1}{n}\sum_{m=1}^{n} T^m x = \frac{1}{n}\sum_{m=1}^{n}(I - T^m)x = \frac{1}{n}\sum_{m=1}^{n}(I-T)[(I + T + \cdots + T^{m-1})x]$$

and thus $x - T_n x \in (I-T)X$. As $x \in Z$, taking limit as $n \to \infty$, we get $x \in \overline{(I-T)X}$. The proof of $\overline{(I-T)X} \cap \ker(I-T) = \{0\}$ is similar to the proof of Lemma 1.2.3. □

We recall that a topological space X is called **sequentially compact** if every sequence in X has a convergent subsequence.

Theorem 1.2.5 (Mean ergodic theorem). *Let $T \in L(X)$ and assume:*

a) $\lim_n \frac{1}{n} T^n x = 0$, *for each $x \in X$, and*

b) for each $x \in X$ the set $\{T_n x \,:\, n \in \mathbb{N}\}$ is relatively $\sigma(X, X')-$sequentially compact.

Then, the following assertions are satisfied:

1. *The limit $Px := \lim_n T_n x$ exists for all $x \in X$ (i.e. T is mean ergodic), and*

2. *$Px \in \ker(I-T)$ for all $x \in X$.*

Remark 1.2.6. Assumption b) can be changed to the set $\{T_n x : n \in \mathbb{N}\}$ being relatively $\sigma(X, X')-$compact, due to the theorem of Eberlein-Šmulian [7, Ch.24].

Proof (of the Theorem 1.2.5). Firstly we note that assertion b) implies that $(T_n x)$ is bounded in X for each $x \in X$, and by the theorem of Banach-Steinhaus, T is Cesàro bounded, and thus we can use the Theorem 1.2.4 during the proof.

Let $x \in X$, then there exists an increasing subsequence $(n_k)_k \subseteq \mathbb{N}$ and $x_0 \in X$ such that $\lim_k T_{n_k} x = x_0$ for the weak topology $\sigma(X, X')$. We want to show that $x_0 = Px$. Now, we have, using the Theorem 1.2.4,

$$\lim_{n\to\infty} T_n(I - T)x = 0.$$

This implies that, for all $f \in X'$,

$$\lim_{k\to\infty} < TT_{n_k} x, f >= \lim_{k\to\infty} < T_{n_k}, f >=< x_0, f > .$$

Thus, for all $f \in X'$,

$$< x_0, f >= \lim_{k\to\infty} < T_{n_k} x, T^t f >=< x_0, T^t f >=< Tx_0, f >$$

and, then, $Tx_0 = x_0$. We shall use this in the rest of the proof. Moreover, when we prove 1., assertion 2. will be satisfied. Now, for $m \in \mathbb{N}$,

$$T^m x = T^m x_0 + T^m(x - x_0) = x_0 + T^m(x - x_0),$$

and then, for every $n \in \mathbb{N}$,

$$T_n x = \frac{1}{n} \sum_{m=1}^{n} T^m x = x_0 + T_n(x - x_0).$$

We also have

$$x - x_0 = x - \sigma - \lim_{k\to\infty} T_{n_k} x = \sigma - \lim_{k\to\infty} (x - T_{n_k} x),$$

where $\sigma-\lim$ indicates the limit for the weak topology $\sigma(X, X')$. But $x - T_{n_k} x \in (I - T)X$, as we saw in the proof of the Theorem 1.2.4, and then $x - x_0 \in \overline{(I - T)X}$ (because the clausure for the norm topology of X coincides with the clausure for the weak topology $\sigma(X, X')$ for convex sets). By the Theorem 1.2.4, we have that $\lim_n T_n(x - x_0) = 0$, and thus,

$$Px = \lim_{n\to\infty} T_n x = x_0.$$

In particular, $TPx = Tx_0 = x_0 = Px$. □

Corollary 1.2.7. *Let X be a reflexive Banach space and $T \in L(X)$ and assume:*

a) $\lim_n \frac{1}{n} T^n x = 0$*, for each* $x \in X$*, and*

b) T *is Cesàro bounded* ($\sup_n \|T_n\| < \infty$)*.*

Then, T *is mean ergodic.*

Proof. As X is reflexive, every bounded set in X is relatively weakly sequentially compact, and we can use Theorem 1.2.5. Fix $x \in X$. By b), the sequence $(T_n x)$ is bounded, and thus it is relatively weakly sequentially compact, hence there exists an increasing subsequence $(n_k)_k \subseteq \mathbb{N}$ and $x_0 \in X$ such that $x_0 = \sigma(X, X') - \lim_n T_{n_k} x$. Then by the Theorem 1.2.5, the limit $Px = \lim_n T_n x$ exists, and P is well-defined. By its definition it is clear that P is linear. Using b) we know that P is also continuous. □

Theorem 1.2.8. *Let $T \in L(X)$ be mean ergodic and denote $Px = \lim_{n\to\infty} T_n x$ for each $x \in X$. Then, $P : X \longrightarrow X$ satisfies:*

1. $P = P^2 = TP = PT$. *In particular P is a projection.*
2. $P(X) = \ker(I - T)$,
3. $\ker P = \overline{(I - T)X} = (I - P)X$.

Moreover, $X = \overline{(I - T)X} \oplus \ker(I - T)$.

Proof. We prove 1. Let $x \in X$, then,

$$(I - T)(Px) = (I - T) \lim_{n\to\infty} T_n x = \lim_{n\to\infty} \frac{1}{n}(T - T^{n+1})x = 0,$$

then $Px = TPx$ and $P = TP$. This gives us that $T^n P = P$ for all $n \in \mathbb{N}$, and $T_n P = P$, for all $n \in \mathbb{N}$, and we get

$$P^2 x = \lim_{n\to\infty} T_n P x = \lim_{n\to\infty} Px = Px,$$

which implies $P^2 = P$. Lastly,

$$P(I - T)x = \lim_{n\to\infty} \frac{1}{n}(T - T^{n+1})x = 0$$

and thus, $P = PT$.

For the proof of 2., let $x \in \ker(I - T)$, then $x = Tx$, and $T^n x = x$ and $T_n x = x$. Thus $Px = \lim_n T_n x = x$, which gives us $x \in P(X)$.

Now let $x \in P(X)$. Then there exists $z \in X$ such that $x = Pz = P^2 z = P(Pz) = Px$. We get

$$Tx = T(Px) = Px = x,$$

which tells us that $x \in \ker(I - T)$.

Now we prove 3. Let $x \in \ker P$, then $Px = 0$. Thus $(I - P)x = x - Px = x$, therefore $x \in (I - P)X$.

Now let $x \in (I - P)X$, then, $x = (I - P)z$ for some $z \in X$. One gets

$$Px = Pz - P^2 z = Pz - Pz = 0$$

and thus $x \in \ker P$. We have proven that $\ker P = (I - P)X$. Now we prove that both are equal to $\overline{(I - T)X}$. Let $x \in (I - P)X$, then $x = (I - P)z$ for some $z \in X$. We have

$$(I - T_n)z = (I - T)\left[\frac{1}{n}\sum_{r=0}^{n-1}\sum_{j=0}^{r} T^j z\right],$$

and then $(I - T_n)z \in (I - T)X$. Now,

$$x = (I - P)z = z - \lim_{n\to\infty} T_n z = \lim_{n\to\infty}(I - T_n)z \in \overline{(I - T)X}.$$

Let $x \in (I-T)X$, then $x = (I-T)z$ for some $z \in X$. Thus, $Px = Pz - TPz = Pz - Pz = 0$ and $z \in \ker P$. Let $w \in \overline{(I - T)X}$, then there exists a sequence $(x_n) \subseteq (I - T)X$ such that $\lim x_n = w$. We conclude,

$$Pw = P(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} 0 = 0,$$

where we have used the continuity of P. We have obtained $w \in \ker P$ and thus, assertion 3.. □

Theorem 1.2.9 (Strong ergodic theorem). *Let $T \in L(X)$, then there exists $P \in L(X)$ such that $\lim_n T_n x = Px$ for all $x \in X$, if and only if the the following assertions are satisfied:*

a) $\lim_n \frac{1}{n} T^n x = 0$ for all $x \in X$, and

b) for each $x \in X$ the set $\{T_n x : n \in \mathbb{N}\}$ is relatively $\sigma(X, X')-$sequentially compact.

Proof. Firstly, we suppose that there exists $P \in L(X)$ such that for all $x \in X$, $\lim_n T_n x = Px$. Then,

$$\frac{1}{n}T^n = T_n - \frac{n-1}{n}T_{n-1},$$

which converges to 0, and thus $a)$ is satisfied. We also have $T_n x \to Px$ for all $x \in X$, and this implies that $(T_n x)$ is relatively sequentially compact in X and then, $(T_n x)$ is relatively sequentially $\sigma(X, X')-$compact, satisfying $b)$.

For the converse, we apply Theorem 1.2.5. □

1.3 Lin's theorem

Lemma 1.3.1. *Let $T \in L(X)$ with $\ker(I - T) = \{0\}$ and with*

$$\lim_{n\to\infty} \left\| \frac{1}{n} T^n \right\| = 0,$$

then, the following assertions are equivalent:

1. *$I - T_n$ is surjective for some $n \in \mathbb{N}$.*
2. *$I - T$ is surjective.*
3. *$\lim_{n\to\infty} \|T_n\| = 0$ (T is uniformly mean ergodic).*

Proof. $\boxed{3. \Rightarrow 1.}$ Because of the limit being 0, there exists an $n \in \mathbb{N}$ such that $\|T_n\| < 1$. This implies that $I - T_n$ is an isomorphism by Theorem 1.2.2 and, in particular, it is surjective.

$\boxed{1. \Rightarrow 2.}$ Let $y \in X$, by 1. there exist $x \in X$ such that $(I - T_n)x = y$. Thus,

$$y = (I - T_n)x = (I - T) \left[\frac{1}{n} \sum_{r=0}^{n-1} \sum_{j=0}^{r} T^j x \right].$$

and $(I - T)$ is surjective.

$\boxed{2. \Rightarrow 3.}$ We have that $I - T : X \longrightarrow X$ is injective by hypothesis and it is onto by 2., also it is continuous. By the open mapping theorem its inverse $(I - T)^{-1} : X \longrightarrow X$ is continuous. If B is the closed unit ball of X, then $C = (I - T)^{-1} B$ is bounded. Let $M = \sup_{x\in C} \|x\|$. We have

$$\begin{aligned} \|T_n\| &= \sup_{z\in B} \|T_n z\| = \sup_{x\in C} \|(I - T)T_n x\| = \sup_{x\in C} \left\| \frac{1}{n}(T - T^{n+1})x \right\| \\ &\le \frac{1}{n} \sup_{x\in C} \|Tx\| + \frac{n+1}{n} \sup_{x\in C} \left\| \frac{1}{n+1} T^{n+1} x \right\| \le \frac{M}{n} \|T\| + 2M \frac{\|T^{n+1}\|}{n+1} \end{aligned}$$

and, thus $\lim_n \|T_n\| = 0$. □

Theorem 1.3.2 (Lin). *Let $T \in L(X)$ satisfy*

$$\lim_{n\to\infty} \frac{\|T^n\|}{n} = 0.$$

The following assertions are equivalent:

1. *T is uniformly mean ergodic.*
2. *$(I-T)X$ is closed and $X = \ker(I-T) \oplus (I-T)X$.*
3. *$(I-T)^2X$ is closed.*
4. *$(I-T)X$ is closed.*
5. *$(I-T)^kX$ is closed for every $k \in \mathbb{N}$.*
6. *$(I-T)^kX$ is closed for some $k > 2$.*

Proof. In the proof we use the following notations: $Y = \overline{(I-T)X}$ and $S = T_{|Y}$.

$\boxed{1. \Rightarrow 2.}$ There exist $P \in L(X)$ such that

$$\lim_{n\to\infty} \|T_n - P\| = 0.$$

By the Theorem 1.2.4, we have $X = \ker(I-T) \oplus Y$. We want to show that $T(Y) \subseteq Y$, so we can iterate S. To do so we shall prove first that $T(I-T)X \subseteq (I-T)X$. Let $x \in X$, then

$$T(I-T)x = (I-T)(Tx) \in (I-T)X.$$

Using this, we have

$$T(Y) = T(\overline{(I-T)X}) \subseteq \overline{T(I-T)X} \subseteq \overline{(I-T)X} = \overline{Y} = Y,$$

as we wanted. By the Theorem 1.2.4 we have $Y = \ker P$. Due to $\lim_n \|T_n - P\| = 0$, we conclude that $\lim_n \|S_n\| = 0$. Now, S satisfies the assumptions and hence the conclusion 3. of the Lemma 1.3.1, so $(I-S)Y = Y$. Finally,

$$Y = (I-S)Y = (I-T)Y \subseteq (I-T)X \subseteq Y$$

gives us that $Y = (I-T)X$ and $(I-T)X$ is closed.

$\boxed{2. \Rightarrow 3.}$ We have $Y = (I-T)X$, we want to show that $(I-T)^2X = Y$. The inclusion $(I-T)^2X = (I-T)[(I-T)X] \subseteq Y$ is direct. To show the other inclusion, let $y \in Y$. Then $y = (I-T)x$ for some $x \in X$. By 2. we can write x as

$$x = x_0 + x_1, \quad \text{with } Tx_0 = x_0 \text{ and } x_1 \in Y$$

and we get $y = (I-T)x = (I-T)x_1 \in (I-T)Y = (I-T)^2X$ and $Y = (I-T)^2X$, which is closed.

$\boxed{3. \Rightarrow 4.}$ Recall that $Y = \overline{(I-T)X}$. We want to show that $(I-T)X = Y$. Firstly we check that $(I-T)Y = (I-T)^2X$. The first inclusion is:

$$(I-T)^2X = (I-T)(I-T)X \subseteq (I-T)\overline{(I-T)X} = (I-T)Y.$$

On the other hand, we can apply 3. to get

$$(I-T)Y = (I-T)\overline{(I-T)X} \subseteq \overline{(I-T)^2X} = (I-T)^2X.$$

This also tells us that $(I-T)Y = \overline{(I-T)Y}$ because $(I-T)^2X$ is closed.

Since $T(Y) \subseteq Y$, $S : Y \longrightarrow Y$ is well defined. Every $y \in (I-T)X$ satisfies that $\lim_n S_n y = 0$ as we saw in the proof of Yosida's Theorem 1.2.4.

Now we show that $(I-T)X \subseteq \overline{(I-S)Y}$. Let $y \in Y$ with $\lim_n S_n y = 0$. Then

$$y - S_n y = (I-S)[\frac{1}{n}\sum_{m=1}^{n}(I + S + \cdots + S^{m-1})y] \in (I-S)Y,$$

for each $n \in \mathbb{N}$ then we have $y = \lim_n (y - S_n y) \in \overline{(I-S)Y}$. And, finally, $(I-T)X \subseteq \overline{(I-S)Y} = (I-S)Y = (I-T)^2X$. We also know that $(I-T)^2X \subseteq (I-T)X$, then both are equal and $(I-T)X$ is closed.

$\boxed{4. \Rightarrow 1.}$ In this case we have $Y = (I-T)X$, and it is a Banach space. The operator $I - T : X \longrightarrow Y$ is surjective and continuous, then, by the open mapping theorem, $(I-T) : X \longrightarrow Y$ is open. If we apply the Lemma 1.2.1, we find that there exists $K > 0$ such that for each $y \in Y$, there exists $z \in X$ such that

$$(I-T)z = y \text{ and } \|z\| \leq K\|y\|.$$

Now, let $y \in Y$ and select $z \in X$ as above, then we have

$$\|T_n y\| = \|T_n(I-T)z\| \leq \frac{1}{n}\|T - T^{n+1}\|\|z\| \leq \frac{K}{n}(\|T\| + \|T^{n+1}\|)\|y\|.$$

Recall that $S = T_{|Y} : Y \longrightarrow Y$ is well-defined and continuous. By the previous inequalities, taking supremum over $\|y\| \leq 1$, one finds that $\lim_n \|S_n\| = 0$.

Applying the Lemma 1.2.3, we find

$$(I-T)X \cap \ker(I-T) = Y \cap \ker(I-T) = \{0\}$$

and therefore $\ker(I-S) = \{0\}$. Now, applying Lemma 1.3.1 to $S \in L(Y)$, we conclude that $I - S : Y \longrightarrow Y$ is surjective. Thus, $I - S$ is an isomorphism. Also, $(I-T)X =$

$Y = (I-S)Y = (I-T)^2X$. Therefore, for each $x \in X$ there exists $y \in Y$ such that $(I-T)x = (I-T)y$ and thus $y = (I-S)^{-1}[(I-T)x]$. Since $(I-S)^{-1} : Y \longrightarrow Y$ is continuous,

$$\|y\| \leq \|(I-S)^{-1}\|\|(I-T)x\|.$$

We also have that $(I-T)(x-y) = 0$, and thus $T(x-y) = x-y$ and $T^m(x-y) = x-y$, for each $m \in \mathbb{N}$, which gives us $T_m(x-y) = x-y$, for all $m \in \mathbb{N}$.

We define the map $P : X \longrightarrow X$ by $Px = x-y$, where y is the unique element defined as $y = (I-S)^{-1}[(I-T)x]$. This map is a well-defined and continuous operator. Our aim now is to show that

$$\lim_{n\to\infty} \|T_n - P\| = 0.$$

To do so, let $x \in X$ and use y selected by P,

$$\begin{aligned}\|(T_n-P)x\| = \|T_nx - Px\| &= \|T_nx - (x-y)\| = \|T_nx - T_n(x-y)\| = \|T_ny\| \\ &= \|T_n(I-S)^{-1}(I-T)x\| \leq \|(I-S)^{-1}\|\|T_n(I-T)x\| \\ &\leq \frac{K}{n}\|(I-S)^{-1}\|(\|T\| + \|T^{n+1}\|)\|x\|.\end{aligned}$$

If we now take supremum over $\|x\| \leq 1$ for $x \in X$, we get

$$\|(T_n - P)\| \leq \frac{K}{n}\|(I-S)^{-1}\|(\|T\| + \|T^{n+1}\|)$$

and this converges to 0 as $n \to \infty$ and thus T_n converges to P and T_n is uniformly mean ergodic.

$\boxed{2. \Rightarrow 5.}$ Let $k \in \mathbb{N}$ with $k \geq 2$. We want to show that $(I-T)^kX = (I-T)^{k-1}X$. It is clear that $(I-T)^kX \subseteq (I-T)^{k-1}X$. Now let $z \in (I-T)^{k-1}X$, then, there exists $x \in X$ such that $z = (I-T)^{k-1}x$ and by 2. $x = x_0 + x_1$ with $x_0 = Tx_0$ and $x_1 \in (I-T)X$. Then, we have

$$z = (I-T)^{k-1}x = (I-T)^{k-2}(I-T)x = (I-T)^{k-1}x_1 \in (I-T)^{k-1}(I-T)X.$$

This way we have shown that $(I-T)^{k-1}X = (I-T)^kX$. Iterating this process, we finally have

$$(I-T)^kX = (I-T)^{k-1}X = \cdots = (I-T)^2X = (I-T)X$$

and $(I-T)^kX$ is closed for every $k \in \mathbb{N}$.

$\boxed{6. \Rightarrow 4.}$ Our aim is to show that $(I-T)^{k-1}X$ is closed. First we shall see that $(I-T)^{k-1}X + \ker(I-T)$ is closed. Indeed, let $y_n = (I-T)^{k-1}x_n + z_n$ with $\lim_n y_n = y$, and $(x_n)_n \subseteq X$ and $z_n = Tz_n$ for each $n \in \mathbb{N}$. Then $(I-T)y_n = (I-T)^kx_n$ converges to $(I-T)y$

as $n \to \infty$. But $((I-T)^k x_n) \subseteq (I-T)^k X$, which is closed, then $(I-T)y \in (I-T)^k X$. Then there exists $w \in X$ with $(I-T)y = (I-T)^k w$ and thus $y-(I-T)^{k-1}w \in \ker(I-T)$ and we have

$$y = (I-T)^{k-1}w + (y-(I-T)^{k-1}w) \in (I-T)^{k-1}X + \ker(I-T),$$

which shows that $(I-T)^{k-1}X + \ker(I-T)$ is closed.

Now, using that $(I-T)^{k-1}X + \ker(I-T)$ is closed, $\ker(I-T)$ is closed and $(I-T)^{k-1}X \cap \ker(I-T) = \{0\}$, we get, using [16, Theorem 5.8.], that $(I-T)^{k-1}X$ is closed. Iterating this argument, we find that $(I-T)X$ is closed.

5. $\Rightarrow$ 6. Trivial. $\square$

Chapter 2

Ergodicity of operators in Banach-Grothendieck spaces with the Dunford-Pettis property

2.1 Definitions and general results

The main objective of this chapter is to find some results analogous to Yosida's Theorem 1.2.4, but for uniform mean ergodicity instead of just mean ergodicity. To do so we restrict the space in which we work. We are still in Banach spaces, but we ask them to be Grothendieck spaces with the Dunford-Pettis property. Even if the results given are stated for Banach spaces X, they actually work for Fréchet spaces, with similar proofs. The results in the chapter are mainly taken from Heinrich P. Lotz's article [11]. In this section T_j will not necessarily mean the j-th ergodic mean of $T \in L(X)$, unless it is otherwise stated.

We start with some definitions.

Definition 2.1.1. A Banach space X is a **Grothendieck space** if any sequence $(x'_j) \subset X'$ which is convergent to 0 for the weak topology $\sigma(X', X)$ is also convergent to 0 for the weak topology $\sigma(X', X'')$.

Definition 2.1.2. A Banach space X has the **Dunford-Pettis property** if for any sequence $(x_j) \subset X$ which is convergent to 0 for the weak topology $\sigma(X, X')$ and any sequence $(x'_j) \subset X'$ which is convergent to 0 for the weak topology $\sigma(X', X'')$ one gets

$$\lim_{j\to\infty} < x_j, x'_j >= 0.$$

Firsly we prove some lemmata, which are true for sequences having some properties, which actually the sequence of ergodic means satisfies.

Lemma 2.1.3. *Let X be a Banach space, which is a Grothendieck space and let $(T_j) \subset L(X)$ be a sequence with $\lim_{j\to\infty} T_j x = 0$ for all $x \in X$. Then for each bounded sequence $(x'_j) \subset X'$, we have $\lim_{j\to\infty} T_j^t x'_j = 0$ for the weak topology $\sigma(X', X'')$.*

Proof. Let $(x'_k) \subset X'$ be bounded. There is $M > 0$ such that $\|x'_k\| \leq M$ for each $k \in \mathbb{N}$. Given $j \in \mathbb{N}$ and $x \in X$, we have $| < T_j x, x'_k > | \leq M\|T_j x\|$ for each $k \in \mathbb{N}$. Thus $\sup_k | < T_j x, x'_k > | \leq M\|T_j x\|$. As $T_j x \to 0$ for each $x \in X$, we have

$$\lim_{j\to\infty} \sup_{k\in\mathbb{N}} | < T_j x, x'_k > | = 0$$

for each $x \in X$. Thus, for each $x \in X$, $| < T_j x, x'_j > | \to 0$, and then $T_j^t x'_j \to 0$ for $\sigma(X', X)$. But X is a Grothendieck space, thus, $T_j^t x'_j \to 0$ for $\sigma(X', X'')$. □

Lemma 2.1.4. *Let X be a Banach space, which is a Grothendieck space and let $(T_j) \subset L(X)$ be a sequence with:*

1. $T_j T_k = T_k T_j$,
2. $\lim_{j\to\infty} T_j x = 0$ *for all* $x \in X$, *and*
3. $\lim_{j\to\infty} \|(I - T_m)T_j\| = 0$ *for all* $m \in \mathbb{N}$.

Then, the dual space of X verifies that

$$X' = \{u \in X' : \lim_{j\to\infty} \|T_j^t u\| = 0\}.$$

Proof. We denote $S_j := I - T_j$. Then $S_j^t : X' \longrightarrow X'$ is continuous (with the norm topology in both sides). Also denote

$$H = \{u \in X' : \lim_{j\to\infty} \|S_j^t u - u\| = 0\} = \{u \in X' : \lim_{j\to\infty} \|T_j^t u\| = 0\}.$$

Our aim is to show that H is both dense in X' and closed for the norm topology, and thus, $H = X'$.

We begin showing that H is closed. Let $(x'_k) \subset H$ be a sequence such that

$$\lim_{k\to\infty} \|x'_k - x'_0\| = 0,$$

for an $x'_0 \in X'$.

Fix $\varepsilon > 0$, let B be the closed unit ball of X, and let

$$C = B \cup \bigcup_{j=1}^{\infty} S_j(B).$$

Since $\{S_j\}$ is norm bounded due 2., and B is bounded, C is bounded. Using this and the convergence to x_0', there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and for all $c \in C$,

$$| < x_k' - x_0', c > | \leq \frac{\varepsilon}{3}.$$

Also, as $x_{k_0}' \in H$, there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,

$$\sup_{b \in B} |x_{k_0}'(b) - S_j^t x_{k_0}'(b)| \leq \frac{\varepsilon}{3}.$$

Finally we show that $x_0' \in H$. We have for $j \geq j_0$ and $b \in B$,

$$\begin{aligned} |x_0'(b) - S_j^t x_0'(b)| &\leq |x_0'(b) - x_{k_0}'(b)| + |x_{k_0}'(b) - S_j^t x_{k_0}(b)| + |S_j^t x_{k_0}'(b) - S_j^t x_0'(b)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + |(x_{k_0}' - x_0')(S_j b)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus H is closed for the norm topology. Now we show that H is dense. Firstly we see that

$$\bigcup_{k=1}^{\infty} S_k^t X' \subset H.$$

Fix $k \in \mathbb{N}$. By 1. and 3., we know that

$$\lim_{j \to \infty} \|T_j S_k\| = \lim_{j \to \infty} \|S_k T_j\| = 0.$$

Thus, for all $u \in X'$,

$$\lim_{j \to \infty} \|(S_k^t T_j^t) u\| = \lim_{j \to \infty} \|u(T_j S_k)\| = 0.$$

Then, for each $x' \in X'$

$$\lim_{j \to \infty} \|S_j^t(S_k^t x') - S_k^t x'\| = \lim_{j \to \infty} \|S_k^t(I - T_j^t)x' - S_k^t x'\| = \lim_{j \to \infty} \|S_k^t T_j^t x'\| = 0,$$

which implies that $S_k^t x' \in H$. Thus, $S_k^t X' \subseteq H$ for any $k \in \mathbb{N}$.

By Lemma 2.1.3, for each $u \in X'$, $S_k^t u$ converges to u for the weak topology $\sigma(X', X'')$. Then, for any element of X', there exists a sequence in H which converges weakly to the element, and therefore, H is weakly dense in X'. As H is a subspace, we conclude that H is also dense in X' for the norm topology.

As H is closed and dense in X' for the norm topology, we have that $H = X'$ and we finish. □

Lemma 2.1.5. *Let X be a Banach space, which is a Grothendieck space and let $(T_j) \subset L(X)$ be a sequence satisfying:*

1. $T_j T_k = T_k T_j$,
2. $\lim_{j\to\infty} T_j x = 0$ *for all* $x \in X$, *and*
3. $\lim_{j\to\infty} \|(I - T_m)T_j\| = 0$ *for all* $m \in \mathbb{N}$.

Then for each bounded sequence $(x_j)_j \subset X$, we have $\lim_{j\to\infty} T_j x_j = 0$ for the weak topology $\sigma(X, X')$.

Proof. Fix $(x_k)_k \subset X$ bounded. Take $x' \in X'$. By the Lemma 2.1.4 we know that $T_j^t \to 0$ for the norm topology. Thus, we have

$$\lim_{j\to\infty} \sup_{k\in\mathbb{N}} |T_j^t(x')x_k| = 0.$$

Finally,

$$\lim_{j\to\infty} | < T_j x_j, x' > | = \lim_{j\to\infty} | < x_j, T_j^t x' > | = \lim_{j\to\infty} |T_j^t x'(x_j)| = 0.$$ □

2.2 Lotz's theorem

In this section T_n does mean the n-th ergodic mean of the operator T.

Lemma 2.2.1. *Let X be a Banach space. Let $T \in L(X)$ and let*

$$T_n = \frac{1}{n}\sum_{m=1}^{n} T^m.$$

Assume $\lim_n \left\|\frac{T^n}{n}\right\| = 0$. Then, for each $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \|(I - T_m)T_n\| = 0.$$

Proof. Fix $n, m \in \mathbb{N}$. Then,

$$\begin{aligned} I - T_m &= \frac{1}{m}(mI - T - T^2 - \cdots - T^m) = \frac{1}{m}((I-T) + (I - T^2) + \cdots + (I - T^m)) \\ &= \frac{1}{m}\left[(I-T) + (I+T)(I-T) + \cdots + (I + T + \cdots + T^{m-1})(I-T)\right] \\ &= \frac{1}{m}\left[I + (I+T) + \cdots + (I + T + \cdots + T^{m-1})\right](I-T) = g_m(T)(I-T). \end{aligned}$$

Now, $(I - T_m)T_n = g_m(T)(I - T)T_n = g_m(T)\frac{1}{n}(T - T^{n+1})$. Using that $\lim_n \left\| \frac{T^n}{n} \right\| = 0$, we get

$$\lim_{n\to\infty} \|(I - T_m)T_n\| = 0. \qquad \square$$

The following theorem is the main result of this chapter and the Dunford-Pettis property is used in its proof.

Theorem 2.2.2 (Lotz). *Let X be a Banach space, which is a Grothendieck space with the Dunford-Pettis property. Let $T \in L(X)$ and let*

$$T_n = \frac{1}{n}\sum_{m=1}^{n} T^m.$$

Assume:

1. $\lim_n \left\| \frac{T^n}{n} \right\| = 0.$
2. *For each $x \in X$ the set $\{T_n x : n \in \mathbb{N}\}$ is relatively $\sigma(X, X')-$compact.*

Then, there exists a projection $P \in L(X)$ such that

$$\lim_{n\to\infty} \|T_n - P\| = 0,$$

and T is uniformly mean ergodic.

PROOF. By Yosida's Theorem (Theorems 1.2.4 and 1.2.8), there exists a projection $P \in L(X)$ such that for all $x \in X$, $\lim_n T_n x = Px$ and also $P = P^2 = TP = PT$, $F := P(X) = \ker(I - T)$, $H := \ker P = \overline{(I - T)X}$ and $X = F \oplus H$.

To prove the theorem we prove firstly that $\lim_n \|T_n^2 - P\| = 0$. To do so we see that $(T_n)_{|F} = I_{|F}$ and that $\lim_n \|(T_n^2)_{|H}\| = 0$.

Let $x \in F$, then $x = Px = \lim_n T_n x$. Thus, using Lemma 2.2.1, $(I - T_k)x = \lim_n (I - T_k)T_n x = 0$. Therefore $(I - T_k)x = 0$ and $(T_n)_{|F} = I_{|F}$.

Now we move to the second part. Note that by Yosida's Theorem 1.2.4, $H = \{x \in X : T_n x \to 0\}$. Set

$$A = \bigcup_{k=1}^{\infty} (I - T_k)X$$

and denote by $\overline{\overline{A}}$ its closure for the weak topology $\sigma(X, X')$. We want to see that $H = \overline{\overline{A}}$.

Let $k \in \mathbb{N}$ and $x \in X$, then $(I - T_k)x \in A$ and by Lemma 2.2.1, $(I - T_k)x \in H$. But H is closed for the weak topology, thus $\overline{\overline{A}} \subseteq H$.

Now let $x \in H$, then $x - T_k x \in A$ for every $k \in \mathbb{N}$, but as $x \in H$, $x = \lim_k x - T_k x$, and thus $x \in \overline{A}$. Then $H \subseteq \overline{A} \subset \overline{\overline{A}}$. Thus $H = \overline{\overline{A}}$.

Suppose that $((T_n^2)_{|F})$ does not converge to 0 for the topology of the norm. Then there exists $\varepsilon > 0$ such that $\|(T_n^2)_{|H}\| > \varepsilon$. We can choose a sequence $(x_s) \subset H$ with $\|x_s\| \leq 1$ and a sequence $j_1 < j_2 < \cdots$ so that $\|T_{j_s}^2 x_s\| > \varepsilon$. Then by Hahn Banach Theorem, there exists another sequence $(x'_s) \subset X'$ with $\|x'_s\| \leq 1$ such that $| < T_{j_s}^2 x_s, x'_s > | > \varepsilon$. Thus for each $s \in \mathbb{N}$,

$$| < T_{j_s} x_s, T_{j_s}^t x'_s > | > \varepsilon.$$

Now we note that H is a complemented subspace of a Grothendieck space with the Dunford-Pettis property, and thus H itself is a Grothendieck space with the Dunford-Pettis property. Therefore, T and H verify the conditions of Lemmata 2.1.3 and 2.1.5, due to the definition of H and Lemma 2.2.1, and we can apply them.

Since (x_s) is bounded, $T_{j_s} x_s$ converges to 0 for the weak topology $\sigma(X, X')$ by Lemma 2.1.5. Also (x'_s) is bounded and thus $T_{j_s}^t x'_s$ converges to 0 for the weak topology $\sigma(X', X'')$ by Lemma 2.1.3. Since H has the Dunford-Pettis property,

$$\lim_{s \to \infty} | < T_{j_s} x_s, T_{j_s}^t x'_s > | = 0,$$

which is a contradiction. Therefore, $((T_n^2)_{|H})$ must converge to 0 for the topology of the norm.

Now adding the results for F and H, we get that T_n^2 converges to P for the topology of the norm, as we wanted to check.

Now we complete the proof of the Theorem. As $P_{|F} = I_F$, we only have to show that $\lim_j \|(T_j)_{|H}\| = 0$. As $\lim_j \|(T_j^2)_{|H}\| = 0$, there exists $m \in \mathbb{N}$ such that $\|T_m^2\| < 1$ and thus $I - T_m^2$ is invertible in $L(H)$, by Theorem 1.2.2. We have

$$(I - T_m)(I + T_m)(I - T_m^2) = I,$$

and thus, $I - T_m$ is invertible in $L(H)$. Using Lemma 2.2.1,

$$\begin{aligned}\lim_{n \to \infty} \|(T_n)_{|H}\| &= \lim_{n \to \infty} \|((I - T_m)^{-1}(I - T_m)T_n)_{|H}\| \\ &\leq \lim_{n \to \infty} \|(I - T_m)^{-1}_{|H}\| \lim_{n \to \infty} \|((I - T_m)T_n)_{|H}\| = 0,\end{aligned}$$

and thus T_n converges to P for the topology of the norm. □

Corollary 2.2.3. *Let X be a Banach space, which is a Grothendieck space with the Dunford-Pettis property. Let $T \in L(X)$ be a power bounded operator. Then T is mean ergodic if and only if T is uniformly mean ergodic.*

Chapter 3

Multiplication operators on spaces of holomorphic functions

3.1 Definitions and general results

The objective of this chapter is to investigate the (uniform) mean ergodicity of multiplication operators in some weighted Banach spaces of holomorphic functions. This chapter mainly follows the article [5].

We work with some spaces of holomorphic functions defined on the complex unit disc $\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\}$. The notation for the space of holomorphic functions is $H(\mathbb{D})$ and $\|\cdot\|_\infty$ denotes the supremum norm. We denote the bounded holomorphic functions by $H^\infty(\mathbb{D}) = \{f \in H(\mathbb{D}) : \|f\|_\infty < \infty\}$. We say that a function $v : \mathbb{D} \longrightarrow (0, \infty)$ is a **weight function** if it is continuous, radial ($v(z) = v(|z|)$ for all $z \in \mathbb{D}$) and satisfies $\lim_{r\to 1^-} v(r) = 0$.

The spaces in which we are interested are the following weighted Banach spaces:

$$H_v^\infty = \{f \in H(\mathbb{D}) : \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}$$

and

$$H_v^0 = \{f \in H(\mathbb{D}) : \lim_{|z|\to 1^-} v(z)|f(z)| = 0\}.$$

Some facts of these spaces that are used are the following (see e.g. [4]):

- They are both Banach spaces when endowed with the norm $\|\cdot\|_v$.
- H_v^0 is a closed subspace of H_v^∞.

- The set of polynomials is dense in H_v^0.

From now on, we fix a weight function $v : \mathbb{D} \longrightarrow (0, \infty)$.

If we choose a function $\varphi \in H(\mathbb{D})$, we can define the multiplication operators $T_\varphi : H_v^\infty \longrightarrow H_v^\infty$ and $S_\varphi : H_v^0 \longrightarrow H_v^0$ defined by $T_\varphi(f) = \varphi f$ and $S_\varphi(f) = \varphi f$. In Lemma 3.1.3 we check $T_\varphi \in L(H_v^\infty)$ and $S_\varphi \in L(H_v^0)$ if $\varphi \in H^\infty$. Our aim in this chapter is finding when these operators are mean ergodic, uniformly ergodic, power bounded and Cesàro bounded.

Lemma 3.1.1. *Given $z \in \mathbb{D}$, the evaluation functional $\delta_z : H_v^\infty \longrightarrow \mathbb{C}$ defined by*

$$< f, \delta_z >= f(z)$$

is linear and continuous ($\delta_z \in (H_v^\infty)'$). Moreover,

$$| < f, \delta_z > | \leq \frac{\|f\|_v}{v(z)}.$$

Also, $\delta_z \in (H_v^0)'$.

Lemma 3.1.2. *Given $\varphi \in H(\mathbb{D})$, if $T_\varphi \in L(H_v^\infty)$, then $\varphi \in H^\infty(\mathbb{D})$. The same holds if $S_\varphi \in L(H_v^0)$.*

Proof. We use the adjoint operator, fix $z \in \mathbb{D}$, then for all $f \in H_v^\infty$,

$$< T_\varphi^t(\delta_z), f >=< \delta_z, \varphi f >= \varphi(z) f(z) = (\varphi(z)\delta_z)(f)$$

and we get $T_\varphi^t(\delta_z) = \varphi(z)\delta_z$. As T_φ is continuous, also T_φ^t is continuous. We have

$$|\varphi(z)|\|\delta_z\| = \|\varphi(z)\delta_z\| = \|T_\varphi^t(\delta_z)\| \leq \|T_\varphi^t\|\|\delta_z\|,$$

which tells us that $|\varphi(z)| \leq \|T_\varphi^t\|$ for all $z \in \mathbb{D}$, and thus φ is bounded and $\varphi \in H^\infty$.

The proof for S_φ is the same one. □

Lemma 3.1.3. *If $\varphi \in H^\infty(\mathbb{D})$, then $T_\varphi \in L(H_v^\infty)$ and $S_\varphi \in L(H_v^0)$ with*

$$\|T_\varphi\| = \|\varphi\|_\infty = \|S_\varphi\|.$$

Proof. We prove $\|S_\varphi\| \leq \|T_\varphi\| \leq \|\varphi\|_\infty \leq \|S_\varphi\|$. The first one is direct:

$$\|S_\varphi\| = \sup_{f \in H_v^0} \frac{\|T_\varphi\|_v}{\|f\|_v} \leq \sup_{f \in H_v^\infty} \frac{\|T_\varphi\|_v}{\|f\|_v} = \|T_\varphi\|.$$

We have seen that $\|S_\varphi\| \leq \|T_\varphi\|$. Now we check the second inequality.

$$\|T_\varphi\| = \sup_{\|f\|_v=1} \|T_\varphi f\|_v = \sup_{\|f\|_v=1} \|\varphi f\|_v = \sup_{\|f\|_v=1} \sup_{z\in\mathbb{D}} v(z)|\varphi(z)f(z)|$$
$$\leq \sup_{z\in\mathbb{D}} |\varphi(z)| \sup_{\|f\|_v=1} \sup_{z\in\mathbb{D}} v(z)|f(z)| = \sup_{z\in\mathbb{D}} |\varphi(z)| \sup_{\|f\|_v=1} \|f\|_v = \|\varphi\|_\infty.$$

Then we get $\|T_\varphi\| \leq \|\varphi\|_\infty$ and thus, the continuity of the operators. We finally check $\|S_\varphi\| \geq \|\varphi\|_\infty$. We use the adjoint operator S^t_φ of S_φ. Fix $z \in \mathbb{D}$.

$$\|S_\varphi\| = \|S^t_\varphi\| = \sup_{P\in(H^0_v)'} \frac{\|S^t_\varphi P\|_{(H^0_v)'}}{\|P\|_{(H^0_v)'}} \geq \frac{\|S^t_\varphi \delta_z\|_{(H^0_v)'}}{\|\delta_z\|_{(H^0_v)'}} = \frac{1}{\|\delta_z\|_{(H^0_v)'}} \sup_{f\in H^0_v, \|f\|_v\leq 1} |< f, S^t_\varphi \delta_z >|$$
$$= \frac{1}{\|\delta_z\|_{(H^0_v)'}} \sup_{f\in H^0_v, \|f\|_v\leq 1} |< S_\varphi f, \delta_z >| = \frac{1}{\|\delta_z\|_{(H^0_v)'}} \sup_{f\in H^0_v, \|f\|_v\leq 1} |< \varphi f, \delta_z >|$$
$$= \frac{|\varphi(z)|}{\|\delta_z\|_{(H^0_v)'}} \sup_{f\in H^0_v, \|f\|_v\leq 1} |< f, \delta_z >| = |\varphi(z)|.$$

Thus, $\|S_\varphi\| \geq |\varphi(z)|$ for each $z \in \mathbb{D}$, and then $\|S_\varphi\| \geq \|\varphi\|_\infty$, which concludes the proof.□

Lemma 3.1.4. *T_φ is an isomorphism if and only if $1/\varphi \in H^\infty$. Equivalently for S_φ.*

PROOF. If $1/\varphi \in H^\infty$, then $T_{1/\varphi}$ is continuous and satisfies

$$T_\varphi T_{1/\varphi} = T_{1/\varphi} T_\varphi = I,$$

thus T_φ is an isomorphism.

Suppose that T_φ is an isomorphism, then there exists $M \in L(H^\infty_v)$ such that $T_\varphi M = I$. If we evaluate this equality at the constant function 1 and a point $z \in \mathbb{D}$, we get

$$\varphi(z)(M1)(z) = 1, \qquad (M1)(z) = \frac{1}{\varphi(z)}$$

and as $M1 \in H(\mathbb{D})$, also $1/\varphi \in H(\mathbb{D})$. If we now evaluate at any function $f \in H^\infty_v$ and any point $z \in \mathbb{D}$, we get

$$(Mf)(z) = \frac{1}{\varphi(z)} f(z)$$

and thus $M = T_{1/\varphi}$. By Lemma 3.1.2, $1/\varphi \in H^\infty$.

The proof for S_φ is the same. □

3.2 Mean ergodicity and power boundedness

Our main interest is knowing properties of the iterated operators. Information about that is seen next.

Remark 3.2.1. Clearly the identities $T_\varphi^n = T_{\varphi^n}$ and $S_\varphi^n = S_{\varphi^n}$ hold for each $n \in \mathbb{N}$. Using them and the Lemma 3.1.3, we find that for each $n \in \mathbb{N}$

$$\|T_\varphi^n\| = \|\varphi^n\|_\infty = \|\varphi\|_\infty^n = \|S_\varphi^n\|.$$

Proposition 3.2.2. *For each $\varphi \in H^\infty(\mathbb{D})$, the following assertions are equivalent:*

1. $\|\varphi\|_\infty \leq 1$,
2. $T_\varphi \in L(H_v^\infty)$ *is power bounded,*
3. $S_\varphi \in L(H_v^0)$ *is power bounded.*

PROOF. Remark 3.2.1 yields 1. $\Rightarrow$ 2. and 1. $\Rightarrow$ 3.

For the proof of 2. $\Rightarrow$ 1., fix $n \in \mathbb{N}$. We have, using the Remark 3.2.1,

$$\|\varphi\|_\infty^n = \|T_\varphi^n\| \leq \sup_{m\in\mathbb{N}} \|T_\varphi^m\| =: C.$$

As T_φ is power bounded, $C < \infty$ and thus $\|\varphi\|_\infty < C^{1/n}$. This holds for each $n \in \mathbb{N}$, then $\|\varphi\|_\infty \leq 1$.

The proof of 3. $\Rightarrow$ 1. mimics the previous one. $\square$

Remark 3.2.3. Let $\varphi \in H^\infty(\mathbb{D})$. Then for each $f \in H_v^\infty$ and $n \in \mathbb{N}$,

$$((T_\varphi)_n f)(z) = \frac{f(z)}{n} \sum_{m=1}^{n} (\varphi(z))^m, \quad \text{for each } z \in \mathbb{D}.$$

But in the case when $\varphi(z) \neq 1$, we get

$$((T_\varphi)_n f)(z) = \frac{\varphi(z) f(z)}{n} \frac{1-(\varphi(z))^n}{1-\varphi(z)}, \quad \text{for each } z \in \mathbb{D} \setminus \ker(1-\varphi).$$

If we take $f \in H_v^0$, these formulae also hold with S_φ instead of T_φ.

As we saw in Propositions 1.1.7 and 1.1.8 and the Example following them, mean ergodicity does not imply power boundedness in general. However it does in our current case.

Proposition 3.2.4. *Let $\varphi \in H^\infty(\mathbb{D})$. If $T_\varphi \in L(H_v^\infty)$ (resp. $S_\varphi \in L(H_v^0)$) is mean ergodic, then T_φ (resp. S_φ) is power bounded.*

PROOF. By mean ergodicity of T_φ and Lemma 1.1.5, we have the pointwise limit

$$\lim_{n\to\infty} \frac{1}{n} T_\varphi^n = 0.$$

In particular, if we evaluate in the constant function 1 we find

$$\lim_{n\to\infty} \left\| \frac{1}{n}\varphi^n \right\|_v = 0.$$

For a fixed $z \in \mathbb{D}$, the inequality

$$\left\| \frac{1}{n}\varphi^n \right\|_v \geq v(z) \left| \frac{(\varphi(z))^n}{n} \right|$$

leads us to

$$\lim_{n\to\infty} \frac{(\varphi(z))^n}{n} = 0$$

in $\mathbb{C}$. Clearly we have $|\varphi(z)| \leq 1$ and thus $\|\varphi\|_\infty \leq 1$. By Proposition 3.2.2, T_φ is power bounded.

Considering that the constant function 1 belongs also to H_v^0, the proof for S_φ follows the same ideas. □

Lemma 3.2.5. *The compact-open topology and the weak topology $\sigma(H_v^\infty, H_v^0)$ coincide on the bounded sets of H_v^∞.*

Proof. Proof in [3, Lemma 13]. □

For the case of the operators S_φ, we can extend Proposition 3.2.4, proving the converse.

Proposition 3.2.6. *Let $\varphi \in H^\infty(\mathbb{D})$. Then $S_\varphi \in L(H_v^0)$ is mean ergodic, if and only if S_φ is power bounded, if and only if $\|\varphi\|_\infty \leq 1$.*

Proof. Using Propositions 3.2.2 and 3.2.4 we prove most of the Proposition. It only remains to show that $\|\varphi\|_\infty \leq 1$ implies mean ergodicity of S_φ. We work in two different cases.

Firstly suppose that there exists $z_0 \in \mathbb{D}$ with $|\varphi(z_0)| = 1$. Applying the Maximum Principle, there exists $w \in \mathbb{C}$ with $|w| = 1$ and $\varphi(z) = w$ for all $z \in \mathbb{D}$. Clearly, $S_\varphi = wI$. If $w = 1$, then $(S_\varphi)_n = I$ for each $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|(S_\varphi)_n - I\| = \lim_{n\to\infty} 0 = 0.$$

If $w \neq 1$, we use Remark 3.2.3, and we get

$$(S_\varphi)_n = \frac{w(1-w^n)}{n(1-w)} I,$$

for each $n \in \mathbb{N}$, and clearly

$$\lim_{n\to\infty} \|(S_\varphi)_n\| = 0.$$

In both cases we find S_φ is (uniformly) mean ergodic.

The other case is that $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$. We would like to apply the Mean Ergodic Theorem 1.2.5. The first assumption is satisfied by using that S_φ is power bounded. The second one is satisfied if we can show that for each $f \in H_v^0$

$$\lim_{n\to\infty} (S_\varphi)_n f = 0$$

for the weak topology $\sigma(H_v^0, (H_v^0)')$. Indeed, fix $f \in H_v^0$ and $z \in \mathbb{D}$. Using Remark 3.2.3 and $\|\varphi\|_\infty \leq 1$, for each $n \in \mathbb{N}$,

$$|((S_\varphi)_n f)(z)| \leq |f(z)|.$$

Thus, $\|(S_\varphi)_n f\|_v \leq \|f\|_v$ for each $n \in \mathbb{N}$ and $\{(S_\varphi)_n f,\, n \in \mathbb{N}\}$ is bounded. Now using that $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$ and Remark 3.2.3, we find

$$\lim_{n\to\infty} ((S_\varphi)_n f)(z) = 0,$$

for each $z \in \mathbb{D}$. We can even see that the convergence holds for the compact open topology. Indeed, given $0 < r < 1$, and $m = \max_{|z|\leq r} |\varphi(z)| < 1$, we have for $|z| < r$

$$|((S_\varphi)_n f)(z)| \leq \frac{|\varphi(z) f(z)|}{n} \frac{|1-(\varphi(z))^n|}{|1-\varphi(z)|} \leq \frac{2}{n(1-m)} \sup_{|z|\leq r} |f(z)|,$$

which converges to 0 for the compact open topology. If we now apply Lemma 3.2.5, we find that

$$\lim_{n\to\infty} (S_\varphi)_n f = 0$$

for the weak topology $\sigma(H_v^0, (H_v^0)')$.

We are now able to use the Theorem 1.2.5 and we find that $(S_\varphi)_n \to 0$ pointwise and thus S_φ is mean ergodic. □

Remark 3.2.7. Proposition 3.2.6 not only tells us that S_φ is mean ergodic, it also tells us the value of the limit projection of the ergodic means. This value is exactly the operator 0 for every case but one. This one case is $\varphi = 1$, which gives us the identity operator I for the mean.

3.3 Uniform mean ergodicity

Proposition 3.3.1. *Let $\varphi \in H^\infty(\mathbb{D})$. Then $S_\varphi \in L(H_v^0)$ is uniformly mean ergodic if and only if $\|\varphi\|_\infty \leq 1$ and either*

1. *there is $w \in \mathbb{C}$ with $|w| = 1$ such that $\varphi(z) = w$ for all $z \in \mathbb{D}$, or*

2. $\frac{1}{1-\varphi} \in H^\infty(\mathbb{D})$.

Proof. It was shown in the proof of the Proposition 3.2.6 that if φ satisfies 1. and $\|\varphi\|_\infty \leq 1$, then S_φ is uniformly ergodic.

Now suppose that $\|\varphi\|_\infty \leq 1$ and φ satisfies 2. Then there exists $\varepsilon > 0$ such that $|1 - \varphi(z)| \geq \varepsilon$ for all $z \in \mathbb{D}$, then $\varphi(z) \neq 1$ for each $z \in \mathbb{D}$. Thus, applying Remark 3.2.3, for each $f \in H_v^0$, each $z \in \mathbb{D}$ and each $n \in \mathbb{N}$, we get

$$|((S_\varphi)_n f)(z)| \leq \frac{2}{n\varepsilon}\|\varphi\|_\infty |f(z)|.$$

Then, taking suprema over $f \in H_v^0$ and $z \in \mathbb{D}$

$$\|(S_\varphi)_n\| \leq \frac{2}{n\varepsilon}\|\varphi\|_\infty$$

and hence

$$\lim_{n\to\infty} \|(S_\varphi)_n\| = 0,$$

which tells us that S_φ is uniformly mean ergodic.

For the converse, suppose that S_φ is uniformly mean ergodic. Then, by Proposition 3.2.6, $\|\varphi\|_\infty \leq 1$. Suppose 1. does not hold, then by the Maximum Principle, $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$. By Proposition 3.2.6 and Remark 3.2.7, we have the pointwise limit

$$\lim_{n\to\infty} (S_\varphi)_n = 0.$$

Our aim is to use Lemma 1.3.1. Firstly $\ker(I - S_\varphi) = \ker(S_{1-\varphi}) = \{0\}$. By Proposition 3.2.2, S_φ is power bounded, and thus

$$\lim_{n\to\infty} \left\| \frac{1}{n} S_\varphi^n \right\| = 0.$$

By Lemma 1.3.1, S_φ is uniformly mean ergodic if and only if $I - S_\varphi = S_{1-\varphi}$ is an isomorphism. Now by Lemma 3.1.4, $\frac{1}{1-\varphi} \in H^\infty$. □

Until now, most of the results were the same for T_φ and S_φ. However the last result showed some differences, which are confirmed in the next proposition.

Proposition 3.3.2. *Let $\varphi \in H^\infty(\mathbb{D})$ with $\|\varphi\| \leq 1$. Then, the following assertions are equivalent:*

a) T_φ is mean ergodic.

b) T_φ *is uniformly mean ergodic.*

c) Either

1. *there is* $w \in \mathbb{C}$ *with* $|w| = 1$ *such that* $\varphi(z) = w$ *for all* $z \in \mathbb{D}$*, or*
2. $\frac{1}{1-\varphi} \in H^\infty(\mathbb{D})$.

Proof. $b) \Leftrightarrow c)$ is proved the same way we proved Proposition 3.3.1.

$b) \Rightarrow a)$ is direct by the definitions.

$a) \Rightarrow b)$: By Proposition 3.2.4, T_φ is power bounded and thus

$$\lim_{n\to\infty} \left\| \frac{1}{n} T_\varphi^n \right\| = 0.$$

By Lusky [13] H_v^∞ is isomorphic to either l^∞ or $H^\infty(\mathbb{D})$, which are Grothendieck spaces with the Dunford-Pettis property [12]. It follows from Corollary 2.2.3 that T_φ is uniformly mean ergodic, since T_φ is mean ergodic and power bounded. □

Chapter 4

Multiplication operators on spaces of continuous functions

4.1 Definitions and general results

The objective of this chapter is to show some results about the ergodicity of the multiplication operator on spaces of continuous functions of a topological space. The results are analogous to those in Chapter 3, and some proofs coincide.

Let X be a topological space that is Hausdorff, locally compact, σ-compact and connected. We denote the set of continuous functions from X to $\mathbb{C}$ as $C(X)$.

A function $f \in C(X)$ **vanishes at infinity** if for each $\varepsilon > 0$, there exists $K \subset X$ compact such that $|f(x)| < \varepsilon$, for each $x \in X \setminus K$.

We say that $v : X \longrightarrow (0, \infty)$ is a **weight function** if it is continuous and vanishes at infinity. This definition coincides with the one given for holomorphic functions in Chapter 3, in the case that $X = \mathbb{D}$, if we add the condition of v being radial.

If we fix a weight function v, we can define the following spaces, analogous to the ones for holomorphic functions:

$$C_v = \{f \in C(X) \, : \, \|f\|_v = \sup_{x \in X} v(x)|f(x)| < \infty\}$$

and

$$C_v^0 = \{f \in C(X) \, : \, v|f| \text{ vanishes at infinity}\}.$$

These are Banach spaces when endowed with the norm $\|\cdot\|_v$. Observe that for $v \equiv 1$, C_v is the Banach space $CB(X)$ of bounded continuous functions endowed with the supremum norm, and C_v^0 is the space $C_0(X)$ of continuous functions vanishing at infinity, also endowed with the supremum norm.

If a function $\varphi \in C(X)$ is fixed, we can also define the multiplication operator on these spaces as $T_\varphi : C_v \longrightarrow C(X)$ and $S_\varphi : C_v^0 \longrightarrow C(X)$ defined by $T_\varphi(f) = f\varphi$ and $S_\varphi(f) = f\varphi$.

Most of the properties of these operators are the same that the ones for the operators defined for holomorphic functions. For that reason, most of the proofs of the results are not given, as they mimic the other ones.

Lemma 4.1.1. *The operator T_φ is continuous if and only if the operator S_φ is continuous if and only if φ is bounded. Moreover,*

$$\|T_\varphi\| = \|S_\varphi\| = \|\varphi\|.$$

Lemma 4.1.2. *The operator T_φ is power bounded if and only if the operator S_φ is power bounded if and only if $\|\varphi\|_\infty \leq 1$.*

Proposition 4.1.3. *If the operator T_φ (resp. S_φ) is mean ergodic, then $\|\varphi\|_\infty \leq 1$, and thus T_φ (resp. S_φ) is power bounded.*

4.2 Ergodic results

The following theorems show the difference between the results for holomorphic functions and those for continuous functions, or at least the difference between their proofs. Firstly we have a characterization for mean ergodicity of S_φ and later another for uniform mean ergodicity of S_φ.

Remark 4.2.1. Let $\varphi \in C(X)$. Then for each $f \in C_v$ and $n \in \mathbb{N}$,

$$((T_\varphi)_n f)(x) = \frac{f(x)}{n} \sum_{m=1}^{n} (\varphi(x))^m, \quad \forall x \in X.$$

In the case when $\varphi(x) \neq 1$, we get

$$((T_\varphi)_n f)(x) = \frac{\varphi(x) f(x)}{n} \frac{1 - (\varphi(x))^n}{1 - \varphi(x)}, \quad \forall x \in X \setminus \ker(1 - \varphi).$$

If we take $f \in C_v^0$, these formulae also hold with S_φ instead of T_φ.

Proposition 4.2.2. *Assume that φ is not identically 1 and that $\|\varphi\|_\infty \leq 1$. Then S_φ is mean ergodic if and only if $\varphi(x) \neq 1$ for each $x \in X$.*

PROOF. Firstly suppose that $\varphi(x) \neq 1$ for each $x \in X$. Fix $f \in C_v^0$ and fix a compact set $K \subset X$. Then, there exists $0 < \varepsilon < 1$ such that $|1 - \varphi(x)| > \varepsilon$ for every $x \in K$. Then for any $x \in K$, we have

$$|((S_\varphi)_n f)(x)| = \frac{|\varphi(x) f(x)|}{n} \frac{|1 - (\varphi(x))^n|}{|1 - \varphi(x)|} \leq \frac{2}{n\varepsilon} \sup_{y \in K} |f(y)|,$$

which converges to 0 as $n \to \infty$. Thus $(S_\varphi)_n f$ converges to 0 for the compact open topology, and thus $(S_\varphi)_n f$ converges to 0 for the weak topology $\sigma(C_v^0, (C_v^0)')$ (which can be proven using Riesz's representation theorem). By the mean ergodic Theorem 1.2.4, S_φ is mean ergodic.

Assume now that S_φ is mean ergodic, and let $A = \{x \in X \ : \ \varphi(x) = 1\}$. As S_φ is mean ergodic, there exists $h \in C(X)$ such that $\lim_{n \to \infty}((S_\varphi)_n 1)(x) = h(x)$ for each $x \in X$, where 1 denotes the constant function 1. As we saw before, $h(x) = 0$ if $x \notin A$, and $h(x) = 1$ if $x \in A$. But h is continuous and X is connected. Therefore, $A = X$ or $A = \emptyset$, the first one is not possible because it would mean that φ is identically 1, thus the second one holds and $\varphi(x) \neq 1$ for each $x \in X$. $\square$

Proposition 4.2.3. *Assume that φ is not identically 1 and that $\|\varphi\|_\infty \leq 1$. Then S_φ is uniformly mean ergodic if and only if $\inf_{x \in X} |\varphi(x) - 1| > 0$ (i.e. $\frac{1}{1-\varphi}$ is bounded).*

PROOF. Suppose that $\varepsilon = \inf_{x \in X} |\varphi(x) - 1| > 0$, then,

$$v(z)|(S_\varphi)_n f)(x)| = \frac{v(x)|f(x)||\varphi(x)|}{n} \frac{|1 - (\varphi(x))^n|}{|1 - \varphi(x)|} \leq \frac{\|f\|_v}{n} \frac{2}{\varepsilon}.$$

Thus, $\|(S_\varphi)_n\|$ converges to 0 as $n \to \infty$, and S_φ is uniformly mean ergodic.

Assume now that S_φ is uniformly mean ergodic, then by Lin's Theorem 1.3.2, $(I - S_\varphi)C_v^0$ is closed. Let $f \in C_v^0$, with $(I - S_\varphi)f = 0$, then $(1 - \varphi)f = 0$. As S_φ is uniformly mean ergodic, it is mean ergodic, and by Proposition 4.2.2, f must be identically 0, hence $\ker(I - S_\varphi) = \{0\}$ and $I - S_\varphi$ is injective. We can use the decomposition

$$C_v^0 = \ker(I - S_\varphi) \oplus \overline{(I - S_\varphi)C_v^0} = \overline{(I - S_\varphi)C_v^0} = (I - S_\varphi)C_v^0$$

to find that $I - S_\varphi$ is surjective, and thus it is bijective.

As $I - S_\varphi = S_{1-\varphi}$, using a result analogous to Lemma 3.1.4, $S_{1-\varphi}$ is bijective if and only if $\frac{1}{1-\varphi}$ is bounded. $\square$

Now we focus our interest in T_φ, with similar results of those for S_φ.

Proposition 4.2.4. *Assume that φ is not identically 1 and that $\|\varphi\|_\infty \leq 1$. If T_φ is mean ergodic then $\varphi(x) \neq 1$ for each $x \in X$.*

PROOF. Just follow one direction of the proof of Proposition 4.2.2. □

Remark 4.2.5. We cannot use the same argument used in Proposition 4.2.2 to prove the converse of Proposition 4.2.4, because convergence of a sequence for the compact open topology in C_v does not necessarily imply convergence of the sequence for the weak topology $\sigma(C_v, C_v')$.

Proposition 4.2.6. *Assume that φ is not identically 1 and that $\|\varphi\|_\infty \leq 1$. Then T_φ is uniformly mean ergodic if and only if $\inf_{x\in X} |\varphi(x) - 1| > 0$ (i.e. $\frac{1}{1-\varphi}$ is bounded).*

PROOF. Just follow the proof of Proposition 4.2.3. □

Proposition 4.2.7. *Assume that φ is not identically 1. If T_φ is mean ergodic, then $\inf_{x\in X} |1 - \varphi(x)| > 0$.*

PROOF. We already know that $|\varphi(x)| \leq 1$ for all $x \in X$ and that $\varphi(x) \neq 1$ for all $x \in X$. Just proceed as we did for S_φ.

If $T_\varphi : C_v(X) \to C_v(X)$ is mean ergodic, then for each $f \in C_v(X)$ there is $h \in C_v(X)$ such that $\lim_{n\to\infty}(T_\varphi)_n f = h$ in $C_v(X)$. Since, for each $x \in X$, the sequence

$$(T_\varphi)_n f(x) = \frac{\varphi(x)f(x)}{n}\frac{1-\varphi(x)^n}{1-\varphi(x)}$$

converges to 0 as n goes to ∞ (even uniformly on compact subsets of X), it follows that $h = 0$. That means that for each $f \in C_v(X)$, $(T_\varphi)_n f$ converges to 0 for the norm topology in C_v.

In particular, this must hold for $f := 1/v \in C_v(X)$. Therefore the sequence

$$\sup_{x\in X} v(x)|(T_\varphi)_n(1/v)(x)| = \sup_{x\in X} \frac{|\varphi(x)|}{n}\frac{|1-\varphi(x)^n|}{|1-\varphi(x)|}$$

tends to 0 as n goes to ∞.

Now we proceed by contradiction and assume that the conclusion does not hold. Then, for each $n \in \mathbb{N}$, $n > 2$, there is $x_n \in X$ such that $|1 - \varphi(x_n)| < 1/n$. Using several times that $1 - |a| \leq |1 - a|$, we get $|\varphi(x_n)| > 1 - (1/n)$, and

$$|1 - \varphi(x_n)^n| \geq 1 - |\varphi(x_n)|^n > 1 - (1 - (1/n))^n.$$

Hence

$$\frac{|\varphi(x_n)|}{n}\frac{|1-\varphi(x_n)^n|}{|1-\varphi(x_n)|} > \frac{1}{2}(1 - (1 - (1/n))^n),$$

which tends to $\frac{1}{2}(1 - \frac{1}{e})$ as n goes to ∞, and thus the sequence

$$\sup_{x\in X} \frac{|\varphi(x)|}{n}\frac{|1-\varphi(x)^n|}{|1-\varphi(x)|}$$

does not tend to 0 as n goes to ∞.

This implies that $\inf_{x\in X} |1-\varphi(x)| > 0$ and we conlude. □

Remark 4.2.8. The proof of Proposition 4.2.7 cannot be used for S_φ since $1/v \notin C_v^0$. Neither can it be used for holomorphic functions since $1/v$ is continuous but not holomorphic.

Corollary 4.2.9. *The operator $T_\varphi : C_v(X) \to C_v(X)$ is mean ergodic if and only if it is uniformly mean ergodic.*

Chapter 5

The Cesàro operator

5.1 Definitions

This chapter presents some results about the Cesàro operator, which is studied in different sequence spaces. We check if it is well-defined, if it is continuous and its ergodic properties (power boundedness, mean ergodicity and uniform mean ergodicity).

Let $\omega = \mathbb{C}^{\mathbb{N}}$. The Cesàro operator is defined from ω to itself and it sends each sequence to a sequence such that the n-th term is the mean of the n first terms of the original sequence, i.e. the Cesàro operator $C : \omega \longrightarrow \omega$ is defined as

$$C((x_i)_i) = (x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \ldots).$$

The Cesàro operator is actually a bijection, with inverse defined as

$$C^{-1}((y_i)_i) = (jy_j - (j-1)y_{j-1})_j, \quad y_0 = 0.$$

We can endow ω with the topology given by the seminorms q_k defined for each $k \in \mathbb{N}$ as

$$q_k((x_i)_i) = \max_{1 \le i \le k} |x_i|.$$

With this topology one can prove that C is continuous and power bounded, as well as other interesting results [2], but those are not in this work as we are focusing only in Banach spaces.

5.2 Continuity and self-mapping

Our first objective is to check whether $C(X) \subset X$ if X is l^∞, c, c_0 or l^p, with $1 \le p < \infty$. In case this holds, we also calculate the norm of $C_{|X}$. Our aim is to study the ergodic properties of $C : X \longrightarrow X$.

Lemma 5.2.1. *The Cesàro operator verifies $C(l^\infty) \subset l^\infty$. The operator $C^{(\infty)} : l^\infty \longrightarrow l^\infty$, defined as $C^{(\infty)}(x) = C(x)$, is continuous and $\|C^{(\infty)}\| = 1$.*

PROOF. Let $x = (x_n) \in l^\infty$ and denote $y = (y_i)_i = C(x)$. We know that the average of any amount of positive numbers is always lesser or equal than the largest of those numbers, and thus $|y_i| \leq \max\{|x_1|, \ldots, |x_i|\}$. Taking suprema we find $\|y\|_\infty \leq \|x\|_\infty$ and therefore, $y \in l^\infty$ and $C(l^\infty) \subset l^\infty$. Also, $C^{(\infty)}$ is continuous by the closed graph theorem.

The vector $e_1 = (1, 0, 0, \ldots)$ belongs to l^∞ and $C^{(\infty)}(e_1) = (1, 1/2, 1/3, \ldots)$, thus $\|C^{(\infty)}\| = 1$. □

Lemma 5.2.2. *The Cesàro operator verifies $C(c) \subset c$. The operator $C^{(c)} : c \longrightarrow c$, defined as $C^{(c)}(x) = C(x)$, is continuous and $\|C^{(c)}\| = 1$. Furthermore,*

$$\lim Cx = \lim x,$$

for each $x \in c$.

PROOF. It suffices to show the property of the limit, since it also proves the first property. For the continuity and the value of the norm, apply the proof of the Lemma 5.2.1 (considering that $e_1 \in c$).

Let $x = (x_i)_i \in c$ and $L = \lim_{i\to\infty} x_i$. Fix $\varepsilon > 0$. Choose $j_0 \in \mathbb{N}$ such that for each $j \geq j_0$, $|x_j - L| < \frac{\varepsilon}{2}$. Also choose $j_1 > j_0$ such that for each $j \geq j_1$,

$$\sum_{i=1}^{j_0} |x_i - L| < j\frac{\varepsilon}{2}.$$

Then, for each $j \geq j_1 > j_0$,

$$\left|\frac{1}{j}\sum_{i=1}^{j} x_i - L\right| \leq \sum_{i=1}^{j_0} \frac{|x_i - L|}{j} + \sum_{i=j_0+1}^{j} \frac{|x_i - L|}{j} < \frac{\varepsilon}{2} + \frac{j - j_0}{j}\frac{\varepsilon}{2} < \varepsilon.$$

Therefore, $\lim Cx = \lim x$ and we conclude. □

Lemma 5.2.3. *The Cesàro operator verifies $C(c_0) \subset c_0$. The operator $C^{(0)} : c_0 \longrightarrow c_0$, defined as $C^{(0)}(x) = C(x)$, is continuous and $\|C^{(0)}\| = 1$.*

PROOF. Use Lemmata 5.2.1 and 5.2.2 considering that $e_1 \in c_0$. □

Remark 5.2.4. The Cesàro operator C fails to send l^1 to itself since $C(e_1) = (1, 1/2, 1/3, \ldots) \notin l^1$.

To show the case of l^p with $1 < p < \infty$, we need the following lemma.

Theorem 5.2.5 (Hardy). *If $1 < p < \infty$ and $1/p + 1/q = 1$, then $C(l^p) \subset l^p$. The operator $C^{(p)} : l^p \longrightarrow l^p$ is continuous with $\|Cx\|_p \leq q\|x\|_p$ for each $x \in l^p$. Furthermore, $\|C^{(p)}\| = q$.*

PROOF. Use Young and Hölder's inequalities to prove the inequality. To show $\|C^{(p)}\| = q$ use the sequence $(a^N)_N \subset l^p$ defined as

$$a^N = (1, \frac{1}{2^{1/p}}, \frac{1}{3^{1/p}}, \ldots, \frac{1}{N^{1/p}}, 0, \ldots).$$ □

5.3 Spectral properties

To study ergodic properties of the Cesàro operator in these spaces, we must determine the spectrum.

Lemma 5.3.1 (Leibowitz). *The spectrum of the Cèsaro operator C verifies the following:*

- $\sigma(C, l^\infty) = \sigma(C, c_0) = \{\lambda \in \mathbb{C} \,:\, \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$,
- $\sigma(C, l^p) = \{\lambda \in \mathbb{C} \,:\, \left|\lambda - \frac{q}{2}\right| \leq \frac{q}{2}\}$, *where $1 < p < \infty$ and $1/p + 1/q = 1$.*

Lemma 5.3.2. *The operator $C : \omega \longrightarrow \omega$ verifies the following:*

1. $\ker(I - C) = \mathrm{span}\{\mathbf{1}\}$, *where $\mathbf{1} = (1, 1, \ldots)$,*
2. $(I - C)(\omega) = \{x \in \omega \,:\, x_1 = 0\}$,
3. *the eigenvalues of C are the elements of the set $\{1/k \,:\, k \in \mathbb{N}\}$, and*
4. *the eigenvectors of $\lambda = 1/m$ are of the form*

$$x = \alpha(0, \ldots, 0, 1, m, \frac{m(m+1)}{2!}, \frac{m(m+1)(m+2)}{3!}, \ldots),$$

where the first 1 is at the m-th position and $\alpha \in \mathbb{C}$.

PROOF. To show 1. let $0 \neq x \in \omega$ with $Cx = x$, then $(x_1 + x_2)/2 = x_2$ and thus, $x_1 = x_2$. Suppose that $x_1 = \cdots = x_n$, then $(nx_1 + x_{n+1})/(n+1) = x_{n+1}$. Thus, we have seen by induction that $x_1 = x_2 = \cdots = x_n = \cdots$. Since $x \neq 0$, $x = x_1(1, 1, \ldots, 1, \ldots) = x_1\mathbf{1}$. As $\mathbf{1} \in \ker(I - C)$, the equality $\ker(I - C) = \mathrm{span}\{\mathbf{1}\}$ holds.

Now we check 2.. It is direct that the first coordinate of $x - Cx$ is 0, for any $x \in \omega$. So,

$$(I - C)(\omega) \subset \{y \in \omega \,:\, y_1 = 0\}.$$

For the other inclusion let $y = (y_n)$ with $y_1 = 0$. We want to find an $x = (x_n)$ such that $(I - C)x = y$. Clearly $x_1 = 0$. Also, $x_2 - (x_1 + x_2)/2 = y_2$, and thus, $x_2 = 2y_2$. Inductively, $x_n - (x_1 + \cdots + x_n)/n = y_n$. Doing this we get

$$x_n = \frac{n}{n-1} y_n + \frac{1}{n-2} y_{n-1} + \cdots + \frac{1}{3} y_4 + \frac{1}{2} y_3 + y_2$$

and $x - Cx = y$. Therefore

$$(I - C)(\omega) = \{y \in \omega \,:\, y_1 = 0\}.$$

For 3. and 4. take $\lambda \in \mathbb{C} \setminus \{0, 1\}$ and $0 \neq x = (x_n) \in \omega$ such that $(\lambda I - C)x = 0$. Then, $\lambda x_1 = x_1$, $(2\lambda - 1)x_2 = x_1$ and $(n\lambda - 1)x_n = \lambda(n-1)x_{n-1}$ if $n \geq 3$.

Now, take $m = \min\{i \in \mathbb{N} \,:\, x_i \neq 0\}$, which exists since $x \neq 0$. Then $\lambda = 1/m$ must hold. Also

$$x_n = \frac{n-1}{n-m} x_{n-1},$$

for each $n > m$. We get the result of 4. choosing $\alpha = x_m$. □

Lemma 5.3.3 (Reade). *If $\lambda \notin \{1/k \,:\, k \in \mathbb{N}\}$, then $(C - \lambda I)^{-1} : \omega \longrightarrow \omega$ exists. Furthermore, if $(a_{ij})_{ij}$ denotes the matrix of $(C - \lambda I)^{-1}$, then,*

$$a_{ij} = -\frac{1}{i\lambda^2 \prod_{k=j}^{i} (1 - \frac{1}{k\lambda})} = \frac{-\lambda^{i-j-1}}{i \prod_{k=j}^{i} (\lambda - \frac{1}{k})}, \qquad \text{if } 1 \leq j < i,$$

$$a_{ij} = \frac{1}{\frac{1}{i} - \lambda}, \qquad \text{if } i = j$$

and $a_{ij} = 0$ otherwise.

Proof. Check [15]. □

5.4 Ergodic results

Proposition 5.4.1. *The operator $C^{(0)} : c_0 \longrightarrow c_0$ is power bounded but not mean ergodic. Moreover,*

$$\ker(I - C^{(0)}) = \{0\}.$$

Also, $(I - C^{(0)})(c_0)$ is not closed, with

$$\overline{(I - C^{(0)})(c_0)} = \overline{\text{span}}\{e_r\}_{r \geq 2} = \{x \in c_0 \,:\, x_1 = 0\}.$$

PROOF. $C^{(0)}$ is power bounded, since, by Lemma 5.2.3, $\|C^{(0)}\| = 1$.

By Lemma 5.3.2,

$$\ker(I - C^{(0)}) \subset \operatorname{span}\{\mathbf{1}\},$$

however, none of the elements of span$\{\mathbf{1}\}$ is in c_0, besides 0. Thus, we have the assertion.

To show that

$$\overline{(I - C^{(0)})(c_0)} = \overline{\operatorname{span}}\{e_r\}_{r\geq 2} = \{x \in c_0 \, : \, x_1 = 0\},$$

use the argument given to prove 2. in Lemma 5.3.2 and check that the results stay in c_0.

If we assume that $C^{(0)}$ is mean ergodic, we can apply Theorem 1.2.8 to have the decomposition

$$c_0 = \ker(I - C^{(0)}) \oplus \overline{(I - C^{(0)})(c_0)} = \{0\} \oplus \overline{\operatorname{span}}\{e_r\}_{r\geq 2}.$$

But that is not true, therefore, $C^{(0)}$ cannot be mean ergodic.

Finally, $(I - C^{(0)})(c_0)$ cannot be closed. If it were, together with the fact that $\|(C^{(0)})^n\|/n \to 0$ (because it is power bounded), using Theorem 1.3.2, we would have that $C^{(0)}$ is uniformly mean ergodic, and thus it would be mean ergodic, which we proved false. □

Proposition 5.4.2. *The operator $C^{(\infty)} : l^\infty \longrightarrow l^\infty$ is power bounded but not mean ergodic. Moreover,*

$$\ker(I - C^{(\infty)}) = \operatorname{span}\{\mathbf{1}\}.$$

PROOF. $C^{(\infty)}$ is power bounded, since, by Lemma 5.2.1, $\|C^{(\infty)}\| = 1$.

Using Lemma 5.3.2, and considering that span$\{\mathbf{1}\} \subset l^\infty$, we get

$$\ker(I - C^{(\infty)}) = \operatorname{span}\{\mathbf{1}\}.$$

If $C^{(\infty)}$ were mean ergodic, then the restriction of $C^{(\infty)}$ to any closed $C^{(\infty)}$-invariant subspace of l^∞ would be also mean ergodic. However $C^{(0)}$ is not mean ergodic by Proposition 5.4.1, and c_0 is a closed $C^{(\infty)}$-invariant subspace of l^∞. This is a contradiction, hence, $C^{(\infty)}$ cannot be mean ergodic. □

Proposition 5.4.3. *The operator $C^{(c)} : c \longrightarrow c$ is power bounded but not mean ergodic. Moreover,*

$$\ker(I - C^{(c)}) = \operatorname{span}\{\mathbf{1}\}.$$

PROOF. Follow the proof of Proposition 5.4.2, using that span$\{\mathbf{1}\} \subset c$ and that c_0 is a closed subspace of c. □

Proposition 5.4.4. *If $1 < p < \infty$, the operator $C^{(p)} : l^p \longrightarrow l^p$ is neither power bounded nor mean ergodic. Moreover,*

$$\ker(I - C^{(p)}) = \{0\}.$$

Also,

$$(I - C^{(p)})(l^p) = \overline{\text{span}}\{e_r\}_{r\geq 2} = \{x \in l^p \ : \ x_1 = 0\}.$$

Proof. Let $1/p + 1/q = 1$. By the spectral mapping theorem, $q^n \in \sigma((C^{(p)})^n)$, since $q \in \sigma(C^{(p)})$. Thus, by [9, Th. 7.3-4], $q^n \leq r((C^{(p)})^n) \leq \|(C^{(p)})\|$, and

$$\sup_{n\in\mathbb{N}} \frac{\|(C^{(p)})^n\|}{n} \geq \sup_{n\in\mathbb{N}} \frac{q^n}{n} = \infty.$$

Therefore, $C^{(p)}$ is not power bounded.

Considering that $\mathbf{1} \notin l^p$ and that $(I - C^{(p)})(l^p)$ is closed [6], apply the proofs of Propositions 5.4.1 and 5.4.2 to conclude. □

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