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Analysis of Normal Lines For Structural Grids

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Abstract

For free-forms the structural grid affects the design of the envelope as well as the design of the structure. Due to the complex geometry of the surface the directions of the normal lines will vary. The diversity of the directions of the normal lines affects the design of the envelope and structure, especially in case the surface is connected directly, without secondary structure, to the members of the structural frame and the structural members have to be curved and twisted. Comparing several structural grids shows the effect of the grid for the directions of the normal lines and the design of the structural elements.

Keywords: Normal lines, structural grid, geometric complex surface, free-form, ellipsoid, design.

1 Introduction

For buildings with a complex geometry the choice of the grid of the surface affects both architectural and structural design, especially in case the surface is attached to the structural members. For conventional buildings the structural elements, supporting floors and roofs, are positioned mostly with the main axis of the section perpendicular to the surface. For example, the wooden beams supporting a pitched roof are constructed by preference perpendicular to the roof, so the paneling can be fixed easily and the moment of inertia of the sections of the beams is maximal perpendicular to the surface. For free-forms the directions of the normal lines will vary, consequently the bars, connected perpendicular to the surface, must be curved and twisted to follow the complex geometry of the surface. For single curved surfaces as conical and cylindrical roofs the grid can be constructed of single curved untwisted bars. Even the members of a grid for a double curved spherical dome can be single curved and untwisted, just because the normal lines of a spherical surface intersect at the center. For any two points on the surface of a sphere a single plane can be drawn through the center. The normal lines of these two points intersect at the center, so these normal lines are a part of the plane too. A member of a structure following the intersection line is single curved and the curvature is constant. The normal lines are within the constructed face and the sections of the single curved member can be constructed perpendicular to the surface without twisting. This paper will show that for free-forms the normal lines do not intersect. Consequently the members of the structure must be twisted in case the sections are attached perpendicular to the curved surface. An interesting example of a spatial frame composed of twisted members was designed for the DO-Bubble (Veltkamp [3]).

Envelopes of blobs can be designed by transforming a regular form. Many blobs are created by flattening, stretching, truncating a spherical volume. In the past many spatial frames were designed for spherical surfaces. For example, Engel describes the Schwedler, the hexagonal, the lamella and the geodesic dome (Engel [1]). These frames can be constructed of single curved untwisted bars. Probably this advantage vanishes if the structural grid, originally designed for spherical domes, is transformed. To answer this question this research focuses on a single transformation by stretching a sphere into an ellipsoid. In spite of the symmetry the directions of the normal lines of an ellipsoid vary and will not intersect at one central point. Analyzing and evaluating the structural grids for ellipsoids will show the consequences for the form of the members.

2 The directions of the normal lines of an ellipsoid

The surface of the analyzed ellipsoid can be described with the general expression:

$$(x/a)^n + (y/b)^n + (z/c)^n = 1 \qquad (1)$$

The ellipsoid is stretched along the Y-axis. The following parameters are chosen: n = 2, c = a and b = 2a. Substituting the parameters gives the following expression:

$$(x/a)^2 + \tfrac{1}{4}\,(y/a)^2 + (z/a)^2 = 1 \qquad (2)$$

For any point of a surface exactly one line can be constructed which is perpendicular to the surface. This normal line is perpendicular to all tangential lines on the surface through this point. Conform mathematics the normal line in a certain point P (p_1, p_2, p_3) of the ellipsoidal surface is described as a vector including the parameter λ:

$$(x, y, z) = (p_1, p_2, p_3) + \lambda\,(1, \tfrac{1}{4}\, p_2 / p_1\,,\; p_3 / p_1) \qquad (3)$$

The normal lines of two points of the surface of the ellipsoid are positioned in one face if these normal lines are parallel or intersect. For several grids the directions of the normal lines are analyzed. Next these grids are evaluated and compared concerning the form of the structural elements.

3 The globe-grid

The globe-grid is composed of vertical meridians jointed in the top (0, 0, a) and horizontal parallels. This grid is identical to the grids constructed on a globe to define any position on the surface with the altitude and latitude. The vertical meridian in The XZ-face is circular, the radius of the circle is equal to r. Assume the angle with the Z-axis is named ϕ. The meridian is partitioned into equal parts r.dϕ, with r = a. The parallels are defined by intersection of the surface with the parallel faces with z = a. cos ϕ. Next the parallel in the ground face is defined. Assume the angle of a vector in the ground face with respect to the X-axis is equal to θ. The coordinates of the parallel constructed by the intersection of the ellipsoid with the ground face are: x = a cos θ; y = 2a sin θ; z = 0.

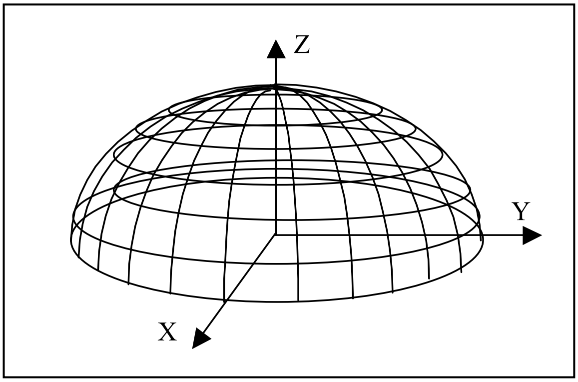

Figure 1: The globe-grid. The meridians are jointed at the top and the parallels are ellipses constructed in faces parallel the ground face.

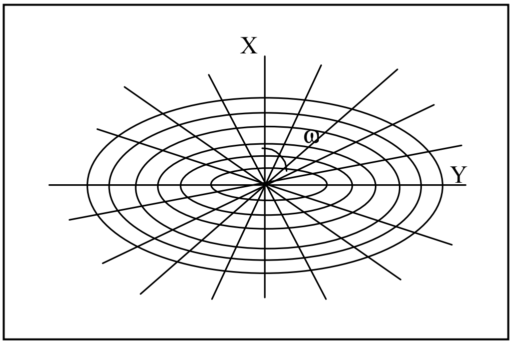

Figure 2: Plan of the globe-grid

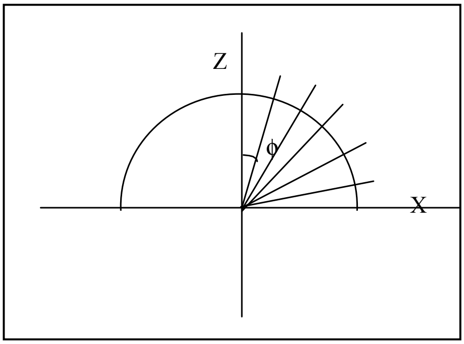

Figure 3: section of the globe-grid projected in the X-Z face.

Assume the angle between the X-axis and a vector directing to a point on the ellipse in the ground face is equal to ω. The tangent of the angle ω follows from:

$$\tan \omega = y/x = 2a \sin \theta / (a \cos \theta) = 2.\tan \theta. \tag{4}$$

Substituting $z = a. \cos \phi$ into expression (2) gives the following expression for the parallels:

$$(x/a)^2 + ¼ (y/a)^2 = 1- \cos^2 \phi \tag{5}$$

The form of the parallels does not change, the parallels are similar ellipses. Consequently $\tan \omega$ is independent with respect to the altitude and the meridians can be drawn for a certain value of ω in a vertical face. The coordinates of the meridians can be calculated with:

$$x = a (1- \cos^2 \phi)^{0.5}.\cos \theta \quad \text{and} \quad y = 2a (1- \cos^2 \phi)^{0.5}.\sin \theta. \tag{6}$$

The normal in any point P is defined with expression (3):

$$(x, y, z) = (p_1, p_2, p_3) + \lambda (1, ¼ p_2 / p_1 , p_3 / p_1) \tag{3}$$

Two points on a parallel with $\phi = 30^0$ are chosen with $\theta = 45^0$ and $\theta = 60^0$. Next the normal lines are defined for these two points:

For $\phi = 30^0$ en $\theta = 45^0$:

$$x = ¼ \sqrt{2} + 1 .\lambda_1$$
$$y = ½ \sqrt{2} + ½ .\lambda_1$$
$$z = ½ \sqrt{3} + 2\sqrt{3}/\sqrt{2}.\lambda_1$$

For $\phi = 30^0$ en $\theta = 60^0$.

$$x = ¼ + 1.\ \lambda_2$$
$$y = ½ \sqrt{3} + ½\sqrt{3}.\ \lambda_2$$
$$z = ½ \sqrt{3} + 2 \sqrt{3}.\ \lambda_2$$

The normal lines intersect if there is a point (x, y, z) that fulfills both expressions:

$$¼ \sqrt{2} + 1 .\lambda_1 = ¼ + 1 .\lambda_2$$
$$½ \sqrt{2} + ½ .\lambda_1 = ½\sqrt{3} + ½ \sqrt{3}.\ \lambda_2$$
$$½ \sqrt{3} + 2\sqrt{3}/\sqrt{2}.\ .\lambda_1 = ½ \sqrt{3} + 2 \sqrt{3}.\ \lambda_2$$

The Gauss-decomposition shows that the matrix is singular, the lines cross. It is impossible to create a face containing both normal lines through the chosen points of the parallel.

Next two points on a meridian are chosen, with $\theta = 45^0$ with $\phi = 30^0$ en $\phi = 45^0$. The normal lines in these two points are:

For $\theta = 45^0$ en $\phi = 30^0$:
$$x = \tfrac{1}{4}\sqrt{2} + 1.\ \lambda_1$$
$$y = \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\ \lambda_1$$
$$z = \tfrac{1}{2}\sqrt{3} + 2\sqrt{3}/\sqrt{2}\ \lambda_1$$

For $\theta = 45^0$ en $\phi = 45^0$:
$$x = \tfrac{1}{2} + 1.\ \lambda_2$$
$$y = 1 + \tfrac{1}{2}\ .\ \lambda_2$$
$$z = \tfrac{1}{2}\sqrt{2} + \sqrt{2}.\ \lambda_2$$

The normal lines intersect if there is a point (x, y, z) that fulfills both expressions:

$$\tfrac{1}{4}\sqrt{2} + \lambda_1\ .1 = \tfrac{1}{2} + 1.\ \lambda_2$$
$$\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\lambda_1 = 1 + \tfrac{1}{2}\ .\lambda_2$$
$$\tfrac{1}{2}\sqrt{3} + 2\sqrt{3}/\sqrt{2}.\ \lambda_1 = \tfrac{1}{2}\sqrt{2} + \sqrt{2}.\ \lambda_2$$

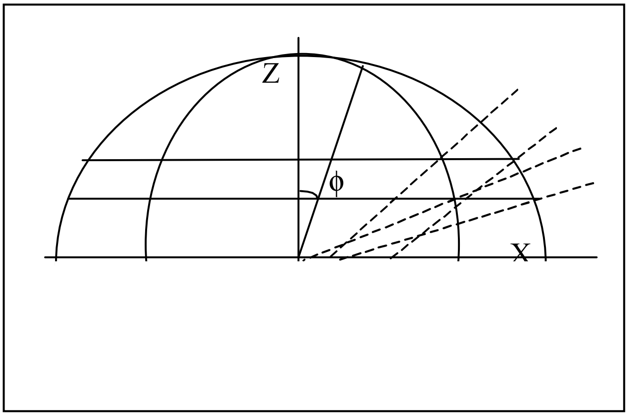

Figure 4: The normal lines drawn for four points of the surface projected on the X-Z face.

The Gauss-decomposition shows the matrix is singular, the lines cross. It is not possible to create a face through the normal lines of the chosen points. Figure 4 shows the normal lines for two points on a meridian and two points on a parallel projected on the X-Z face. For the constructed points and normal lines the angles ϕ and angle θ vary, so these lines do not intersect.

4 Parallel-grid

The parallel-grid is constructed with horizontal and vertical parallels. The volume is intersected by parallel faces perpendicular to respectively the Z– and Y-axis, with $z = a.\gamma$ for $\gamma \leq 1$ and $y = 2a.\beta$ for $\beta \leq 1$. The intersections of the horizontal faces with the surface are ellipses, the intersections of the vertical faces with the surface are circular. The angle of a vector with the Z-axis is ϕ. For the vertical parallels the coordinates are equal to:

$$x = \pm a \sin\phi\, (1- \tfrac{1}{4}\, y^2/a^2)^{\frac{1}{2}} \quad \text{and} \quad z = \pm a \cos\phi\, (1- \tfrac{1}{4}\, y^2/a^2)^{\frac{1}{2}} \qquad (7)$$

The vertical parallel positioned at the XZ-face with y = 0 is partitioned into equal parts with length $r.d\phi$, r is the radius of the circle with r is equal to a. The horizontal parallels are constructed on a height $z = a. \cos\phi$. with $\phi = n.d\phi$. The angle of the vector with the X-axis is named θ**.** For the horizontal parallels the coordinates are equal to:

$$x = \pm a \cos\theta\, (1-z^2/a^2)^{\frac{1}{2}} \quad \text{and} \quad y = \pm 2\, a \sin\theta\, (1-z^2/a^2)^{\frac{1}{2}} \qquad (8)$$

The position of the horizontal and vertical parallels is chosen so the vertical and horizontal parallels tough in the XZ-face by tuning $\theta = n.d\theta$ and $\phi = n.d\phi$, so $n.d\theta = n.d\phi$. Thus the center to center distances of the parallels vary.

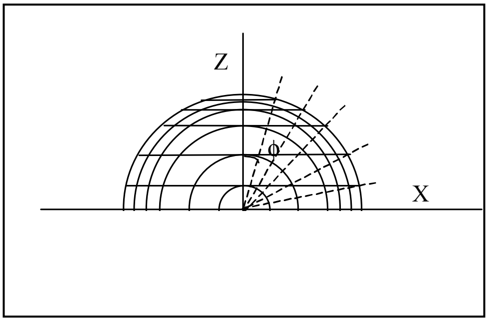

Figure 5: Section parallel X- axis

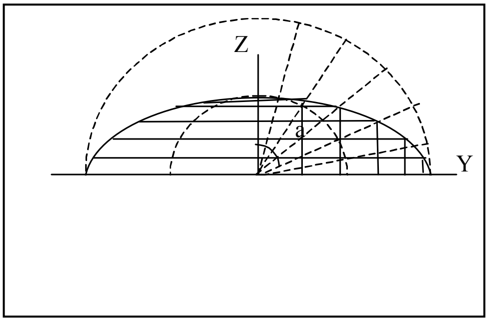

Figure 6: Section parallel Y – axis

The coordinates of the vertical parallels are constructed as follows, at first the position on the y-axis is found with: $y = 2a \sin\theta$, next the height is found with $z = a \cos\phi$. Finally the x coordinate is calculated with:

$$(x/a)^2 = 1- (z/c)^2 - (y/b)^2 \qquad (9)$$

Substitute y = 2a sin θ and z = a cos ϕ in (8):

$$(x/a)^2 = 1- (c.\cos\phi /c)^2 - (b.\sin\theta /b)^2 \quad \rightarrow \quad x = \pm a\,(1- \cos^2\phi - \sin^2\theta)^{½} \tag{10}$$

The normal in any point P is defined with expression (3):

$$(x,y,z) = (p_1,p_2,p_3) + \lambda\ (1, ¼\ p_2 /p_1\ ,\ p_3 / p_1) \tag{3}$$

The normal lines of two points on the surface are in a common face in case the normal lines are parallel or intersect. For two values of θ and ϕ the normal lines will be defined and analyzed.

For a parallel with $\phi = 60^0$ two points are chosen with $\theta = 30^0$ and $\theta = 45^0$. Next the expressions are constructed for the normal lines in these two points.

For $\phi = 60^0$, $\theta = 30^0$:

$$x = ½\sqrt{2} + 1\ .\ \lambda_1$$
$$y = 1 \quad + ¼\,\lambda_1\ \sqrt{2}$$
$$z = ½ \quad + ½\ \lambda_1\ .\sqrt{2}$$

For $\phi = 60^0$, $\theta = 45^0$:

$$x = ½ \quad + 1.\ \lambda_2$$
$$y = \sqrt{2} \quad + ½\sqrt{2}\,\lambda_2$$
$$z = ½ \quad + 1.\ \lambda_2$$

The normal lines intersect if any point (x, y, z) fulfills both expressions:

$$½\sqrt{2} + 1.\ \lambda_1 = ½ + 1.\ \lambda_2$$
$$1 + ¼\ \lambda_1 /\sqrt{2} = \sqrt{2} + ½\sqrt{2}.\lambda_2$$
$$½ + ½\,\lambda_1\ .\sqrt{2} = ½ + \lambda_2$$

The Gauss-decomposition of the matrix shows that the matrix is singular. The lines cross. It is not possible to create a face containing both normal lines on the chosen points of the parallel. In the same way we can prove that for the circular parallels the normal lines do not intersect too. For a parallel with $\theta = 60^0$ two points are chosen with $\theta = 30^0$ and $\theta = 45^0$. Next the expressions are constructed for the normal lines in these two points.

For $\theta = 60^0$, $\phi = 45^0$:

$$x = \tfrac{1}{4} + 1.\lambda_1$$
$$y = 1 + 1.\lambda_1$$
$$z = \tfrac{1}{2}.\sqrt{2} + 2\lambda_1.\sqrt{2}$$

For $\theta = 60^0$, $\phi = 60^0$:

$$x = \tfrac{1}{2} + 1.\lambda_2$$
$$y = 1 + \tfrac{1}{2}\lambda_2$$
$$z = \tfrac{1}{2} + 1.\lambda_2$$

The normal lines intersect if any point (x, y, z) fulfills both expressions:

$$\tfrac{1}{4} + 1.\lambda_1 = \tfrac{1}{2} + \lambda_2$$
$$1 + \lambda_1 = 1 + \tfrac{1}{2}.\lambda_2$$
$$\tfrac{1}{2}.\sqrt{2} + 2\lambda_1.\sqrt{2} = \tfrac{1}{2} + \lambda_2$$

The Gauss-decomposition shows the matrix is singular, the lines cross. It is not possible to create a face containing both normal lines on the chosen points of the parallel.

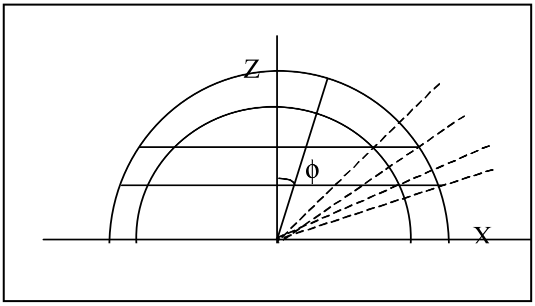

Figure 7: Projection of the grid on the XZ-face.

The results of the analysis are visualized with figure 7 in which two sections are projected on X Z – face. The angle ϕ of the normal lines varies for the nodes on a horizontal parallel. The ellipses decrease from the ground face bottom to the top so the angle θ will chance too for the nodes on a vertical parallel. The normal lines projected on the X Z – face seem to intersect at the center but perpendicular to the X Z -face the angle θ will vary, so the lines do not intersect.

5 Zeppelin grid

This grid is composed of circular curves positioned in faces parallel the XZ-face and perpendicular to the Y–axis for $y = 2a.\beta$ and $\beta = 0, \tfrac{1}{4}., \tfrac{1}{2}., \tfrac{3}{4}.,1$ and elliptical meridians which are jointed in the bow and rear at the point with the coordinates (0, 2a, 0). The center to center distances between the parallels are constant. This grid was applied for the rigid airships known as Zeppelins. The elliptical meridian in the ground face is defined as follows. Assume the angle of a vector in the ground face with the X-axis is equal to θ. The coordinates of the ellipse in the ground face can be constructed with: $x = a \cos \theta$; $y = 2a \sin \theta$; $z = 0$. Assume the angle with the Z-axis is named ϕ. The meridian in the XZ-face is partitioned into equal parts $r.d\phi$, r is the radius of the circle, $r = a$. The radius of the parallel follows from $x = a \cos \theta$. Any other parallel with $y > 0$ is partitioned in the same way. The meridians are constructed by connecting the nodes constructed on the parallels. A meridian can be constructed also by intersecting of the ellipsoid with a face through the Z-axis and the vector from the center to the node on the circular parallel in the XZ-face. The coordinates of the parallels can be calculated with:

$$x^2 / a^2 + z^2 / c^2 = (1 - \tfrac{1}{4} (y/a)^2) \qquad (11)$$

For a certain angle ϕ the coordinates can be calculated also with:

$$x = a(1 - \tfrac{1}{4} (y/a)^2)^{0,5} \sin \phi \quad \text{and } z = b(1 - \tfrac{1}{4} (y/a)^2)^{0,5} \cos \phi . \qquad (12)$$

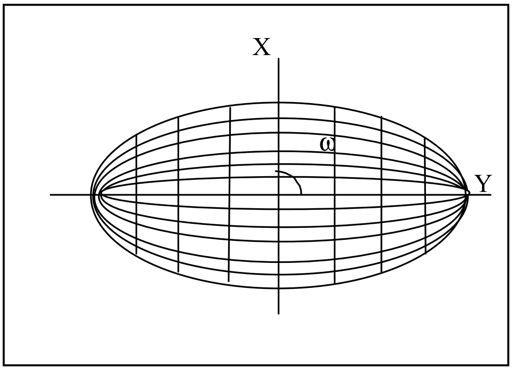

Figure 8: Plan of the Zeppelin-grid.

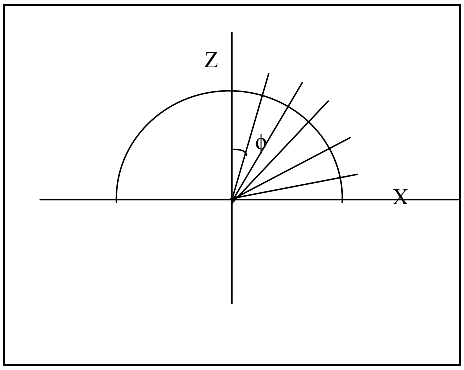

Figure 9: section parallel X-Z face.

The normal line in a point P is defined with expression (3):

$$(x, y, z) = (p_1, p_2, p_3) + \lambda .(1, \tfrac{1}{4}\ p_2 / p_1 ,\ p_3 / p_1) \qquad (3)$$

The normal lines of two points of the surface are in a face if these normal lines are parallel or intersect. Two points are chosen on the surface with a meridian for $\phi = 30^0$ and the parallels with y = a and y = 3/2 a. Next the expressions of the normal lines are defined:

For $\phi = 30^0$ en y = 1:

$$x = \tfrac{1}{4}\sqrt{3} + 1 . \lambda_1$$
$$y = 1 + \lambda_1 /\sqrt{3}$$
$$z = \tfrac{3}{4} + \lambda_1 .\sqrt{3}$$

For $\phi = 30^0$ en y = 3/2.

$$x = {}^1/_8 . \sqrt{7} + 1. \lambda_2$$
$$y = 3/2 + 3. \lambda_2 /\sqrt{7}$$
$$z = {}^1/_8 \sqrt{21} + \sqrt{3}.\lambda_2$$

The normal lines intersect if there is a point (x, y, z) that fulfills both expressions:

$$\tfrac{1}{4}\sqrt{3} + 1. \lambda_1 = {}^1/_8 . \sqrt{7} + 1. \lambda_2$$
$$1 + \lambda_1 /\sqrt{3} = {}^3/_2 + 3. \lambda_2 /\sqrt{7}$$
$$\tfrac{3}{4} + \lambda_1 .\sqrt{3} = {}^1/_8 . \sqrt{21} + \lambda_2 .\sqrt{3}$$

The expressions can be solved for $\lambda_2 = -1.00452$ and $\lambda_1 = -1.1068138$. A face can be constructed through both normal lines. Figure 10 shows the normal lines of two meridians projected on the X-Z face.

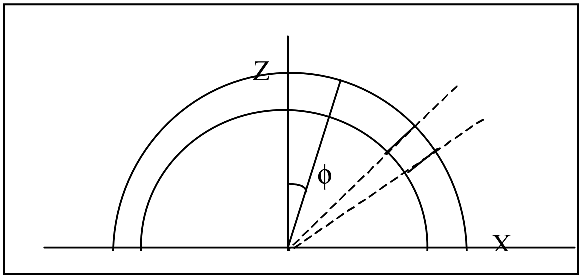

Figure 10: Section parallel to the XZ-face. For the meridians the angles ϕ of the normal lines with the Z-axis do not vary.

The zeppelin-grid can be constructed by rotating a meridian around the horizontal axis. The vertical parallels are perpendicular to the Y-axis and cross the meridians oblique. For every parallel the angle between the normal lines of the points of a parallel and the meridians is constant. The normal lines of the begin and end nodes of the elements following the meridians and the parallels of the zeppelin grid intersect so these elements do not have to be twisted if these members are attached with the section perpendicular to the surface. To be perpendicular to the surface the sections of the vertical parallels must be oblique to the vertical face. The sections of the parallel frames can be positioned parallel to the vertical faces too, then these sections are oblique with respect to the surface

6 Evaluation

The analysis of the normal lines shows that for an ellipsoid the normal lines do not intersect unless the grid is constructed according to the symmetry. Generally free-forms do not have an axis of symmetry and can not be constructed by rotating a curve around an axis. Consequently it will be very unlikely that the normal lines of a member intersect, so the elements of the structure must be twisted in case the surface is double curved and connected to the structural elements directly. Twisting the elements will be affecting the design of the structure. To twist an element the torsion stiffness must be small, open profiles will be preferred over closed sections. Elements with a triangular or square section are often to stiff to rotate, then these elements must be composed of twisted strips. To avoid twisted members the following options rise:

- Apply tubular sections, thanks to the symmetry the section of a tube will be always perpendicular to the surface.
- Facet the surface of the free-form and compose the structure of straight bars. Now the normal lines of the faces and members do not coincide. The faces are oblique with respect to the sections of the supporting members. At the joints the angle between the surface elements and the members of the structure vary.
- Support the surface at the vertices of the frame only.

Figure 11: Sweet water pavilion, Neeltje-Jans, The Netherlands

Figure 11 shows the structure of a free-form composed of vertical frames supporting twisted beams. The normal lines of the joints of the beams and frames vary. The beams are twisted to support the corrugated steel plates perpendicular to the surface.

Figure 12 shows the structure of a free-form composed of radial frames. The main decision of this structure was to align the structure and envelope without a secondary structure

(Kocaturk [2]). The diagonal and horizontal members are twisted to support the envelope, composed of double curved plates of hylite, directly.

Figure 12: The I-Web, Delft University, The Netherlands.

At the start of the process of design the effect of decisions is at most. The geometry of the surface and grid will affect the load transfer as well as the form of the structure. Especially for smooth double curved surfaces the choice of the elements of the envelope will affect the form of the structure. By preference the structural engineer has to investigate the features of the envelope at the start of the process, to choose a structural grid and design the structure optimal according to the load transfer, construction and architectural concept.

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