Expressions for generalized inverses of square matrices

Julio Benítez* and Xiaoji Liu†

Abstract

We find expressions for many types of generalized inverses of an arbitrary square complex matrix by using two representations given in [J. Benítez, A new decomposition for square matrices, Electronic Journal of Linear Algebra, 20 (2010) 207-225] and in [R.E. Hartwig, K. Spindelböck, Matrices for which A^* and $A^\dagger$ commute, Linear and Multilinear Algebra, 14 (1984) 241-256].
Keywords: Generalized inverses.
AMS-Subject Classification: 15A09.

1 Introduction

Let $\mathbb{C}_{m,n}$ be the set of $m \times n$ matrices. The symbols $\mathbf{A}^*$, $\mathcal{R}(\mathbf{A})$, $\mathcal{N}(\mathbf{A})$, and $\mathrm{rank}(\mathbf{A})$ denote the conjugate transpose, range (column space), null space, and rank, respectively, of $\mathbf{A} \in \mathbb{C}_{m,n}$. Additionally, $\mathbf{I}_n$ stands for the identity matrix of order n. Furthermore, let $\mathbf{A}^\dagger$ be the Moore-Penrose inverse of $\mathbf{A}$, i.e., the unique matrix satisfying the equations

$$(1)\ \mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}, \quad (2)\ \mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger, \quad (3)\ \mathbf{A}\mathbf{A}^\dagger = (\mathbf{A}\mathbf{A}^\dagger)^*, \quad (4)\ \mathbf{A}^\dagger\mathbf{A} = (\mathbf{A}^\dagger\mathbf{A})^*.$$

For any $\mathbf{A} \in \mathbb{C}_{m,n}$, let $\mathbf{A}\{i, j, \ldots, l\}$ denote the set of matrices $\mathbf{X} \in \mathbb{C}_{n,m}$ that satisfy equations $(i), (j), \ldots, (l)$ of (1), (2), (3), (4). A matrix $\mathbf{X} \in \mathbf{A}\{i, j, \ldots, l\}$ is called an $\{i, j, \ldots, l\}$-inverse of $\mathbf{A}$.

In this paper, we find expressions for generalized inverses by using two known decompositions (see Theorems 1 and 2 below). The results given here generalize the ones established in [3], where expressions for generalized inverses of normal matrices are given. We will not assume the normality of the involved matrices.

Hartwig and Spindelböck arrived at the following result ([4, Corollary 6]).

*Departamento de Matemática Aplicada, Instituto de Matemática Multidisciplinar, Universidad Politécnica de Valencia, Camino de Vera s/n, 46022 Valencia, Spain.

†School of Science, Guangxi University for Nationalities and Guangxi Key Laborarory of Hybrid Computational and IC Design Analysis, Nanning 530006, P.R. China.

Theorem 1. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be of rank* r*. Then there exists unitary* $\mathbf{U} \in \mathbb{C}_{n,n}$ *such that*

$$\mathbf{A} = \mathbf{U} \begin{pmatrix} \mathbf{\Sigma K} & \mathbf{\Sigma L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^*, \tag{1}$$

where $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1 \mathbf{I}_{r_1}, \ldots, \sigma_t \mathbf{I}_{r_t})$ *is the diagonal matrix of singular values of* $\mathbf{A}$*,* $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$*,* $r_1 + r_2 + \cdots + r_t = r$*, and* $\mathbf{K} \in \mathbb{C}_{r,r}$*,* $\mathbf{L} \in \mathbb{C}_{r,n-r}$ *satisfy*

$$\mathbf{KK}^* + \mathbf{LL}^* = \mathbf{I}_r. \tag{2}$$

From (1) it follows that

$$\mathbf{A}^* = \mathbf{U} \begin{pmatrix} \mathbf{K}^*\mathbf{\Sigma} & \mathbf{0} \\ \mathbf{L}^*\mathbf{\Sigma} & \mathbf{0} \end{pmatrix} \mathbf{U}^* \quad \text{and} \quad \mathbf{A}^\dagger = \mathbf{U} \begin{pmatrix} \mathbf{K}^*\mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{L}^*\mathbf{\Sigma}^{-1} & \mathbf{0} \end{pmatrix} \mathbf{U}^*. \tag{3}$$

A related decomposition was given in [1]. The symbols $\mathbf{1}_n$ and $\mathbf{0}_n$ will denote the $n \times 1$ row vectors all of whose components are 1 and 0, respectively.

Theorem 2. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be of rank* r*, and let* $\theta_1, \ldots, \theta_p$ *be the canonical angles between* $\mathcal{R}(\mathbf{A})$ *and* $\mathcal{R}(\mathbf{A}^*)$ *belonging to* $]0, \pi/2[$*. Denote by* x *and* y *the multiplicities of the angles* 0 *and* $\pi/2$ *as a canonical angle between* $\mathcal{R}(\mathbf{A})$ *and* $\mathcal{R}(\mathbf{A}^*)$*, respectively. There exists a unitary matrix* $\mathbf{V} \in \mathbb{C}_{n,n}$ *such that*

$$\mathbf{A} = \mathbf{V} \begin{pmatrix} \mathbf{MC} & \mathbf{MS} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^*, \tag{4}$$

where $\mathbf{M} \in \mathbb{C}_{r,r}$ *is nonsingular,*

$$\mathbf{C} = \mathrm{diag}(\mathbf{0}_y, \cos\theta_1, \ldots, \cos\theta_p, \mathbf{1}_x),$$

$$\mathbf{S} = \begin{pmatrix} \mathrm{diag}(\mathbf{1}_y, \sin\theta_1, \ldots, \sin\theta_p) & \mathbf{0}_{p+y,n-(r+p+y)} \\ \mathbf{0}_{x,p+y} & \mathbf{0}_{x,n-(r+p+y)} \end{pmatrix},$$

and $r = y + p + x$*. Furthermore,* x *and* $y + n - r$ *are the multiplicities of the singular values 1 and 0 in* $P_{\mathcal{R}(\mathbf{A})} P_{\mathcal{R}(\mathbf{A}^*)}$*, respectively.*

For $\mathbf{C}$ in Theorem 2, $\mathbf{C} = \mathbf{C}^*$. A simple (but useful) expression is the following:

$$\mathbf{C}^2 + \mathbf{SS}^* = \mathbf{I}_r. \tag{5}$$

Although the decomposition given in Theorem 1 can be derived from the singular value decomposition (s.v.d.) of $\mathbf{A}$, we will show how Theorem 1 can be deduced from Theorem 2 clarifying the relation between these two decompositions.

Let $\mathbf{A} \in \mathbb{C}_{n,n}$ be decomposed as in (4). Let $\mathbf{M} = \mathbf{W}_1 \mathbf{\Sigma} \mathbf{W}_2^*$ be the singular value decomposition of $\mathbf{M}$. Observe that $\mathbf{\Sigma}$ is nonsingular because $\mathbf{M}$ is nonsingular. Define the unitary matrix $\mathbf{U} = \mathbf{V}(\mathbf{W}_1 \oplus \mathbf{I}_{n-r})$. Now we have

$$\begin{aligned}
\mathbf{A} = \mathbf{V} \begin{pmatrix} \mathbf{MC} & \mathbf{MS} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^* = \mathbf{V} \begin{pmatrix} \mathbf{W}_1 \mathbf{\Sigma} \mathbf{W}_2^* \mathbf{C} & \mathbf{W}_1 \mathbf{\Sigma} \mathbf{W}_2^* \mathbf{S} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^* = \\
= \mathbf{V} \begin{pmatrix} \mathbf{W}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix} \begin{pmatrix} \mathbf{\Sigma} \mathbf{W}_2^* \mathbf{C} & \mathbf{\Sigma} \mathbf{W}_2^* \mathbf{S} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{W}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix} \begin{pmatrix} \mathbf{W}_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix} \mathbf{V}^* \\
= \mathbf{U} \begin{pmatrix} \mathbf{\Sigma} \mathbf{W}_2^* \mathbf{C} \mathbf{W}_1 & \mathbf{\Sigma} \mathbf{W}_2^* \mathbf{S} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^*.
\end{aligned}$$

Let us denote $\mathbf{K} = \mathbf{W}_2^* \mathbf{C} \mathbf{W}_1$ and $\mathbf{L} = \mathbf{W}_2^* \mathbf{S}$. In order to prove that we have obtained the decomposition of Theorem 1, we need to prove that the nonzero singular values of $\mathbf{A}$ are the singular values of $\mathbf{M}$ and (2) holds. By using the representation (4) and (5) one has $\mathbf{A}\mathbf{A}^* = \mathbf{V}(\mathbf{M}\mathbf{M}^* \oplus \mathbf{0})\mathbf{V}^*$, which reveals that the nonzero singular values of $\mathbf{A}$ are the singular values of $\mathbf{M}$. Since $\mathbf{W}_1$ and $\mathbf{W}_2$ are unitary, $\mathbf{C} = \mathbf{C}^*$, and (5), we have that (2) holds. To summarize: For any $\mathbf{A} \in \mathbb{C}_{n,n}$ represented as in (4), if we set

$$\mathbf{M} = \mathbf{W}_1 \mathbf{\Sigma} \mathbf{W}_2^* \text{ be the s.v.d. of } \mathbf{M}, \quad \mathbf{K} = \mathbf{W}_2^* \mathbf{C} \mathbf{W}_1, \quad \mathbf{L} = \mathbf{W}_2^* \mathbf{S}, \quad \mathbf{U} = \mathbf{V}(\mathbf{W}_1 \oplus \mathbf{I}_{n-r}), \tag{6}$$

then the decomposition of Theorem 1 is obtained.

Another useful formula (it can be verified by checking the four equations of the Moore-Penrose inverse or by applying (3) and (6)) is the following: If $\mathbf{A}$ is represented as in (4), then

$$\mathbf{A}^\dagger = \mathbf{V} \begin{pmatrix} \mathbf{C}\mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{S}^*\mathbf{M}^{-1} & \mathbf{0} \end{pmatrix} \mathbf{V}^*. \tag{7}$$

2 Expressions for $\{1\}$-inverses

In [5, Lemma 2] the authors gave the following general expression for $\mathbf{A}\{1\}$ when $\mathbf{A}$ is represented as in (1).

Theorem 3. *Let* $\mathbf{A}$ *be given as in (1) and let*

$$\mathbf{B} = \mathbf{U} \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{pmatrix} \mathbf{U}^*, \qquad \mathbf{B}_1 \in \mathbb{C}_{r,r}. \tag{8}$$

Then $\mathbf{B} \in \mathbf{A}\{1\} \iff \mathbf{\Sigma}\mathbf{K}\mathbf{B}_1 + \mathbf{\Sigma}\mathbf{L}\mathbf{B}_3 = \mathbf{I}_r$.

Theorem 3 is restated as follows.

Corollary 1. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be represented as in (1). Then*

$$\mathbf{A}\{1\} = \left\{ \mathbf{A}^\dagger + \mathbf{U}(\mathbf{D}_1 \mid \mathbf{D}_2)\mathbf{U}^* : \mathbf{D}_1 \in \mathbb{C}_{n,r}, \mathbf{D}_2 \in \mathbb{C}_{n,n-r}, (\mathbf{K} \mid \mathbf{L})\mathbf{D}_1 = \mathbf{0} \right\}.$$

Proof. $\subset$: Pick any $\mathbf{B} \in \mathbf{A}\{1\}$ and let us represent $\mathbf{B}$ as in (8). By using the right identity of (3) it follows that

$$\mathbf{B} - \mathbf{A}^\dagger = \mathbf{U} \begin{pmatrix} \mathbf{B}_1 - \mathbf{K}^*\mathbf{\Sigma}^{-1} & \mathbf{B}_2 \\ \mathbf{B}_3 - \mathbf{L}^*\mathbf{\Sigma}^{-1} & \mathbf{B}_4 \end{pmatrix} \mathbf{U}^*.$$

Observe that by Theorem 3 one has $\mathbf{K}\mathbf{B}_1 + \mathbf{L}\mathbf{B}_3 = \mathbf{\Sigma}^{-1}$. Hence by (2),

$$(\mathbf{K} \mid \mathbf{L}) \begin{pmatrix} \mathbf{B}_1 - \mathbf{K}^*\mathbf{\Sigma}^{-1} \\ \mathbf{B}_3 - \mathbf{L}^*\mathbf{\Sigma}^{-1} \end{pmatrix} = \mathbf{K}\mathbf{B}_1 - \mathbf{K}\mathbf{K}^*\mathbf{\Sigma}^{-1} + \mathbf{L}\mathbf{B}_3 - \mathbf{L}\mathbf{L}^*\mathbf{\Sigma}^{-1} = \mathbf{0}.$$

$\supset$: Choose any $\mathbf{D}_1 \in \mathbb{C}_{n,r}$ such that $(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1 = \mathbf{0}$ and pick any $\mathbf{D}_2 \in \mathbb{C}_{n,n-r}$. To prove this inclusion, it is sufficient to prove $\mathbf{A}\mathbf{U}(\mathbf{D}_1 \mid \mathbf{D}_2)\mathbf{U}^*\mathbf{A} = \mathbf{0}$. This last equality follows from the computation

$$\begin{aligned} &\begin{pmatrix} \mathbf{\Sigma}\mathbf{K} & \mathbf{\Sigma}\mathbf{L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} (\mathbf{D}_1 \mid \mathbf{D}_2) \begin{pmatrix} \mathbf{\Sigma}\mathbf{K} & \mathbf{\Sigma}\mathbf{L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \\ &\quad = \begin{pmatrix} \mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1 & \mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{\Sigma}\mathbf{K} & \mathbf{\Sigma}\mathbf{L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \mathbf{0}. \end{aligned}$$

The proof is finished. □

For a matrix $\mathbf{A} \in \mathbb{C}_{n,n}$ as given in (4) we have the following result.

Theorem 4. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be represented as in (4). Then*

$$\begin{aligned} &\mathbf{A}\{1\} = \\ &\left\{ \mathbf{V} \begin{pmatrix} \mathbf{X}\mathbf{M}^{-1} & \mathbf{Y} \\ \mathbf{Z}\mathbf{M}^{-1} & \mathbf{T} \end{pmatrix} \mathbf{V}^* : \ \mathbf{X} \in \mathbb{C}_{r,r}, \mathbf{Y} \in \mathbb{C}_{r,n-r}, \mathbf{Z} \in \mathbb{C}_{n-r,r}, \mathbf{T} \in \mathbb{C}_{n-r,n-r}, \mathbf{C}\mathbf{X} + \mathbf{S}\mathbf{Z} = \mathbf{I}_r \right\}. \end{aligned}$$

Proof. Let $\mathbf{A}$ have the form as in (4). Define $\mathbf{\Sigma}$, $\mathbf{K}$, $\mathbf{L}$, $\mathbf{U}$, $\mathbf{W}_1$, and $\mathbf{W}_2$ as in (6).

If $\mathbf{B}$ is an arbitrary element of $\mathbf{A}\{1\}$ represented as in (8), by Theorem 3, it follows that $\mathbf{\Sigma}\mathbf{K}\mathbf{B}_1 + \mathbf{\Sigma}\mathbf{L}\mathbf{B}_3 = \mathbf{I}_r$ holds. Now, by using (6) we have

$$\begin{aligned} \mathbf{B} &= \mathbf{V} \begin{pmatrix} \mathbf{W}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix} \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{pmatrix} \begin{pmatrix} \mathbf{W}_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix} \mathbf{V}^* \\ &= \mathbf{V} \begin{pmatrix} \mathbf{W}_1\mathbf{B}_1\mathbf{W}_1^* & \mathbf{W}_1\mathbf{B}_2 \\ \mathbf{B}_3\mathbf{W}_1^* & \mathbf{B}_4 \end{pmatrix} \mathbf{V}^*. \end{aligned}$$

Let us define

$$\mathbf{X} = \mathbf{W}_1\mathbf{B}_1\mathbf{W}_1^*\mathbf{M}, \qquad \mathbf{Y} = \mathbf{W}_1\mathbf{B}_2, \qquad \mathbf{Z} = \mathbf{B}_3\mathbf{W}_1^*\mathbf{M}, \qquad \mathbf{T} = \mathbf{B}_4.$$

To get the expression in the theorem, it is enough to prove $\mathbf{CX} + \mathbf{SZ} = \mathbf{I}_r$. For this, we use (6) and $\mathbf{KB}_1 + \mathbf{LB}_3 = \mathbf{\Sigma}^{-1}$.

$$\begin{aligned}\mathbf{CX} + \mathbf{SZ} &= \mathbf{W}_2\mathbf{K}\mathbf{W}_1^*\mathbf{W}_1\mathbf{B}_1\mathbf{W}_1^*\mathbf{M} + \mathbf{W}_2\mathbf{L}\mathbf{B}_3\mathbf{W}_1^*\mathbf{M} \\ &= \mathbf{W}_2\left(\mathbf{KB}_1 + \mathbf{LB}_3\right)\mathbf{W}_1^*\mathbf{M} \;=\; \mathbf{W}_2\mathbf{\Sigma}^{-1}\mathbf{W}_1^*\mathbf{M} \;=\; \mathbf{I}_r.\end{aligned}$$

This proves the inclusion '⊂' in the statement of the theorem. The opposite inclusion is trivial. □

The following corollary follows from Theorem 4.

Corollary 2. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be represented as in (4). Then*

$$\mathbf{A}\{1\} = \left\{\mathbf{V}(\widetilde{\mathbf{X}}\mathbf{M}^{-1} \mid \widetilde{\mathbf{Y}})\mathbf{V}^* : \; \widetilde{\mathbf{X}} \in \mathbb{C}_{n,r}, \; \widetilde{\mathbf{Y}} \in \mathbb{C}_{n,n-r}, \; (\mathbf{C} \mid \mathbf{S})\widetilde{\mathbf{X}} = \mathbf{I}_r\right\}.$$

Let $\mathbf{A} \in \mathbb{C}_{n,n}$ be represented as in (1) and as in (2). Then $\mathbf{A}$ is EP (i.e., $\mathbf{AA}^\dagger = \mathbf{A}^\dagger\mathbf{A}$) $\iff$ $\mathbf{S} = \mathbf{0}$ $\iff$ $\mathbf{L} = \mathbf{0}$. Observe that $\mathbf{S} = \mathbf{0}$ implies $\mathbf{C} = \mathbf{I}_r$ (in view of (5)). Thus from Theorem 3 and Theorem 4 we get the following corollary.

Corollary 3. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be EP.*

(i) *If* $\mathbf{A}$ *is of the form (1), then*

$$\mathbf{A}\{1\} = \left\{\mathbf{U}\begin{pmatrix} \mathbf{K}^{-1}\mathbf{\Sigma}^{-1} & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{pmatrix}\mathbf{U}^* : \mathbf{B}_2 \in \mathbb{C}_{r,n-r}, \mathbf{B}_3 \in \mathbb{C}_{n-r,r}, \mathbf{B}_4 \in \mathbb{C}_{n-r,n-r}\right\}.$$

(ii) *If* $\mathbf{A}$ *is of the form (4), then*

$$\mathbf{A}\{1\} = \left\{\mathbf{V}\begin{pmatrix} \mathbf{M}^{-1} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{T} \end{pmatrix}\mathbf{V}^* : \mathbf{Y} \in \mathbb{C}_{r,n-r}, \mathbf{Z} \in \mathbb{C}_{n-r,r}, \mathbf{T} \in \mathbb{C}_{n-r,n-r}\right\}.$$

Recall that the spectral theorem states that $\mathbf{A} \in \mathbb{C}_{n,n}$ is a normal matrix (i.e., $\mathbf{AA}^* = \mathbf{A}^*\mathbf{A}$) if and only if there exist a unitary matrix $\mathbf{U} \in \mathbb{C}_{n,n}$ and a diagonal matrix $\mathbf{D} \in \mathbb{C}_{n,n}$ such that $\mathbf{A} = \mathbf{UDU}^*$. Thus, any normal matrix is EP. Corollary 3 extends item (a) of Theorem 2.2 in [3].

We may restate Corollary 2 as follows.

Corollary 4. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be represented as in (4). Then*

$$\mathbf{A}\{1\} = \left\{\mathbf{A}^\dagger + \mathbf{V}(\widehat{\mathbf{X}} \mid \widehat{\mathbf{Y}})\mathbf{V}^* : \; \widehat{\mathbf{X}} \in \mathbb{C}_{n,r}, \widehat{\mathbf{Y}} \in \mathbb{C}_{n,n-r}, \; (\mathbf{C} \mid \mathbf{S})\widehat{\mathbf{X}} = \mathbf{0}\right\}.$$

Proof. $\subset$: Let $\mathbf{B} \in \mathbf{A}\{1\}$. By Corollary 2, there exist $\widetilde{\mathbf{X}} \in \mathbb{C}_{n,r}$, $\widetilde{\mathbf{Y}} \in \mathbb{C}_{n,n-r}$ such that $(\mathbf{C} \mid \mathbf{S})\widetilde{\mathbf{X}} = \mathbf{I}_r$ and $\mathbf{B} = \mathbf{V}(\widetilde{\mathbf{X}}\mathbf{M}^{-1} \mid \widetilde{\mathbf{Y}})\mathbf{V}^*$. Let us partition $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$ as follows:

$$\widetilde{\mathbf{X}} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Z} \end{pmatrix}, \qquad \widetilde{\mathbf{Y}} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{T} \end{pmatrix},$$

where $\mathbf{X} \in \mathbb{C}_{r,r}$, $\mathbf{Z} \in \mathbb{C}_{n-r,r}$, $\mathbf{Y} \in \mathbb{C}_{r,n-r}$, and $\mathbf{T} \in \mathbb{C}_{n-r,n-r}$. Let us define

$$\widehat{\mathbf{X}} = \begin{pmatrix} (\mathbf{X} - \mathbf{C})\mathbf{M}^{-1} \\ (\mathbf{Z} - \mathbf{S}^*)\mathbf{M}^{-1} \end{pmatrix}.$$

Now we have by (7)

$$\begin{aligned} \mathbf{V}(\widetilde{\mathbf{X}}\mathbf{M}^{-1} \mid \widetilde{\mathbf{Y}})\mathbf{V}^* &= \mathbf{V}\left\{\begin{pmatrix} (\mathbf{X} - \mathbf{C})\mathbf{M}^{-1} & \mathbf{Y} \\ (\mathbf{Z} - \mathbf{S}^*)\mathbf{M}^{-1} & \mathbf{T} \end{pmatrix} + \begin{pmatrix} \mathbf{C}\mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{S}^*\mathbf{M}^{-1} & \mathbf{0} \end{pmatrix}\right\}\mathbf{V}^* \\ &= \mathbf{V}\begin{pmatrix} (\mathbf{X} - \mathbf{C})\mathbf{M}^{-1} & \mathbf{Y} \\ (\mathbf{Z} - \mathbf{S}^*)\mathbf{M}^{-1} & \mathbf{T} \end{pmatrix}\mathbf{V}^* + \mathbf{A}^\dagger = \mathbf{V}(\widehat{\mathbf{X}} \mid \widetilde{\mathbf{Y}})\mathbf{V}^* + \mathbf{A}^\dagger. \end{aligned}$$

Observe that by (5) we have

$$(\mathbf{C} \mid \mathbf{S})\widetilde{\mathbf{X}} = \mathbf{I}_r \iff \mathbf{C}\mathbf{X} + \mathbf{S}\mathbf{Z} = \mathbf{I}_r \iff \mathbf{C}(\mathbf{X} - \mathbf{C}) + \mathbf{S}(\mathbf{Z} - \mathbf{S}^*) = \mathbf{0} \iff (\mathbf{C} \mid \mathbf{S})\widehat{\mathbf{X}} = \mathbf{0}.$$

$\supset$: Pick any $\widehat{\mathbf{X}} \in \mathbb{C}_{n,r}$ such that $(\mathbf{C} \mid \mathbf{S})\widehat{\mathbf{X}} = \mathbf{0}$ and pick any $\widehat{\mathbf{Y}} \in \mathbb{C}_{n,n-r}$. Let us define

$$\widetilde{\mathbf{X}} = \widehat{\mathbf{X}}\mathbf{M} + \begin{pmatrix} \mathbf{C} \\ \mathbf{S}^* \end{pmatrix}.$$

Obviously we have $(\mathbf{C} \mid \mathbf{S})\widetilde{\mathbf{X}} = \mathbf{I}_r$ and

$$\begin{aligned} \mathbf{A}^\dagger + \mathbf{V}(\widehat{\mathbf{X}} \mid \widehat{\mathbf{Y}})\mathbf{V}^* &= \mathbf{A}^\dagger + \mathbf{V}\left(\left[\widetilde{\mathbf{X}} - \begin{pmatrix} \mathbf{C} \\ \mathbf{S}^* \end{pmatrix}\right]\mathbf{M}^{-1} \,\middle|\, \widehat{\mathbf{Y}}\right)\mathbf{V}^* \\ &= \mathbf{A}^\dagger + \mathbf{V}(\widetilde{\mathbf{X}}\mathbf{M}^{-1} \mid \widetilde{\mathbf{Y}})\mathbf{V}^* - \mathbf{V}\begin{pmatrix} \mathbf{C}\mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{S}^*\mathbf{M}^{-1} & \mathbf{0} \end{pmatrix}\mathbf{V}^* = \mathbf{V}(\widetilde{\mathbf{X}}\mathbf{M}^{-1} \mid \widetilde{\mathbf{Y}})\mathbf{V}^*. \end{aligned}$$

By Corollary 2, we are done. $\square$

Observe that to use Corollary 4, we must solve $(\mathbf{C} \mid \mathbf{S})\widehat{\mathbf{X}} = \mathbf{0}$, where $\widehat{\mathbf{X}} \in \mathbb{C}_{n,r}$. Let us define $\mathbf{R} = (\mathbf{C} \mid \mathbf{S}) \in \mathbb{C}_{r,n}$ and thus, any column of $\widehat{\mathbf{X}}$ must satisfy the linear system $\mathbf{R}\mathbf{x} = \mathbf{0}$. Since $\mathbf{R}\mathbf{R}^* = \mathbf{I}_r$, then $\operatorname{rank}(\mathbf{R}) = \operatorname{rank}(\mathbf{R}\mathbf{R}^*) = r$, and therefore, $\dim \mathcal{N}(\mathbf{R}) = n - r$. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_{n-r}\}$ be a basis of $\mathcal{N}(\mathbf{R})$. If $\mathbf{u}_i$ is the i-th column of $\widehat{\mathbf{X}}$ (for $1 \le i \le r$), then there exist $\lambda_{i,1}, \ldots, \lambda_{i,n-r} \in \mathbb{C}$ such that $\mathbf{u}_i = \lambda_{i,1}\mathbf{x}_1 + \cdots + \lambda_{i,n-r}\mathbf{x}_{n-r}$. Hence

$$\widehat{\mathbf{X}} = (\mathbf{u}_1 \mid \cdots \mid \mathbf{u}_r) = (\mathbf{x}_1 \mid \cdots \mid \mathbf{x}_{n-r})\begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{r,1} \\ \vdots & \ddots & \vdots \\ \lambda_{1,n-r} & \cdots & \lambda_{r,n-r} \end{pmatrix}.$$

We rewrite Corollary 4 as follows.

Corollary 5. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be as in (4),* $\{\mathbf{x}_1, \cdots, \mathbf{x}_{n-r}\}$ *be a basis of* $\mathcal{N}((\mathbf{C} \mid \mathbf{S}))$*, and* $\mathbf{P} = (\mathbf{x}_1 \mid \cdots \mid \mathbf{x}_{n-r})$*. Then*

$$\mathbf{A}\{1\} = \{\mathbf{A}^\dagger + \mathbf{V}(\mathbf{P}\mathbf{\Lambda} \mid \mathbf{Q})\mathbf{V}^* : \mathbf{\Lambda} \in \mathbb{C}_{n-r,r}, \mathbf{Q} \in \mathbb{C}_{n,n-r}\}.$$

Note that in the above Corollary 5, matrices $\mathbf{\Lambda}$ and $\mathbf{Q}$ are completely arbitrary. To find a matrix $\mathbf{P}$ satisfying the hypotheses of former corollary, it is sufficient to solve the easy linear system $(\mathbf{C} \mid \mathbf{S})\mathbf{x} = \mathbf{0}$, $\mathbf{x} \in \mathbb{C}_{n,1}$. Observe that the matrix $\mathbf{C}$ is diagonal and $\mathbf{S}$ is "almost diagonal".

Although Corollary 4 can be used to prove the following result (by means of $\mathbf{A}^\dagger + \mathbf{V}(\widehat{\mathbf{X}} \mid \mathbf{0})\mathbf{V}^* + \mathbf{V}(\mathbf{0} \mid \widehat{\mathbf{Y}})\mathbf{V}^*$), the singular value decomposition leads to a simpler proof.

Corollary 6. *If* $\mathbf{A} \in \mathbb{C}_{n,n}$*, then*

(i) $\mathbf{A}\{1\} = \left\{\mathbf{A}^\dagger + \mathbf{B}_1 + \mathbf{B}_2 : \mathbf{B}_1, \mathbf{B}_2 \in \mathbb{C}_{n,n}, \mathbf{A}\mathbf{B}_1 = \mathbf{0}, \mathbf{B}_1 = \mathbf{B}_1\mathbf{A}\mathbf{A}^\dagger, \mathbf{B}_2\mathbf{A} = \mathbf{0}\right\}$.

(ii) $\mathbf{A}\{1\} = \left\{\mathbf{A}^\dagger + \mathbf{B}_1 + \mathbf{B}_2 : \mathbf{B}_1, \mathbf{B}_2 \in \mathbb{C}_{n,n}, \mathbf{A}\mathbf{B}_1 = \mathbf{0}, \mathbf{B}_2\mathbf{A} = \mathbf{0}\right\}$.

(iii) *If* $\mathbf{B} \in \mathbf{A}\{1\}$*, then* $\mathbf{A}\{1\} = \{\mathbf{B} + \mathbf{B}_1 + \mathbf{B}_2 : \mathbf{B}_1, \mathbf{B}_2 \in \mathbb{C}_{n,n}, \mathbf{A}\mathbf{B}_1 = \mathbf{0}, \mathbf{B}_2\mathbf{A} = \mathbf{0}\}$.

Proof. Let $\mathbf{A} = \mathbf{U}_1(\mathbf{D} \oplus \mathbf{0})\mathbf{U}_2^*$ be the singular value decomposition of $\mathbf{A}$, with $\mathbf{D} \in \mathbb{C}_{r,r}$ being nonsingular. Let

$$\mathcal{A}_1 = \left\{\mathbf{A}^\dagger + \mathbf{B}_1 + \mathbf{B}_2 : \mathbf{B}_1, \mathbf{B}_2 \in \mathbb{C}_{n,n}, \mathbf{A}\mathbf{B}_1 = \mathbf{0}, \mathbf{B}_1 = \mathbf{B}_1\mathbf{A}\mathbf{A}^\dagger, \mathbf{B}_2\mathbf{A} = \mathbf{0}\right\}$$

and

$$\mathcal{A}_2 = \left\{\mathbf{A}^\dagger + \mathbf{B}_1 + \mathbf{B}_2 : \mathbf{B}_1, \mathbf{B}_2 \in \mathbb{C}_{n,n}, \mathbf{A}\mathbf{B}_1 = \mathbf{0}, \mathbf{B}_2\mathbf{A} = \mathbf{0}\right\}.$$

We shall prove $\mathbf{A}\{1\} \subset \mathcal{A}_1$: Let $\mathbf{A}^- \in \mathbf{A}\{1\}$. It is well known that $\mathbf{A}^\dagger = \mathbf{U}_2\left(\begin{smallmatrix}\mathbf{D}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{smallmatrix}\right)\mathbf{U}_1^*$. Also, it is simple to prove that $\mathbf{A}^-$ can be written as $\mathbf{A}^- = \mathbf{U}_2\left(\begin{smallmatrix}\mathbf{D}^{-1} & \mathbf{X}\\ \mathbf{Y} & \mathbf{Z}\end{smallmatrix}\right)\mathbf{U}_1^*$, where $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are matrices of suitable size. If we define $\mathbf{B}_1 = \mathbf{U}_2\left(\begin{smallmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{Y} & \mathbf{0}\end{smallmatrix}\right)\mathbf{U}_1^*$ and $\mathbf{B}_2 = \mathbf{U}_2\left(\begin{smallmatrix}\mathbf{0} & \mathbf{X}\\ \mathbf{0} & \mathbf{Z}\end{smallmatrix}\right)\mathbf{U}_1^*$, then we have $\mathbf{A}^- = \mathbf{A}^\dagger + \mathbf{B}_1 + \mathbf{B}_2$. The equalities $\mathbf{A}\mathbf{B}_1 = \mathbf{0}$, $\mathbf{B}_1\mathbf{A}\mathbf{A}^\dagger = \mathbf{B}_1$, and $\mathbf{B}_2\mathbf{A} = \mathbf{0}$ are trivial to verify.

The inclusion $\mathcal{A}_1 \subset \mathcal{A}_2$ is evident. Finally, we prove $\mathcal{A}_2 \subset \mathbf{A}\{1\}$. To this end, take any $\mathbf{B}_1, \mathbf{B}_2 \in \mathbb{C}_{n,n}$ such that $\mathbf{A}\mathbf{B}_1 = \mathbf{B}_2\mathbf{A} = \mathbf{0}$. Now $\mathbf{A}(\mathbf{A}^\dagger + \mathbf{B}_1 + \mathbf{B}_2)\mathbf{A} = \mathbf{A}\mathbf{A}^\dagger\mathbf{A} + \mathbf{A}\mathbf{B}_1\mathbf{A} + \mathbf{A}\mathbf{B}_2\mathbf{A} = \mathbf{A}$. Hence we have proved (i) and (ii).

To prove (iii, $\subset$), pick any $\mathbf{A}^- \in \mathbf{A}\{1\}$. By item (ii), there exist $\mathbf{D}_1, \mathbf{D}_2 \in \mathbb{C}_{n,n}$ such that $\mathbf{A}^- = \mathbf{A}^\dagger + \mathbf{D}_1 + \mathbf{D}_2$ and $\mathbf{A}\mathbf{D}_1 = \mathbf{D}_2\mathbf{A} = \mathbf{0}$. Since $\mathbf{B} \in \mathbf{A}\{1\}$, there exist $\mathbf{E}_1, \mathbf{E}_2 \in \mathbb{C}_{n,n}$ such that $\mathbf{B} = \mathbf{A}^\dagger + \mathbf{E}_1 + \mathbf{E}_2$ and $\mathbf{A}\mathbf{E}_1 = \mathbf{E}_2\mathbf{A} = \mathbf{0}$. We get $\mathbf{A}^- = \mathbf{B} + (\mathbf{D}_1 - \mathbf{E}_1) + (\mathbf{D}_2 - \mathbf{E}_2)$ and $\mathbf{A}(\mathbf{D}_1 - \mathbf{E}_1) = (\mathbf{D}_2 - \mathbf{E}_2)\mathbf{A} = \mathbf{0}$. The proof of (iii, $\supset$) is similar to that of $\mathcal{A}_2 \subset \mathbf{A}\{1\}$. □

3 Expressions for $\{2\}$-inverses

In this section we give expressions for $\mathbf{A}\{2\}$ when $\mathbf{A} \in \mathbb{C}_{n,n}$ is represented as in (1) or in (4).

Theorem 5. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$ be as in (1). Then*

$$\mathbf{A}\{2\} = \\ \{\mathbf{U}(\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_2)\mathbf{U}^* : \mathbf{D}_1 \in \mathbb{C}_{n,r}, \mathbf{D}_2 \in \mathbb{C}_{n,n-r}, \mathbf{D}_1(\mathbf{K} \mid \mathbf{L}) \textit{ is idempotent}, \mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 = \mathbf{D}_2\}.$$

Proof. Let $\mathbf{B} \in \mathbf{A}\{2\}$ be represented as follows:

$$\mathbf{B} = \mathbf{U}(\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_2)\mathbf{U}^*, \qquad \mathbf{D}_1 \in \mathbb{C}_{n,r},\ \mathbf{D}_2 \in \mathbb{C}_{n,n-r}.$$

We have

$$\mathbf{BAB} = \mathbf{U}(\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2)\mathbf{U}^*.$$

From $\mathbf{BAB} = \mathbf{B}$ we get

$$\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1 = \mathbf{D}_1 \qquad \text{and} \qquad \mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 = \mathbf{D}_2. \tag{9}$$

Postmultiplying the first equality of (9) by $(\mathbf{K} \mid \mathbf{L})$ leads to the idempotency of $\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})$. We have proved the "$\subset$" inclusion.

We now prove the opposite inclusion. Let $\mathbf{D}_1 \in \mathbb{C}_{n,r}$ and $\mathbf{D}_2 \in \mathbb{C}_{n,n-r}$ such that $\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})$ is idempotent and the second equality of (9) holds. By postmultiplying $\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1(\mathbf{K} \mid \mathbf{L}) = \mathbf{D}_1(\mathbf{K} \mid \mathbf{L})$ by $\left(\begin{smallmatrix}\mathbf{K}^* \\ \mathbf{L}^*\end{smallmatrix}\right)$ and using (2), we have

$$\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1 = \mathbf{D}_1. \tag{10}$$

To check $\mathbf{U}(\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_2)\mathbf{U}^* \in \mathbf{A}\{2\}$, we use the second equality of (9) and (10).

$$\begin{aligned}
&\mathbf{U}(\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_2)\mathbf{U}^*\mathbf{A}\mathbf{U}(\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_2)\mathbf{U}^* \\
&\quad = \mathbf{U}(\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_2)\begin{pmatrix} \mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L}) \\ \mathbf{0} \end{pmatrix}(\mathbf{D}_1\mathbf{\Sigma}^{-1} | \mathbf{D}_2)\mathbf{U}^* \\
&\quad = \mathbf{U}\mathbf{D}_1 (\mathbf{K} \mid \mathbf{L}) (\mathbf{D}_1\mathbf{\Sigma}^{-1} | \mathbf{D}_2)\mathbf{U}^* \\
&\quad = \mathbf{U}\left(\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2\right)\mathbf{U}^* \\
&\quad = \mathbf{U}\left(\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_2\right)\mathbf{U}^*.
\end{aligned}$$

□

For matrix $\mathbf{A} \in \mathbb{C}_{n,n}$ in (4), we can also give a general expression for $\mathbf{A}\{2\}$.

Theorem 6. *If* $\mathbf{A} \in \mathbb{C}_{n,n}$ *is represented as in (4), then*

$$\mathbf{A}\{2\} = \left\{ \mathbf{V}(\widetilde{\mathbf{X}}\mathbf{M}^{-1} \mid \widetilde{\mathbf{Y}})\mathbf{V}^* : \widetilde{\mathbf{X}} \in \mathbb{C}_{n,r}, \widetilde{\mathbf{Y}} \in \mathbb{C}_{n,n-r}, \widetilde{\mathbf{X}}(\mathbf{C} \mid \mathbf{S}) \text{ is idempotent, } \widetilde{\mathbf{X}}(\mathbf{C} \mid \mathbf{S})\widetilde{\mathbf{Y}} = \widetilde{\mathbf{Y}} \right\}.$$

Proof. $\subset$: If $\mathbf{B} \in \mathbf{A}\{2\}$, then by Theorem 5, there exist $\mathbf{D}_1 \in \mathbb{C}_{n,r}$ and $\mathbf{D}_2 \in \mathbb{C}_{n,n-r}$ such that $\mathbf{B} = \mathbf{U}(\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_2)\mathbf{U}^*$, $\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})$ is idempotent and $\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 = \mathbf{D}_2$. Let us define

$$\widetilde{\mathbf{X}} = (\mathbf{W}_1 \oplus \mathbf{I}_{n-r})\mathbf{D}_1\mathbf{W}_2^* \qquad \text{and} \qquad \widetilde{\mathbf{Y}} = (\mathbf{W}_1 \oplus \mathbf{I}_{n-r})\mathbf{D}_2.$$

By (6) we get

$$\begin{aligned} \mathbf{B} &= \mathbf{U}(\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_2)\mathbf{U}^* \\ &= \mathbf{V}(\mathbf{W}_1 \oplus \mathbf{I}_{n-r})(\mathbf{D}_1\mathbf{W}_2^*\mathbf{M}^{-1}\mathbf{W}_1 \mid \mathbf{D}_2)(\mathbf{W}_1^* \oplus \mathbf{I}_{n-r})\mathbf{V}^* \\ &= \mathbf{V}(\widetilde{\mathbf{X}}\mathbf{M}^{-1}\mathbf{W}_1 \mid \widetilde{\mathbf{Y}})(\mathbf{W}_1^* \oplus \mathbf{I}_{n-r})\mathbf{V}^* \\ &= \mathbf{V}(\widetilde{\mathbf{X}}\mathbf{M}^{-1} \mid \widetilde{\mathbf{Y}})\mathbf{V}^*. \end{aligned}$$

Now we prove that $\widetilde{\mathbf{X}}(\mathbf{C} \mid \mathbf{S})$ is idempotent and $\widetilde{\mathbf{X}}(\mathbf{C} \mid \mathbf{S})\widetilde{\mathbf{Y}} = \widetilde{\mathbf{Y}}$.

$$\begin{aligned} \widetilde{\mathbf{X}}(\mathbf{C} \mid \mathbf{S}) &= (\mathbf{W}_1 \oplus \mathbf{I}_{n-r})\mathbf{D}_1\mathbf{W}_2^*(\mathbf{W}_2\mathbf{K}\mathbf{W}_1^* \mid \mathbf{W}_2\mathbf{L}) \\ &= (\mathbf{W}_1 \oplus \mathbf{I}_{n-r})\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})(\mathbf{W}_1^* \oplus \mathbf{I}_{n-r}). \end{aligned} \tag{11}$$

Since $\mathbf{W}_1 \oplus \mathbf{I}_{n-r}$ is unitary and $\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})$ is idempotent, $\widetilde{\mathbf{X}}(\mathbf{C} \mid \mathbf{S})$ is also idempotent. From (11), $\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 = \mathbf{D}_2$, and the definition of $\widetilde{\mathbf{Y}}$ we easily get $\widetilde{\mathbf{X}}(\mathbf{C} \mid \mathbf{S})\widetilde{\mathbf{Y}} = \widetilde{\mathbf{Y}}$.

$\supset$: The proof of this inclusion is similar to the proof of "$\supset$" in Theorem 6. □

In the proof of the following corollary, we utilize the fact that every idempotent matrix $\mathbf{X} \in \mathbb{C}_{n,n}$ is diagonalizable (see e.g., [2, Theorem 4.1]), and thus there exists a nonsingular matrix $\mathbf{R} \in \mathbb{C}_{n,n}$ such that

$$\mathbf{X} = \mathbf{R}(\mathbf{I}_r \oplus \mathbf{0})\mathbf{R}^{-1}, \tag{12}$$

where $r = \text{rank}(\mathbf{X})$. Clearly, $0 \leq r \leq n$ and if $r = n$ ($r = 0$), then the latter (the first) of the summands in representation (12) vanishes. The following result generalizes item (b) of Theorem 2.1 in [3].

Corollary 7. *If* $\mathbf{A} \in \mathbb{C}_{n,n}$ *is EP and represented as in (4), then*

$$\mathbf{A}\{2\} = \left\{ \mathbf{V} \begin{pmatrix} \mathbf{R}(\mathbf{I}_s \oplus \mathbf{0})\mathbf{R}^{-1}\mathbf{M}^{-1} & \mathbf{R}\binom{\mathbf{Y}_1}{\mathbf{0}} \\ (\mathbf{Z}_1 \mid \mathbf{0})\mathbf{R}^{-1}\mathbf{M}^{-1} & \mathbf{Z}_1\mathbf{Y}_1 \end{pmatrix} \mathbf{V}^* : \right.$$
$$\left. \mathbf{R} \in \mathbb{C}_{r,r} \text{ is nonsingular}, 0 \leq s \leq r, \mathbf{Y}_1 \in \mathbb{C}_{s,n-r}, \mathbf{Z}_1 \in \mathbb{C}_{n-r,s} \right\}. \tag{13}$$

Proof. If $\mathbf{A}$ is EP and represented as in (4), then $\mathbf{S} = \mathbf{0}$ and $\mathbf{C} = \mathbf{I}_r$.

$\subset$: Let $\mathbf{B} \in \mathbf{A}\{2\}$. By Theorem 6, $\mathbf{S} = \mathbf{0}$, and $\mathbf{C} = \mathbf{I}_r$, there exist $\mathbf{X} \in \mathbb{C}_{r,r}$, $\mathbf{Z} \in \mathbb{C}_{n-r,r}$, $\mathbf{Y} \in \mathbb{C}_{r,n-r}$, $\mathbf{T} \in \mathbb{C}_{n-r,n-r}$ such that

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \mathbf{X}\mathbf{M}^{-1} & \mathbf{Y} \\ \mathbf{Z}\mathbf{M}^{-1} & \mathbf{T} \end{pmatrix} \mathbf{V}^*,$$

$\left(\begin{smallmatrix} \mathbf{X} \\ \mathbf{Z} \end{smallmatrix}\right)(\mathbf{I}_r \mid \mathbf{0})$ is idempotent and $\left(\begin{smallmatrix} \mathbf{X} \\ \mathbf{Z} \end{smallmatrix}\right)(\mathbf{I}_r \mid \mathbf{0})\left(\begin{smallmatrix} \mathbf{Y} \\ \mathbf{T} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathbf{Y} \\ \mathbf{T} \end{smallmatrix}\right)$. These conditions are equivalent to

$$\mathbf{X}^2 = \mathbf{X}, \qquad \mathbf{Z}\mathbf{X} = \mathbf{Z}, \qquad \mathbf{X}\mathbf{Y} = \mathbf{Y}, \qquad \mathbf{Z}\mathbf{Y} = \mathbf{T}. \tag{14}$$

Since $\mathbf{X}$ is idempotent, by the decomposition (12), there exist a nonsingular $\mathbf{R} \in \mathbb{C}_{r,r}$ and $s \in \{0, 1, \dots, r\}$ such that $\mathbf{X} = \mathbf{R}(\mathbf{I}_s \oplus \mathbf{0})\mathbf{R}^{-1}$. Now, let us define $\mathbf{Y}_1 \in \mathbb{C}_{s,n-r}$, $\mathbf{Y}_2 \in \mathbb{C}_{r-s,n-r}$, $\mathbf{Z}_1 \in \mathbb{C}_{n-r,s}$, $\mathbf{Z}_2 \in \mathbb{C}_{n-r,r-s}$ in such a way that $\mathbf{Y} = \mathbf{R}\left(\begin{smallmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{smallmatrix}\right)$ and $\mathbf{Z} = (\mathbf{Z}_1 \mid \mathbf{Z}_2)\mathbf{R}^{-1}$. From the second equality of (14) we get $\mathbf{Z}_2 = \mathbf{0}$. The third equality of (14) leads to $\mathbf{Y}_2 = \mathbf{0}$. The fourth equality of (14) implies $\mathbf{Z}_1\mathbf{Y}_1 = \mathbf{T}$.

$\supset$: Pick $\mathbf{B} \in \mathbb{C}_{n,n}$ belonging to the set of the right hand side of (13). Let us denote $\mathbf{P} = \mathbf{R}(\mathbf{I}_s \oplus \mathbf{0})\mathbf{R}^{-1}$, $\widetilde{\mathbf{Y}} = \mathbf{R}\left(\begin{smallmatrix} \mathbf{Y}_1 \\ \mathbf{0} \end{smallmatrix}\right)$, $\widetilde{\mathbf{Z}} = (\mathbf{Z}_1 \mid \mathbf{0})\mathbf{R}^{-1}$. We have

$$\begin{aligned} \mathbf{BAB} &= \mathbf{V} \begin{pmatrix} \mathbf{P}\mathbf{M}^{-1} & \widetilde{\mathbf{Y}} \\ \widetilde{\mathbf{Z}}\mathbf{M}^{-1} & \mathbf{Z}_1\mathbf{Y}_1 \end{pmatrix} \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{P}\mathbf{M}^{-1} & \widetilde{\mathbf{Y}} \\ \widetilde{\mathbf{Z}}\mathbf{M}^{-1} & \mathbf{Z}_1\mathbf{Y}_1 \end{pmatrix} \mathbf{V}^* \\ &= \mathbf{V} \begin{pmatrix} \mathbf{P}^2\mathbf{M}^{-1} & \mathbf{P}\widetilde{\mathbf{Y}} \\ \widetilde{\mathbf{Z}}\mathbf{P}\mathbf{M}^{-1} & \widetilde{\mathbf{Z}}\widetilde{\mathbf{Y}} \end{pmatrix} \mathbf{V}^* = \mathbf{B} \end{aligned}$$

because $\mathbf{P}^2 = \mathbf{P}$, $\mathbf{P}\widetilde{\mathbf{Y}} = \widetilde{\mathbf{Y}}$, $\widetilde{\mathbf{Z}}\mathbf{P} = \widetilde{\mathbf{Z}}$, and $\widetilde{\mathbf{Z}}\widetilde{\mathbf{Y}} = \mathbf{Z}_1\mathbf{Y}_1$ hold. □

Now we give some expressions for $\mathbf{A}\{1, 2\}$.

Corollary 8. *If $\mathbf{A} \in \mathbb{C}_{n,n}$ is represented as in (1), then*

$$\begin{aligned} &\mathbf{A}\{1, 2\} \\ &\quad = \left\{\mathbf{U}(\mathbf{D}_1\boldsymbol{\Sigma}^{-1} \mid \mathbf{D}_2)\mathbf{U}^* : \mathbf{D}_1 \in \mathbb{C}_{n,r}, \mathbf{D}_2 \in \mathbb{C}_{n,n-r},\ (\mathbf{K} \mid \mathbf{L})\mathbf{D}_1 = \mathbf{I}_r,\ \mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 = \mathbf{D}_2\right\}. \end{aligned}$$

Proof. This follows from Theorems 3 and 5. □

Corollary 9. *If $\mathbf{A} \in \mathbb{C}_{n,n}$ is given as in (4), then*

$$\begin{aligned} &\mathbf{A}\{1, 2\} \\ &\quad = \left\{\mathbf{V}(\widetilde{\mathbf{X}}\mathbf{M}^{-1} \mid \widetilde{\mathbf{Y}})\mathbf{V}^* : \widetilde{\mathbf{X}} \in \mathbb{C}_{n,r},\ \widetilde{\mathbf{Y}} \in \mathbb{C}_{n,n-r},\ (\mathbf{C} \mid \mathbf{S})\widetilde{\mathbf{X}} = \mathbf{I}_r,\ \widetilde{\mathbf{X}}(\mathbf{C} \mid \mathbf{S})\widetilde{\mathbf{Y}} = \widetilde{\mathbf{Y}}\right\}. \end{aligned}$$

Proof. This follows from Corollary 2 and Theorem 6. □

Corollary 10. *If* $\mathbf{A} \in \mathbb{C}_{n,n}$ *is EP and represented as in (4), then*

$$\mathbf{A}\{1,2\} = \left\{ \mathbf{V} \begin{pmatrix} \mathbf{M}^{-1} & \mathbf{R}\mathbf{Y}_1 \\ \mathbf{Z}_1\mathbf{R}^{-1}\mathbf{M}^{-1} & \mathbf{Z}_1\mathbf{Y}_1 \end{pmatrix} \mathbf{V}^*, \mathbf{R} \in \mathbb{C}_{r,r} \text{ is nonsingular}, \mathbf{Y}_1 \in \mathbb{C}_{r,n-r},\ \mathbf{Z}_1 \in \mathbb{C}_{n-r,r} \right\}.$$

Proof. This follows from Corollaries 3 and 7. □

4 Expressions for $\{1,3\}$, $\{1,4\}$, and $\{1,3,4\}$ inverses

Next, we investigate the elements of $\mathbf{A}\{1,3\}$ and $\mathbf{A}\{1,4\}$ when $\mathbf{A} \in \mathbb{C}_{n,n}$ is given as in (1) or in (4).

Theorem 7. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be represented as in (1). Then*

(i) $\mathbf{A}\{1,3\} = \left\{\mathbf{A}^\dagger + \mathbf{U}\mathbf{D}\mathbf{U}^* : \mathbf{D} \in \mathbb{C}_{n,n}, (\mathbf{K} \mid \mathbf{L})\mathbf{D} = \mathbf{0}\right\}$.

(ii) $\mathbf{A}\{1,4\} = \{\mathbf{A}^\dagger + \mathbf{U}(\mathbf{0} \mid \mathbf{D}_2)\mathbf{U}^* : \mathbf{D}_2 \in \mathbb{C}_{n,n-r}\}$.

Proof. (i) $\subset$: Let $\mathbf{A}^- \in \mathbf{A}\{1,3\}$. Since $\mathbf{A}^- \in \mathbf{A}\{1\}$, by employing Corollary 1, there exist $\mathbf{D}_1 \in \mathbb{C}_{n,r}$, $\mathbf{D}_2 \in \mathbb{C}_{n,n-r}$ such that

$$\mathbf{A}^- = \mathbf{A}^\dagger + \mathbf{U}(\mathbf{D}_1 \mid \mathbf{D}_2)\mathbf{U}^*, \qquad (\mathbf{K} \mid \mathbf{L})\mathbf{D}_1 = \mathbf{0}. \tag{15}$$

Since $\mathbf{A}^- \in \mathbf{A}\{3\}$, the matrix $\mathbf{A}\mathbf{A}^-$ is Hermitian. But

$$\begin{aligned} \mathbf{A}\mathbf{A}^- &= \mathbf{A}\mathbf{A}^\dagger + \mathbf{U}\begin{pmatrix} \mathbf{\Sigma}\mathbf{K} & \mathbf{\Sigma}\mathbf{L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}(\mathbf{D}_1 \mid \mathbf{D}_2)\,\mathbf{U}^* \\ &= \mathbf{A}\mathbf{A}^\dagger + \mathbf{U}\begin{pmatrix} \mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L}) \\ \mathbf{0} \end{pmatrix}(\mathbf{D}_1 \mid \mathbf{D}_2)\,\mathbf{U}^* \\ &= \mathbf{A}\mathbf{A}^\dagger + \mathbf{U}\begin{pmatrix} \mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1 & \mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^* \\ &= \mathbf{A}\mathbf{A}^\dagger + \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \end{aligned}$$

the hermiticity of $\mathbf{A}\mathbf{A}^-$ and the nonsingularity of $\mathbf{\Sigma}$ lead to $(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 = \mathbf{0}$. It is enough to define $\mathbf{D} = (\mathbf{D}_1 \mid \mathbf{D}_2)$ to get the desired inclusion.

(i) $\supset$: Let $\mathbf{D} \in \mathbb{C}_{n,n}$ such that $(\mathbf{K} \mid \mathbf{L})\mathbf{D} = \mathbf{0}$. It is easy to see that this condition implies $\mathbf{A}\mathbf{U}\mathbf{D} = \mathbf{0}$. Now it is obvious that $\mathbf{A}(\mathbf{A}^\dagger + \mathbf{U}\mathbf{D}\mathbf{U}^*)$ is Hermitian and $\mathbf{A}(\mathbf{A}^\dagger + \mathbf{U}\mathbf{D}\mathbf{U}^*)\mathbf{A} = \mathbf{A}$.

(ii) $\subset$: Let $\mathbf{A}^- \in \mathbf{A}\{1,4\}$. Since $\mathbf{A}^- \in \mathbf{A}\{1\}$, by using Corollary 1, there exist $\mathbf{D}_1 \in \mathbb{C}_{n,r}$ and $\mathbf{D}_2 \in \mathbb{C}_{n,n-r}$ such that (15) holds. Since $\mathbf{A}^- \in \mathbf{A}\{4\}$, the matrix $\mathbf{A}^-\mathbf{A}$ is Hermitian. We have from the first equality of (15)

$$\mathbf{A}^-\mathbf{A} = \mathbf{A}^\dagger\mathbf{A} + \mathbf{U}(\mathbf{D}_1\boldsymbol{\Sigma}\mathbf{K} \mid \mathbf{D}_1\boldsymbol{\Sigma}\mathbf{L})\mathbf{U}^* = \mathbf{A}^\dagger\mathbf{A} + \mathbf{U}\mathbf{D}_1\boldsymbol{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{U}^*.$$

The hermiticity of $\mathbf{A}^-\mathbf{A}$ together with the fact that $\boldsymbol{\Sigma}$ is a real diagonal matrix leads to

$$\mathbf{D}_1\boldsymbol{\Sigma}(\mathbf{K} \mid \mathbf{L}) = \begin{pmatrix} \mathbf{K}^* \\ \mathbf{L}^* \end{pmatrix} \boldsymbol{\Sigma}\mathbf{D}_1^*.$$

Premultiplying this equality by $(\mathbf{K} \mid \mathbf{L})$ and using (2), (15), and the nonsingularity of $\boldsymbol{\Sigma}$ lead to $\mathbf{D}_1 = \mathbf{0}$.

(ii) $\supset$: This inclusion is easy to get since for any $\mathbf{D}_2 \in \mathbb{C}_{n,n-r}$, $\mathbf{U}(\mathbf{0} \mid \mathbf{D}_2)\mathbf{U}^*\mathbf{A} = \mathbf{0}$. $\square$

Theorem 8. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be represented as in (4). Then*

(i) $\mathbf{A}\{1,3\} = \left\{\mathbf{A}^\dagger + \mathbf{V}\mathbf{R}\mathbf{V}^* : \mathbf{R} \in \mathbb{C}_{n,n}, (\mathbf{C} \mid \mathbf{S})\mathbf{R} = \mathbf{0}\right\}$.

(ii) $\mathbf{A}\{1,4\} = \{\mathbf{A}^\dagger + \mathbf{V}(\mathbf{0} \mid \mathbf{R})\mathbf{V}^* : \mathbf{R} \in \mathbb{C}_{n,n-r}\}$.

Proof. (i): By Theorem 7, it is sufficient to prove $\{\mathbf{U}\mathbf{D}\mathbf{U}^* : \mathbf{D} \in \mathbb{C}_{n,n}, (\mathbf{K} \mid \mathbf{L})\mathbf{D} = \mathbf{0}\} = \{\mathbf{V}\mathbf{R}\mathbf{V}^* : \mathbf{R} \in \mathbb{C}_{n,n}, (\mathbf{C} \mid \mathbf{S})\mathbf{R} = \mathbf{0}\}$. We prove only the "$\subset$" inclusion as the opposite is analogous. Pick any $\mathbf{D} \in \mathbb{C}_{n,n}$ such that $(\mathbf{K} \mid \mathbf{L})\mathbf{D} = \mathbf{0}$. By (6) we get $\mathbf{U}\mathbf{D}\mathbf{U}^* = \mathbf{V}(\mathbf{W}_1 \oplus \mathbf{I}_{n-r})\mathbf{D}(\mathbf{W}_1^* \oplus \mathbf{I}_{n-r})\mathbf{V}^*$. If we define $\mathbf{R} = (\mathbf{W}_1 \oplus \mathbf{I}_{n-r})\mathbf{D}(\mathbf{W}_1^* \oplus \mathbf{I}_{n-r})$, then it remains to prove $(\mathbf{C} \mid \mathbf{S})\mathbf{R} = \mathbf{0}$, and to this end, we use again (6).

$$\begin{aligned}&(\mathbf{C} \mid \mathbf{S})\mathbf{R}(\mathbf{W}_1 \oplus \mathbf{I}_{n-r}) \\ &\qquad = (\mathbf{C} \mid \mathbf{S}) \begin{pmatrix} \mathbf{W}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix} \mathbf{D} = (\mathbf{C}\mathbf{W}_1 \mid \mathbf{S})\mathbf{D} = (\mathbf{W}_2\mathbf{K} \mid \mathbf{W}_2\mathbf{L})\mathbf{D} = \mathbf{W}_2(\mathbf{K} \mid \mathbf{L})\mathbf{D} = \mathbf{0}.\end{aligned}$$

This computation yields $(\mathbf{C} \mid \mathbf{S})\mathbf{R} = \mathbf{0}$.

(ii): Again, by Theorem 7, it is enough to prove $\{\mathbf{U}(\mathbf{0} \mid \mathbf{D})\mathbf{U}^* : \mathbf{D} \in \mathbb{C}_{n,n-r}\} = \{\mathbf{V}(\mathbf{0} \mid \mathbf{R})\mathbf{V}^* : \mathbf{R} \in \mathbb{C}_{n,n-r}\}$. We only prove the "$\subset$" inclusion as the opposite is analogous. Let any $\mathbf{D} \in \mathbb{C}_{n,n-r}$ be written as $\mathbf{D} = \left(\begin{smallmatrix} \mathbf{D}_{12} \\ \mathbf{D}_{22} \end{smallmatrix}\right)$, where $\mathbf{D}_{12} \in \mathbb{C}_{r,n-r}$ and $\mathbf{D}_{22} \in \mathbb{C}_{n-r,n-r}$. Now (6) and a simple computation reveals

$$\mathbf{U}(\mathbf{0} \mid \mathbf{D})\mathbf{U}^* = \mathbf{V} \begin{pmatrix} \mathbf{0} & \mathbf{W}_1\mathbf{D}_{12} \\ \mathbf{0} & \mathbf{D}_{22} \end{pmatrix} \mathbf{V}^*.$$

This finishes the proof of the aforementioned inclusion. $\square$

Let us observe that by using Theorem 7 and Theorem 8, we can give representations for $\mathbf{A}\{1,3,4\}$ for any $\mathbf{A} \in \mathbb{C}_{n,n}$ in (1) or as in (4).

Next corollary is trivial in view of Theorem 8. It is noteworthy that this result can be also deduced from the singular value decomposition as in the proof of Corollary 6.

Corollary 11. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$. *Then*

(i) $\mathbf{A}\{1,3\} = \{\mathbf{A}^\dagger + \mathbf{B} : \mathbf{B} \in \mathbb{C}_{n,n}, \mathbf{AB} = \mathbf{0}\}$. *If* $\mathbf{A}^- \in \mathbf{A}\{1,3\}$, *then* $\mathbf{A}\{1,3\} = \{\mathbf{A}^- + \mathbf{B} : \mathbf{B} \in \mathbb{C}_{n,n}, \mathbf{AB} = \mathbf{0}\}$.

(ii) $\mathbf{A}\{1,4\} = \{\mathbf{A}^\dagger + \mathbf{B} : \mathbf{B} \in \mathbb{C}_{n,n}, \mathbf{BA} = \mathbf{0}\}$. *If* $\mathbf{A}^- \in \mathbf{A}\{1,4\}$, *then* $\mathbf{A}\{1,4\} = \{\mathbf{A}^- + \mathbf{B} : \mathbf{B} \in \mathbb{C}_{n,n}, \mathbf{BA} = \mathbf{0}\}$.

(iii) $\mathbf{A}\{1,3,4\} = \{\mathbf{A}^\dagger + \mathbf{B} : \mathbf{B} \in \mathbb{C}_{n,n}, \mathbf{AB} = \mathbf{BA} = \mathbf{0}\}$. *If* $\mathbf{A}^- \in \mathbf{A}\{1,3,4\}$, *then* $\mathbf{A}\{1,3,4\} = \{\mathbf{A}^- + \mathbf{B} : \mathbf{B} \in \mathbb{C}_{n,n}, \mathbf{AB} = \mathbf{BA} = \mathbf{0}\}$.

The following corollary extends of Theorem 3.3 of [3].

Corollary 12. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be EP.*

(i) *If* $\mathbf{A}$ *is of the form (1), then*

(i.a) $\mathbf{A}\{1,3\} = \left\{\mathbf{U}\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{pmatrix}\mathbf{U}^* : \mathbf{D}_{21} \in \mathbb{C}_{n-r,r}, \mathbf{D}_{22} \in \mathbb{C}_{n-r,n-r}\right\}$.

(i.b) $\mathbf{A}\{1,4\} = \left\{\mathbf{U}\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{D}_{12} \\ \mathbf{0} & \mathbf{D}_{22} \end{pmatrix}\mathbf{U}^* : \mathbf{D}_{12} \in \mathbb{C}_{r,n-r}, \mathbf{D}_{22} \in \mathbb{C}_{n-r,n-r}\right\}$.

(i.c) $\mathbf{A}\{1,3,4\} = \left\{\mathbf{U}\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{22} \end{pmatrix}\mathbf{U}^* : \mathbf{D}_{22} \in \mathbb{C}_{n-r,n-r}\right\}$.

(ii) *If* $\mathbf{A}$ *is of the form (4), then*

(ii.a) $\mathbf{A}\{1,3\} = \left\{\mathbf{V}\begin{pmatrix} \mathbf{M}^{-1} & 0 \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{pmatrix}\mathbf{V}^* : \mathbf{R}_{21} \in \mathbb{C}_{n-r,r}, \mathbf{R}_{22} \in \mathbb{C}_{n-r,n-r}\right\}$.

(ii.b) $\mathbf{A}\{1,4\} = \left\{\mathbf{V}\begin{pmatrix} \mathbf{M}^{-1} & \mathbf{R}_{12} \\ \mathbf{0} & \mathbf{R}_{22} \end{pmatrix}\mathbf{V}^* : \mathbf{R}_{12} \in \mathbb{C}_{r,n-r}, \mathbf{R}_{22} \in \mathbb{C}_{n-r,n-r}\right\}$.

(ii.c) $\mathbf{A}\{1,3,4\} = \left\{\mathbf{V}\begin{pmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{22} \end{pmatrix}\mathbf{V}^* : \mathbf{R}_{22} \in \mathbb{C}_{n-r,n-r}\right\}$.

Proof. Since $\mathbf{A}$ is EP and represented as in (1), then $\mathbf{K} = \mathbf{I}_r$ and $\mathbf{L} = \mathbf{0}$. We apply part (i) of Theorem 7 and the last equality of (3).

$$\begin{aligned} \mathbf{A}\{1,3\} &= \left\{\mathbf{U}\left[\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \end{pmatrix}\right]\mathbf{U}^* : \mathbf{D}_1 \in \mathbb{C}_{r,n}, \mathbf{D}_2 \in \mathbb{C}_{n-r,n}, (\mathbf{I}_r \mid \mathbf{0})\begin{pmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \end{pmatrix} = \mathbf{0}\right\} \\ &= \left\{\mathbf{U}\begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{pmatrix}\mathbf{U}^* : \mathbf{D}_{21} \in \mathbb{C}_{n-r,r}, \mathbf{D}_{22} \in \mathbb{C}_{n-r,n-r}\right\}. \end{aligned}$$

This proves (i.a). Item (i.b) trivially follows from part (ii) of Theorem 7 and the last equality of (3). Item (i.c) follows from items (i.a) and (i.b). Item (ii) can be proved in a similar way, but by using (7), $\mathbf{C} = \mathbf{I}_r$, and $\mathbf{S} = \mathbf{0}$. □

5 Expressions for $\{2,3\}$, $\{2,4\}$, and $\{2,3,4\}$-inverses

In this section we investigate the elements of $\mathbf{A}\{2,3\}$, $\mathbf{A}\{2,4\}$, and $\mathbf{A}\{2,3,4\}$, when $\mathbf{A} \in \mathbb{C}_{n,n}$ is represented as in (1) or in (4).

Theorem 9. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$ be of the form (1). Then*

(i) $\mathbf{A}\{2,3\} = \{\mathbf{U}\left(\mathbf{\Lambda P}^* \mid \mathbf{0}\right)\mathbf{U}^* : 0 \leq s \leq r, \mathbf{P} \in \mathbb{C}_{r,s}, \mathbf{\Lambda} \in \mathbb{C}_{n,s}, \mathbf{P}^*\mathbf{P} = \mathbf{I}_s, (\mathbf{K} \mid \mathbf{L})\mathbf{\Lambda} = \mathbf{\Sigma}^{-1}\mathbf{P}\}$.

(ii) $\mathbf{A}\{2,4\} = \{\mathbf{U}\left(\mathbf{P\Lambda}_1\mathbf{\Sigma}^{-1} \mid \mathbf{P\Lambda}_2\right)\mathbf{U}^* : 0 \leq s \leq n, \mathbf{P} \in \mathbb{C}_{n,s}, \mathbf{\Lambda}_1 \in \mathbb{C}_{s,r}, \mathbf{\Lambda}_2 \in \mathbb{C}_{s,n-r}, \mathbf{P}^*\mathbf{P} = \mathbf{I}_s, \mathbf{\Lambda}_1(\mathbf{K} \mid \mathbf{L}) = \mathbf{P}^*\}$.

Proof. (i) $\subset$: Let $\mathbf{A}^- \in \mathbf{A}\{2,3\}$. By Theorem 5 there exist $\mathbf{D}_1 \in \mathbb{C}_{n,r}$ and $\mathbf{D}_2 \in \mathbb{C}_{n,n-r}$ such that $\mathbf{A}^- = \mathbf{U}(\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_2)\mathbf{U}^*$,

$$\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1(\mathbf{K} \mid \mathbf{L}) = \mathbf{D}_1(\mathbf{K} \mid \mathbf{L}) \qquad \text{and} \qquad \mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 = \mathbf{D}_2. \tag{16}$$

Postmultiplying the first equality of (16) by $\begin{pmatrix} \mathbf{K}^* \\ \mathbf{L}^* \end{pmatrix}$ and using (2), we have

$$\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1 = \mathbf{D}_1. \tag{17}$$

Since

$$\mathbf{A}\mathbf{A}^- = \mathbf{U}\begin{pmatrix} \mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L}) \\ \mathbf{0} \end{pmatrix}\left(\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_2\right)\mathbf{U}^* = \mathbf{U}\begin{pmatrix} \mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1\mathbf{\Sigma}^{-1} & \mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

the hermiticity of $\mathbf{A}\mathbf{A}^-$ and the nonsingularity of $\mathbf{\Sigma}$ yield

$$\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1\mathbf{\Sigma}^{-1} \text{ is Hermitian and } (\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 = \mathbf{0}. \tag{18}$$

The second equality of (16) and the last equality of (18) imply $\mathbf{D}_2 = \mathbf{0}$. The first fact of (18) and (17) imply that $\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1\mathbf{\Sigma}^{-1} \in \mathbb{C}_{r,r}$ is an orthogonal projector, and thus there exists a unitary matrix $\mathbf{R} \in \mathbb{C}_{r,r}$ such that

$$\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1\mathbf{\Sigma}^{-1} = \mathbf{R}(\mathbf{I}_s \oplus \mathbf{0})\mathbf{R}^*, \tag{19}$$

where $s = \operatorname{rank}((\mathbf{K} \mid \mathbf{L})\mathbf{D}_1)$. Let us decompose $\mathbf{R} = (\mathbf{P} \mid \mathbf{Q})$, where $\mathbf{P} \in \mathbb{C}_{r,s}$ and $\mathbf{Q} \in \mathbb{C}_{r,r-s}$. With this decomposition, equality (19) can be rewritten as

$$\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1\mathbf{\Sigma}^{-1} = \mathbf{P}\mathbf{P}^*. \tag{20}$$

Since $\mathbf{R}$ is unitary, then

$$\mathbf{P}^*\mathbf{P} = \mathbf{I}_s. \tag{21}$$

Observe that the columns of $\mathbf{P}$ form an orthonormal basis of the eigenspace of $\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1\mathbf{\Sigma}^{-1}$ associated with the eigenvalue 1. Equality (17) and the first fact of (18) yield

$$\begin{aligned}\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1\mathbf{\Sigma}^{-1}(\mathbf{D}_1\mathbf{\Sigma}^{-1})^* &= (\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1\mathbf{\Sigma}^{-1})^*(\mathbf{D}_1\mathbf{\Sigma}^{-1})^* \\ &= (\mathbf{D}_1\mathbf{\Sigma}^{-1}\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1\mathbf{\Sigma}^{-1})^* \ = \ (\mathbf{D}_1\mathbf{\Sigma}^{-1})^*.\end{aligned}$$

Hence any column of $(\mathbf{D}_1\mathbf{\Sigma}^{-1})^*$ can be written as a linear combination of the columns of $\mathbf{P}$, and thus, there exists $\mathbf{\Gamma} \in \mathbb{C}_{s,n}$ such that $(\mathbf{D}_1\mathbf{\Sigma}^{-1})^* = \mathbf{P}\mathbf{\Gamma}$, or equivalently, $\mathbf{D}_1\mathbf{\Sigma}^{-1} = \mathbf{\Gamma}^*\mathbf{P}^*$. Now, this last equality, (20), and (21) lead to $\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{\Gamma}^* = \mathbf{P}$.

(i) $\supset$: Let $s \in \{0, \ldots, r\}$. Pick any $\mathbf{P} \in \mathbb{C}_{r,s}$ such that $\mathbf{P}^*\mathbf{P} = \mathbf{I}_s$ and $\mathbf{\Lambda} \in \mathbb{C}_{n,s}$ such that $(\mathbf{K} \mid \mathbf{L})\mathbf{\Lambda} = \mathbf{\Sigma}^{-1}\mathbf{P}$. We prove that $\mathbf{A}^- = \mathbf{U}(\mathbf{\Lambda}\mathbf{P}^* \mid \mathbf{0})\mathbf{U}^* \in \mathbf{A}\{2,3\}$. From

$$\mathbf{A}\mathbf{A}^- = \mathbf{U}\begin{pmatrix}\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L}) \\ \mathbf{0}\end{pmatrix}(\mathbf{\Lambda}\mathbf{P}^* \mid \mathbf{0})\mathbf{U}^* = \mathbf{U}\begin{pmatrix}\mathbf{P}\mathbf{P}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*$$

we get that $\mathbf{A}\mathbf{A}^-$ is Hermitian. Furthermore, one gets

$$\mathbf{A}^-\mathbf{A}\mathbf{A}^- = \mathbf{A}^-(\mathbf{A}\mathbf{A}^-) = \mathbf{U}(\mathbf{\Lambda}\mathbf{P}^* \mid \mathbf{0})\begin{pmatrix}\mathbf{P}\mathbf{P}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^* = \mathbf{U}(\mathbf{\Lambda}\mathbf{P}^* \mid \mathbf{0})\mathbf{U}^* = \mathbf{A}^-.$$

(ii) $\subset$: Let $\mathbf{A}^- \in \mathbf{A}\{2,4\}$. By Theorem 5 there exist $\mathbf{D}_1 \in \mathbb{C}_{n,r}$ and $\mathbf{D}_2 \in \mathbb{C}_{n,n-r}$ such that $\mathbf{A}^- = \mathbf{U}(\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_2)\mathbf{U}^*$ and (16). Similarly as in the proof of "(i) $\subset$" we get that (17) holds. Since

$$\mathbf{A}^-\mathbf{A} = \mathbf{U}\left(\mathbf{D}_1\mathbf{\Sigma}^{-1} \mid \mathbf{D}_2\right)\begin{pmatrix}\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L}) \\ \mathbf{0}\end{pmatrix}\mathbf{U}^* = \mathbf{U}\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})\mathbf{U}^*$$

and $\mathbf{A}^-\mathbf{A}$ is Hermitian, we get that $\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})$ is Hermitian, and thus $\mathbf{D}_1(\mathbf{K} \mid \mathbf{L}) \in \mathbb{C}_{n,n}$ is an orthogonal projector. Hence there exist an $s \in \{0, 1, \ldots, n\}$ and a unitary matrix $\mathbf{R} \in \mathbb{C}_{n,n}$ such that $\mathbf{D}_1(\mathbf{K} \mid \mathbf{L}) = \mathbf{R}(\mathbf{I}_s \oplus \mathbf{0})\mathbf{R}^*$. Let $\mathbf{P} \in \mathbb{C}_{n,s}$ and $\mathbf{Q} \in \mathbb{C}_{n,n-s}$ be such $\mathbf{R} = (\mathbf{P} \mid \mathbf{Q})$. This decomposition ensures

$$\mathbf{D}_1(\mathbf{K} \mid \mathbf{L}) = \mathbf{P}\mathbf{P}^*. \tag{22}$$

Since $\mathbf{R}$ is unitary we get

$$\mathbf{P}^*\mathbf{P} = \mathbf{I}_s \tag{23}$$

and the s columns of $\mathbf{P}$ form an orthonormal basis of the eigenspace associated with the eigenvalue 1 of the matrix $\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})$. The second equality of (16) and (17) imply that any column of $\mathbf{D}_1$ and $\mathbf{D}_2$ is an eigenvector of the matrix $\mathbf{D}_1(\mathbf{K} \mid \mathbf{L})$ associated with the eigenvalue 1, and thus, there exist $\mathbf{\Lambda}_1 \in \mathbb{C}_{s,r}$ and $\mathbf{\Lambda}_2 \in \mathbb{C}_{s,n-r}$ such that

$$\mathbf{D}_1 = \mathbf{P}\mathbf{\Lambda}_1 \qquad \text{and} \qquad \mathbf{D}_2 = \mathbf{P}\mathbf{\Lambda}_2. \tag{24}$$

Hence we can write $\mathbf{A}^- = \mathbf{U}(\mathbf{P}\boldsymbol{\Lambda}_1\boldsymbol{\Sigma}^{-1} \mid \mathbf{P}\boldsymbol{\Lambda}_2)\mathbf{U}^*$. Furthermore, the first equality of (24), (22), and (23) imply that $\boldsymbol{\Lambda}_1(\mathbf{K} \mid \mathbf{L}) = \mathbf{P}^*$.

(ii) $\supset$: Let $s \in \{0, 1, \dots, n\}$, $\mathbf{P} \in \mathbb{C}_{n,s}$ such that $\mathbf{P}^*\mathbf{P} = \mathbf{I}_s$, and $\boldsymbol{\Lambda}_1 \in \mathbb{C}_{s,r}$ such that $\boldsymbol{\Lambda}_1(\mathbf{K} \mid \mathbf{L}) = \mathbf{P}^*$. Finally, pick any $\boldsymbol{\Lambda}_2 \in \mathbb{C}_{s,n-r}$. Set $\mathbf{A}^- = \mathbf{U}(\mathbf{P}\boldsymbol{\Lambda}_1\boldsymbol{\Sigma}^{-1} \mid \mathbf{P}\boldsymbol{\Lambda}_2)\mathbf{U}^*$. The equality

$$\mathbf{A}^-\mathbf{A} = \mathbf{U}(\mathbf{P}\boldsymbol{\Lambda}_1\boldsymbol{\Sigma}^{-1} \mid \mathbf{P}\boldsymbol{\Lambda}_2)\begin{pmatrix} \boldsymbol{\Sigma}(\mathbf{K} \mid \mathbf{L}) \\ \mathbf{0} \end{pmatrix}\mathbf{U}^* = \mathbf{U}\mathbf{P}\boldsymbol{\Lambda}_1(\mathbf{K} \mid \mathbf{L})\mathbf{U}^* = \mathbf{U}\mathbf{P}\mathbf{P}^*\mathbf{U}^*$$

implies that $\mathbf{A}^-\mathbf{A}$ is Hermitian. Furthermore, by using $\mathbf{P}^*\mathbf{P} = \mathbf{I}_s$ one gets

$$\mathbf{A}^-\mathbf{A}\mathbf{A}^- = (\mathbf{A}^-\mathbf{A})\mathbf{A}^- = \mathbf{U}\mathbf{P}\mathbf{P}^*(\mathbf{P}\boldsymbol{\Lambda}_1\boldsymbol{\Sigma}^{-1} \mid \mathbf{P}\boldsymbol{\Lambda}_2)\mathbf{U}^* = \mathbf{U}(\mathbf{P}\boldsymbol{\Lambda}_1\boldsymbol{\Sigma}^{-1} \mid \mathbf{P}\boldsymbol{\Lambda}_2)\mathbf{U}^* = \mathbf{A}^-.$$

The proof is finished. □

Theorem 10. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be of the form (4). Then*

(i) $\mathbf{A}\{2,3\} = \{\mathbf{V}\,(\boldsymbol{\Delta}\mathbf{Q}^* \mid \mathbf{0})\,\mathbf{V}^* : 0 \le s \le r, \mathbf{Q} \in \mathbb{C}_{r,s}, \boldsymbol{\Delta} \in \mathbb{C}_{n,s}, \mathbf{Q}^*\mathbf{Q} = \mathbf{I}_s, (\mathbf{C} \mid \mathbf{S})\boldsymbol{\Delta} = \mathbf{M}^{-1}\mathbf{Q}\}$.

(ii) $\mathbf{A}\{2,4\} = \{\mathbf{V}(\mathbf{Q}\boldsymbol{\Gamma}_1\mathbf{M}^{-1} \mid \mathbf{Q}\boldsymbol{\Gamma}_2)\mathbf{V}^* : 0 \le s \le n, \mathbf{Q} \in \mathbb{C}_{n,s}, \boldsymbol{\Gamma}_1 \in \mathbb{C}_{s,r}, \boldsymbol{\Gamma}_2 \in \mathbb{C}_{s,n-r}, \mathbf{Q}^*\mathbf{Q} = \mathbf{I}_s, \boldsymbol{\Gamma}_1(\mathbf{C} \mid \mathbf{S}) = \mathbf{Q}^*\}$.

Proof. (i): Let us represent $\mathbf{A}$ as in (1). Let $0 \le s \le r$, $\mathbf{P} \in \mathbb{C}_{r,s}$, and $\boldsymbol{\Lambda} \in \mathbb{C}_{n,s}$ satisfy $\mathbf{P}^*\mathbf{P} = \mathbf{I}_s$ and $(\mathbf{K} \mid \mathbf{L})\boldsymbol{\Lambda} = \boldsymbol{\Sigma}^{-1}\mathbf{P}$. We use (6) and define

$$\mathbf{Q} = \mathbf{W}_1\mathbf{P}, \quad \boldsymbol{\Delta} = (\mathbf{W}_1 \oplus \mathbf{I}_{n-r})\boldsymbol{\Lambda}.$$

Now we have

$$\begin{aligned}(\mathbf{C} \mid \mathbf{S})\boldsymbol{\Delta} &= (\mathbf{W}_2\mathbf{K}\mathbf{W}_1^* \mid \mathbf{W}_2\mathbf{L})(\mathbf{W}_1 \oplus \mathbf{I}_{n-r})\boldsymbol{\Lambda} \\ &= \mathbf{W}_2(\mathbf{K}\mathbf{W}_1^* \mid \mathbf{L})\begin{pmatrix} \mathbf{W}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\boldsymbol{\Lambda} = \mathbf{W}_2(\mathbf{K} \mid \mathbf{L})\boldsymbol{\Lambda} \\ &= \mathbf{W}_2\boldsymbol{\Sigma}^{-1}\mathbf{P} = \mathbf{M}^{-1}\mathbf{W}_1\mathbf{P} = \mathbf{M}^{-1}\mathbf{Q},\end{aligned}$$

and $\mathbf{Q}^*\mathbf{Q} = \mathbf{P}^*\mathbf{W}_1^*\mathbf{W}_1\mathbf{P} = \mathbf{P}^*\mathbf{P} = \mathbf{I}_s$, and furthermore,

$$\begin{aligned}\mathbf{U}(\boldsymbol{\Lambda}\mathbf{P}^* \mid \mathbf{0})\mathbf{U}^* &= \mathbf{U}\left((\mathbf{W}_1^* \oplus \mathbf{I}_{n-r})\boldsymbol{\Delta}\mathbf{Q}^*\mathbf{W}_1 \mid \mathbf{0}\right)\mathbf{U}^* \\ &= \mathbf{U}\begin{pmatrix} \mathbf{W}_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}(\boldsymbol{\Delta}\mathbf{Q}^* \mid \mathbf{0})\begin{pmatrix} \mathbf{W}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^* \\ &= \mathbf{V}(\boldsymbol{\Delta}\mathbf{Q}^* \mid \mathbf{0})\mathbf{V}^*.\end{aligned}$$

The first item of Theorem 10 now follows from item (i) of Theorem 9.

(ii): Let $0 \le s \le n$, $\mathbf{P} \in \mathbb{C}_{n,s}$, $\mathbf{\Lambda}_1 \in \mathbb{C}_{s,r}$, $\mathbf{\Lambda}_2 \in \mathbb{C}_{s,n-r}$ satisfy $\mathbf{P}^*\mathbf{P} = \mathbf{I}_s$ and $\mathbf{\Lambda}_1(\mathbf{K} \mid \mathbf{L}) = \mathbf{P}^*$. We use (6) and define

$$\mathbf{Q} = (\mathbf{W}_1 \oplus \mathbf{I}_{n-r})\mathbf{P} \in \mathbb{C}_{n,s}, \quad \mathbf{\Gamma}_1 = \mathbf{\Lambda}_1\mathbf{W}_2^* \in \mathbb{C}_{s,r}, \quad \mathbf{\Gamma}_2 = \mathbf{\Lambda}_2 \in \mathbb{C}_{s,n-r}.$$

We get $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}_s$ in view of $\mathbf{W}_1^* = \mathbf{W}_1^{-1}$ and $\mathbf{P}^*\mathbf{P} = \mathbf{I}_s$. Now,

$$\begin{aligned}\mathbf{\Gamma}_1(\mathbf{C} \mid \mathbf{S}) &= \mathbf{\Lambda}_1\mathbf{W}_2^*(\mathbf{W}_2\mathbf{K}\mathbf{W}_1^*|\mathbf{W}_2\mathbf{L}) \\ &= \mathbf{\Lambda}_1(\mathbf{K} \mid \mathbf{L})\begin{pmatrix} \mathbf{W}_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r}\end{pmatrix} = \mathbf{P}^*\begin{pmatrix} \mathbf{W}_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r}\end{pmatrix} = \mathbf{Q}^*\end{aligned}$$

and (observe that the first equality of (6) can be rewritten as $\mathbf{M}^{-1}\mathbf{W}_1 = \mathbf{W}_2\mathbf{\Sigma}^{-1}$)

$$\begin{aligned}&\mathbf{U}(\mathbf{P}\mathbf{\Lambda}_1\mathbf{\Sigma}^{-1} \mid \mathbf{P}\mathbf{\Lambda}_2)\mathbf{U}^* \\ &\quad= \mathbf{V}(\mathbf{W}_1 \oplus \mathbf{I}_{n-r})\left((\mathbf{W}_1^* \oplus \mathbf{I}_{n-r})\mathbf{Q}\mathbf{\Gamma}_1\mathbf{W}_2\mathbf{\Sigma}^{-1} \mid (\mathbf{W}_1^* \oplus \mathbf{I}_{n-r})\mathbf{Q}\mathbf{\Gamma}_2\right)\mathbf{U}^* \\ &\quad= \mathbf{V}(\mathbf{Q}\mathbf{\Gamma}_1\mathbf{W}_2\mathbf{\Sigma}^{-1} \mid \mathbf{Q}\mathbf{\Gamma}_2)(\mathbf{W}_1^* \oplus \mathbf{I}_{n-r})\mathbf{V}^* \\ &\quad= \mathbf{V}(\mathbf{Q}\mathbf{\Gamma}_1\mathbf{M}^{-1}\mathbf{W}_1 \mid \mathbf{Q}\mathbf{\Gamma}_2)\begin{pmatrix} \mathbf{W}_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r}\end{pmatrix}\mathbf{V}^* \\ &\quad= \mathbf{V}(\mathbf{Q}\mathbf{\Gamma}_1\mathbf{M}^{-1} \mid \mathbf{Q}\mathbf{\Gamma}_2)\mathbf{V}^*.\end{aligned}$$

By (ii) of Theorem 9, we complete the proof. □

The following corollary extends Theorem 4.1 of [3].

Corollary 13. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be an EP matrix and represented as in (1). Then*

(i) $\mathbf{A}\{2,3\} = \left\{\mathbf{U}\begin{pmatrix} \mathbf{\Sigma}^{-1}\mathbf{P}\mathbf{P}^* & \mathbf{0} \\ \mathbf{Z}\mathbf{P}^* & \mathbf{0}\end{pmatrix}\mathbf{U}^* : 0 \le s \le r, \mathbf{P} \in \mathbb{C}_{r,s}, \mathbf{P}^*\mathbf{P} = \mathbf{I}_s, \mathbf{Z} \in \mathbb{C}_{n-r,s}\right\}$.

(ii) $\mathbf{A}\{2,4\} = \left\{\mathbf{U}\begin{pmatrix} \mathbf{Q}\mathbf{Q}^*\mathbf{\Sigma}^{-1} & \mathbf{Q}\mathbf{Z} \\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^* : 0 \le s \le n, \mathbf{Q} \in \mathbb{C}_{r,s}, \mathbf{Q}^*\mathbf{Q} = \mathbf{I}_s, \mathbf{Z} \in \mathbb{C}_{s,n-r}\right\}$.

Proof. Since $\mathbf{A}$ is EP, we have $\mathbf{K} = \mathbf{I}_r$ and $\mathbf{L} = \mathbf{0}$. We use Theorem 9.

(i): Pick any $s \in \{0, 1, \dots, r\}$, $\mathbf{P} \in \mathbb{C}_{r,s}$, and $\mathbf{\Lambda} \in \mathbb{C}_{n,s}$ such that $\mathbf{P}^*\mathbf{P} = \mathbf{I}_s$ and $(\mathbf{K} \mid \mathbf{L})\mathbf{\Lambda} = \mathbf{\Sigma}^{-1}\mathbf{P}$. By writting $\mathbf{\Lambda} = \begin{pmatrix}\mathbf{Y} \\ \mathbf{Z}\end{pmatrix}$, where $\mathbf{Y} \in \mathbb{C}_{r,s}$ and $\mathbf{Z} \in \mathbb{C}_{n-r,s}$ one gets $(\mathbf{I}_r \mid \mathbf{0})\begin{pmatrix}\mathbf{Y} \\ \mathbf{Z}\end{pmatrix} = \mathbf{\Sigma}^{-1}\mathbf{P}$, and thus, $\mathbf{Y} = \mathbf{\Sigma}^{-1}\mathbf{P}$ and $\mathbf{Z} \in \mathbb{C}_{n-r,s}$ is arbitrary.

(ii): Pick any $s \in \{0, 1, \dots, n\}$, $\mathbf{P} \in \mathbb{C}_{n,s}$, $\mathbf{\Lambda}_1 \in \mathbb{C}_{s,r}$, and $\mathbf{\Lambda}_2 \in \mathbb{C}_{s,n-r}$ satisfying $\mathbf{P}^*\mathbf{P} = \mathbf{I}_s$ and $\mathbf{\Lambda}_1(\mathbf{K} \mid \mathbf{L}) = \mathbf{P}^*$. Let $\mathbf{P} = \begin{pmatrix}\mathbf{Q} \\ \mathbf{R}\end{pmatrix}$, where $\mathbf{Q} \in \mathbb{C}_{r,s}$ and $\mathbf{R} \in \mathbb{C}_{n-r,s}$. One gets $(\mathbf{Q}^* \mid \mathbf{R}^*) =$

$\mathbf{P}^* = \mathbf{\Lambda}_1(\mathbf{I}_r \mid \mathbf{0})$, hence $\mathbf{Q}^* = \mathbf{\Lambda}_1$ and $\mathbf{R} = \mathbf{0}$. From $\mathbf{P}^*\mathbf{P} = \mathbf{I}_s$, $\mathbf{P} = \left(\begin{smallmatrix}\mathbf{Q}\\ \mathbf{R}\end{smallmatrix}\right)$, and $\mathbf{R} = \mathbf{0}$ we get $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}_s$. Furthermore, one has

$$\mathbf{P\Lambda}_1\mathbf{\Sigma}^{-1} = \begin{pmatrix}\mathbf{Q}\\ \mathbf{0}\end{pmatrix}\mathbf{Q}^*\mathbf{\Sigma}^{-1} = \begin{pmatrix}\mathbf{QQ}^*\mathbf{\Sigma}^{-1}\\ \mathbf{0}\end{pmatrix} \quad \text{and} \quad \mathbf{P\Lambda}_2 = \begin{pmatrix}\mathbf{Q}\\ \mathbf{0}\end{pmatrix}\mathbf{\Lambda}_2 = \begin{pmatrix}\mathbf{Q\Lambda}_2\\ \mathbf{0}\end{pmatrix},$$

which imply

$$\mathbf{U}(\mathbf{P\Lambda}_1\mathbf{\Sigma}^{-1} \mid \mathbf{P\Lambda}_2)\mathbf{U}^* = \mathbf{U}\begin{pmatrix}\mathbf{QQ}^*\mathbf{\Sigma}^{-1} & \mathbf{Q\Lambda}_2\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*.$$

By Theorem 9, we finish the proof. □

Theorem 11. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$.

(i) *If* $\mathbf{A}$ *is of the form (1), then* $\mathbf{A}\{2,3,4\} = \{\mathbf{U}\left(\begin{smallmatrix}\mathbf{K}^*\mathbf{\Sigma}^{-1}\mathbf{PP}^* & \mathbf{0}\\ \mathbf{L}^*\mathbf{\Sigma}^{-1}\mathbf{PP}^* & \mathbf{0}\end{smallmatrix}\right)\mathbf{U}^* : 0 \le s \le r, \mathbf{P} \in \mathbb{C}_{r,s}, \mathbf{P}^*\mathbf{P} = \mathbf{I}_s, \mathbf{\Sigma}^2\mathbf{PP}^* = \mathbf{PP}^*\mathbf{\Sigma}^2\}$.

(ii) *If* $\mathbf{A}$ *is of the form (4), then* $\mathbf{A}\{2,3,4\} = \{\mathbf{V}\left(\begin{smallmatrix}\mathbf{CM}^{-1}\mathbf{QQ}^* & \mathbf{0}\\ \mathbf{S}^*\mathbf{M}^{-1}\mathbf{QQ}^* & \mathbf{0}\end{smallmatrix}\right)\mathbf{V}^* : 0 \le s \le r, \mathbf{Q} \in \mathbb{C}_{r,s}, \mathbf{Q}^*\mathbf{Q} = \mathbf{I}_s, \mathbf{MM}^*\mathbf{QQ}^* = \mathbf{QQ}^*\mathbf{MM}^*\}$.

Proof. (i) $\subset$: Let $\mathbf{A}^- \in \mathbf{A}\{2,3,4\}$. Since $\mathbf{A}^- \in \mathbf{A}\{2,3\}$ and by applying (i) of Theorem 9, there exist $s \in \{0,\ldots,r\}$, $\mathbf{P} \in \mathbb{C}_{r,s}$ and $\mathbf{\Lambda} \in \mathbb{C}_{n,s}$ such that

$$\mathbf{A}^- = \mathbf{U}(\mathbf{\Lambda P}^* \mid \mathbf{0})\mathbf{U}^*, \tag{25}$$

$\mathbf{P}^*\mathbf{P} = \mathbf{I}_s$, and

$$(\mathbf{K} \mid \mathbf{L})\mathbf{\Lambda} = \mathbf{\Sigma}^{-1}\mathbf{P}. \tag{26}$$

Moreover, we use the fact that $\mathbf{A}^-\mathbf{A}$ is Hermitian (since $\mathbf{A}^- \in \mathbf{A}\{4\}$). From

$$\mathbf{A}^-\mathbf{A} = \mathbf{U}(\mathbf{\Lambda P}^* \mid \mathbf{0})\begin{pmatrix}\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\\ \mathbf{0}\end{pmatrix}\mathbf{U}^* = \mathbf{U\Lambda P}^*\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{U}^*,$$

we obtain that $\mathbf{\Lambda P}^*\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})$ is Hermitian. I.e., (recall that $\mathbf{\Sigma}$ is a diagonal and real matrix)

$$\mathbf{\Lambda P}^*\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L}) = \begin{pmatrix}\mathbf{K}^*\\ \mathbf{L}^*\end{pmatrix}\mathbf{\Sigma P\Lambda}^*. \tag{27}$$

Postmultiplying this last equality by $\left(\begin{smallmatrix}\mathbf{K}^*\\ \mathbf{L}^*\end{smallmatrix}\right)$ and using (2), (26), we arrive at

$$\mathbf{\Lambda P}^*\mathbf{\Sigma} = \begin{pmatrix}\mathbf{K}^*\\ \mathbf{L}^*\end{pmatrix}\mathbf{\Sigma PP}^*\mathbf{\Sigma}^{-1}. \tag{28}$$

Inserting (28) in (27) leads to

$$\begin{pmatrix} \mathbf{K}^* \\ \mathbf{L}^* \end{pmatrix} \mathbf{\Sigma} \mathbf{P}\mathbf{P}^* \mathbf{\Sigma}^{-1} (\mathbf{K} \mid \mathbf{L}) = \begin{pmatrix} \mathbf{K}^* \\ \mathbf{L}^* \end{pmatrix} \mathbf{\Sigma}^{-1} \mathbf{P}\mathbf{P}^* \mathbf{\Sigma} (\mathbf{K} \mid \mathbf{L}),$$

which, by premultiplying by $(\mathbf{K} \mid \mathbf{L})$ and postmultipliying by $\left(\begin{smallmatrix} \mathbf{K}^* \\ \mathbf{L}^* \end{smallmatrix}\right)$, reduces to

$$\mathbf{\Sigma} \mathbf{P}\mathbf{P}^* \mathbf{\Sigma}^{-1} = \mathbf{\Sigma}^{-1} \mathbf{P}\mathbf{P}^* \mathbf{\Sigma}. \tag{29}$$

Furthermore, observe that (28) and (29) yield $\mathbf{\Lambda}\mathbf{P}^* = \left(\begin{smallmatrix} \mathbf{K}^* \\ \mathbf{L}^* \end{smallmatrix}\right) \mathbf{\Sigma}^{-1}\mathbf{P}\mathbf{P}^*$. This last equality and (25) reveal the required inclusion.

(i) $\supset$: Let $s \in \{0, \ldots, r\}$ and $\mathbf{P} \in \mathbb{C}_{r,s}$ satisfy $\mathbf{P}^*\mathbf{P} = \mathbf{I}_s$ and $\mathbf{\Sigma}^2\mathbf{P}\mathbf{P}^* = \mathbf{P}\mathbf{P}^*\mathbf{\Sigma}^2$. Let us define

$$\mathbf{A}^- = \mathbf{U} \begin{pmatrix} \mathbf{K}^*\mathbf{\Sigma}^{-1}\mathbf{P}\mathbf{P}^* & \mathbf{0} \\ \mathbf{L}^*\mathbf{\Sigma}^{-1}\mathbf{P}\mathbf{P}^* & \mathbf{0} \end{pmatrix} \mathbf{U}^*.$$

Observe that, if we define $\mathbf{\Lambda} = \left(\begin{smallmatrix} \mathbf{K}^*\mathbf{\Sigma}^{-1}\mathbf{P} \\ \mathbf{L}^*\mathbf{\Sigma}^{-1}\mathbf{P} \end{smallmatrix}\right)$, by (i) of Theorem 9, we immediately get $\mathbf{A}^- \in \mathbf{A}\{2,3\}$. Thus, it remains to prove that $\mathbf{A}^-\mathbf{A}$ is Hermitian. The condition $\mathbf{\Sigma}^2\mathbf{P}\mathbf{P}^* = \mathbf{P}\mathbf{P}^*\mathbf{\Sigma}^2$ is equivalent to the hermiticity of the matrix $\mathbf{R}$ defined by $\mathbf{R} = \mathbf{\Sigma}^{-1}\mathbf{P}\mathbf{P}^*\mathbf{\Sigma}$. Since

$$\mathbf{A}^-\mathbf{A} = \mathbf{U} \begin{pmatrix} \mathbf{K}^*\mathbf{\Sigma}^{-1}\mathbf{P}\mathbf{P}^* & \mathbf{0} \\ \mathbf{L}^*\mathbf{\Sigma}^{-1}\mathbf{P}\mathbf{P}^* & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{\Sigma}\mathbf{K} & \mathbf{\Sigma}\mathbf{L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^* = \mathbf{U} \begin{pmatrix} \mathbf{K}^*\mathbf{R}\mathbf{K} & \mathbf{K}^*\mathbf{R}\mathbf{L} \\ \mathbf{L}^*\mathbf{R}\mathbf{K} & \mathbf{L}^*\mathbf{R}\mathbf{L} \end{pmatrix} \mathbf{U}^*,$$

we obviously get that $\mathbf{A}^-\mathbf{A}$ is Hermitian.

(ii): Let $0 \le s \le r$, $\mathbf{P} \in \mathbb{C}_{r,s}$ such that $\mathbf{P}^*\mathbf{P} = \mathbf{I}_s$ and $\mathbf{\Sigma}^2\mathbf{P}\mathbf{P}^* = \mathbf{P}\mathbf{P}^*\mathbf{\Sigma}^2$. Let us define $\mathbf{A}^- = \mathbf{U} \left(\begin{smallmatrix} \mathbf{K}^*\mathbf{\Sigma}^{-1}\mathbf{P}\mathbf{P}^* & \mathbf{0} \\ \mathbf{L}^*\mathbf{\Sigma}^{-1}\mathbf{P}\mathbf{P}^* & \mathbf{0} \end{smallmatrix}\right) \mathbf{U}^*$. Observe that $\mathbf{\Sigma}^2\mathbf{P}\mathbf{P}^* = \mathbf{P}\mathbf{P}^*\mathbf{\Sigma}^2$ is equivalent to the hermiticity of $\mathbf{\Sigma}^{-1}\mathbf{P}\mathbf{P}^*\mathbf{\Sigma}$. Since $\mathbf{K}^*\mathbf{\Sigma}^{-1} = \mathbf{W}_1^*\mathbf{C}\mathbf{M}^{-1}\mathbf{W}_1$ and $\mathbf{L}^*\mathbf{\Sigma}^{-1} = \mathbf{S}^*\mathbf{M}^{-1}\mathbf{W}_1$, we get

$$\begin{aligned} \mathbf{A}^- &= \mathbf{V} \begin{pmatrix} \mathbf{W}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix} \begin{pmatrix} \mathbf{W}_1^*\mathbf{C}\mathbf{M}^{-1}\mathbf{W}_1\mathbf{P}\mathbf{P}^* & \mathbf{0} \\ \mathbf{S}^*\mathbf{M}^{-1}\mathbf{W}_1\mathbf{P}\mathbf{P}^* & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{W}_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix} \mathbf{V}^* \\ &= \mathbf{V} \begin{pmatrix} \mathbf{C}\mathbf{M}^{-1}\mathbf{W}_1\mathbf{P}\mathbf{P}^*\mathbf{W}_1^* & \mathbf{0} \\ \mathbf{S}^*\mathbf{M}^{-1}\mathbf{W}_1\mathbf{P}\mathbf{P}^*\mathbf{W}_1^* & \mathbf{0} \end{pmatrix} \mathbf{V}^*. \end{aligned}$$

Let us denote $\mathbf{Q} = \mathbf{W}_1\mathbf{P} \in \mathbb{C}_{r,s}$. In view of the properties of $\mathbf{W}_1$ and $\mathbf{P}$ we easily get $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}_s$. In addition, since $\mathbf{M}^{-1}\mathbf{Q}\mathbf{Q}^*\mathbf{M} = \mathbf{W}_2\mathbf{\Sigma}^{-1}\mathbf{P}\mathbf{P}^*\mathbf{\Sigma}\mathbf{W}_2^*$ we obtain that $\mathbf{M}^{-1}\mathbf{Q}\mathbf{Q}^*\mathbf{M}$ is Hermitian, or equivalent, $\mathbf{M}\mathbf{M}^*\mathbf{Q}\mathbf{Q}^* = \mathbf{Q}\mathbf{Q}^*\mathbf{M}\mathbf{M}^*$. □

6 Group inverses

Let us recall (see for example Section 4.4 of [6] or Chapter 4 of [7]) that if $\mathbf{A}, \mathbf{X} \in \mathbb{C}_{n,n}$, then $\mathbf{X}$ is called a group inverse of $\mathbf{A}$ if

(1) $\mathbf{AXA} = \mathbf{A}$, (2) $\mathbf{XAX} = \mathbf{X}$, (5) $\mathbf{AX} = \mathbf{XA}$.

It can be proved that for a given $\mathbf{A} \in \mathbb{C}_{n,n}$, the set of matrices $\mathbf{X}$ satisfying (1), (2), and (5) is empty or a singleton. When it is a singleton, it is customary to write its unique element as $\mathbf{A}^{\#}$. If $\mathbf{X}$ satisfies (1) and (5), then $\mathbf{X}$ is called a commuting g-inverse of $\mathbf{A}$ and we denote by $\mathbf{A}\{1,5\}$ the set of commuting g-inverses of $\mathbf{A}$. In this section we study $\mathbf{A}\{1,5\}$ when $\mathbf{A}$ is written as in (1) or as in (4). We shall apply $\exists\, \mathbf{A}^{\#} \iff \exists\, \mathbf{C}^{-1} \iff \exists\, \mathbf{K}^{-1}$ and

$$\mathbf{A}^{\#} = \mathbf{U}\begin{pmatrix} \mathbf{K}^{-1}\boldsymbol{\Sigma}^{-1} & \mathbf{K}^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{K}^{-1}\mathbf{L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^* = \mathbf{V}\begin{pmatrix} \mathbf{C}^{-1}\mathbf{M}^{-1} & \mathbf{C}^{-1}\mathbf{M}^{-1}\mathbf{C}^{-1}\mathbf{S} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^*.$$

See [1, Theorem 3.7] and [5].

Theorem 12. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$. The matrix $\mathbf{A}$ is group invertible if and only if $\mathbf{A}\{1,5\} \neq \varnothing$. Under this situation, one has:*

(i) *If $\mathbf{A}$ is of the form (1), then*

$$\mathbf{A}\{1,5\} = \left\{\mathbf{U}\begin{pmatrix} \mathbf{K}^{-1}\boldsymbol{\Sigma}^{-1} & \mathbf{B}_2 \\ \mathbf{0} & \mathbf{B}_4 \end{pmatrix}\mathbf{U}^* : \mathbf{B}_2 \in \mathbb{C}_{r,n-r}, \mathbf{B}_4 \in \mathbb{C}_{n-r,n-r}, \boldsymbol{\Sigma}(\mathbf{K}\mathbf{B}_2 + \mathbf{L}\mathbf{B}_4) = \mathbf{K}^{-1}\mathbf{L}\right\}.$$

(ii) *If $\mathbf{A}$ of the form (4), then*

$$\mathbf{A}\{1,5\} = \left\{\mathbf{V}\begin{pmatrix} \mathbf{C}^{-1}\mathbf{M}^{-1} & \mathbf{D}_2 \\ \mathbf{0} & \mathbf{D}_4 \end{pmatrix}\mathbf{V}^* : \mathbf{D}_2 \in \mathbb{C}_{r,n-r}, \mathbf{D}_4 \in \mathbb{C}_{n-r,n-r}, \mathbf{M}(\mathbf{C}\mathbf{D}_2 + \mathbf{S}\mathbf{D}_4) = \mathbf{C}^{-1}\mathbf{S}\right\}.$$

Proof. We prove $\mathbf{A}\{1,5\} \neq \varnothing \iff \exists \mathbf{A}^{\#}$. If $\mathbf{X}$ is a commuting g-inverse of $\mathbf{A}$, then $\mathbf{A}(\mathbf{XAX})\mathbf{A} = \mathbf{A}$, $(\mathbf{XAX})\mathbf{A}(\mathbf{XAX}) = \mathbf{XAX}$, and $(\mathbf{XAX})\mathbf{A} = \mathbf{A}(\mathbf{XAX})$, hence $\mathbf{A}$ is group invertible and $\mathbf{A}^{\#} = \mathbf{XAX}$. Conversely, it is obvious that $\mathbf{A}^{\#} \in \mathbf{A}\{1,5\}$.

Pick any $\mathbf{A}^- \in \mathbf{A}\{1,5\}$. By Theorem 3, we can write $\mathbf{A}^- = \mathbf{U}(\mathbf{D}_1 \mid \mathbf{D}_2)\mathbf{U}^*$, where $\mathbf{D}_1 \in \mathbb{C}_{n,r}$ and $\boldsymbol{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_1 = \mathbf{I}_r$. Now, we compute $\mathbf{AA}^-$ and $\mathbf{A}^-\mathbf{A}$:

$$\mathbf{AA}^- = \mathbf{U}\begin{pmatrix} \boldsymbol{\Sigma}(\mathbf{K} \mid \mathbf{L}) \\ \mathbf{0} \end{pmatrix}(\mathbf{D}_1 \mid \mathbf{D}_2)\mathbf{U}^* = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \boldsymbol{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

$$\mathbf{A}^-\mathbf{A} = \mathbf{U}(\mathbf{D}_1 \mid \mathbf{D}_2)\begin{pmatrix} \boldsymbol{\Sigma}(\mathbf{K} \mid \mathbf{L}) \\ \mathbf{0} \end{pmatrix}\mathbf{U}^* = \mathbf{U}\mathbf{D}_1\boldsymbol{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{U}^*.$$

From $\mathbf{AA}^- = \mathbf{A}^-\mathbf{A}$ we get

$$\begin{pmatrix} \mathbf{I}_r & \boldsymbol{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \mathbf{D}_1\boldsymbol{\Sigma}(\mathbf{K} \mid \mathbf{L}). \tag{30}$$

Postmultiplying (30) by $\begin{pmatrix} \mathbf{K}^* \\ \mathbf{L}^* \end{pmatrix}$ leads to $\mathbf{D}_1\boldsymbol{\Sigma} = \begin{pmatrix} \mathbf{K}^*+\boldsymbol{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2\mathbf{L}^* \\ \mathbf{0} \end{pmatrix}$. Thus, having in mind the non-singularity of $\boldsymbol{\Sigma}$, the matrix $\mathbf{D}_1$ is of the form $\mathbf{D}_1 = \begin{pmatrix} \mathbf{D}_{11} \\ \mathbf{0} \end{pmatrix}$, where $\mathbf{D}_{11} \in \mathbb{C}_{r,r}$. From (30) we get $\mathbf{I}_r = \mathbf{D}_{11}\boldsymbol{\Sigma}\mathbf{K}$ and $\boldsymbol{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 = \mathbf{D}_{11}\boldsymbol{\Sigma}\mathbf{L}$. Thus, $\mathbf{D}_{11} = \mathbf{K}^{-1}\boldsymbol{\Sigma}^{-1}$ and

$\mathbf{\Sigma}(\mathbf{K} \mid \mathbf{L})\mathbf{D}_2 = \mathbf{K}^{-1}\mathbf{L}$. We have proved one inclusion of (i). The other direction of the inclusion is trivial.

To prove (ii), we use (6). Choose any $\mathbf{B}_2 \in \mathbb{C}_{r,n-r}$ and $\mathbf{B}_4 \in \mathbb{C}_{n-r,n-r}$ such that $\mathbf{\Sigma}(\mathbf{K}\mathbf{B}_2 + \mathbf{L}\mathbf{B}_4) = \mathbf{K}^{-1}\mathbf{L}$ and let

$$\mathbf{A}^- = \mathbf{U} \begin{pmatrix} \mathbf{K}^{-1}\mathbf{\Sigma}^{-1} & \mathbf{B}_2 \\ \mathbf{0} & \mathbf{B}_4 \end{pmatrix}. \tag{31}$$

Since $\mathbf{K}^{-1}\mathbf{\Sigma}^{-1} = \mathbf{W}_1^*\mathbf{C}^{-1}\mathbf{M}^{-1}\mathbf{W}_1$, we have

$$\mathbf{A}^- = \mathbf{V} \begin{pmatrix} \mathbf{C}^{-1}\mathbf{M}^{-1} & \mathbf{W}_1\mathbf{B}_2 \\ \mathbf{0} & \mathbf{B}_4 \end{pmatrix} \mathbf{V}^*.$$

Thus, if we define $\mathbf{D}_2 = \mathbf{W}_1\mathbf{B}_2$ and $\mathbf{D}_4 = \mathbf{B}_4$, it is sufficient to prove $\mathbf{M}(\mathbf{C}\mathbf{D}_2 + \mathbf{S}\mathbf{D}_4) = \mathbf{C}^{-1}\mathbf{S}$. In fact,

$$\begin{aligned} \mathbf{M}(\mathbf{C}\mathbf{D}_2 + \mathbf{S}\mathbf{D}_4) &= \mathbf{W}_1\mathbf{\Sigma}\mathbf{W}_2^*(\mathbf{W}_2\mathbf{K}\mathbf{W}_1^*\mathbf{W}_1\mathbf{B}_2 + \mathbf{W}_2\mathbf{L}\mathbf{B}_4) \\ &= \mathbf{W}_1\mathbf{\Sigma}(\mathbf{K}\mathbf{B}_2 + \mathbf{L}\mathbf{B}_4) \\ &= \mathbf{W}_1\mathbf{K}^{-1}\mathbf{L} \\ &= \mathbf{W}_1\mathbf{W}_1^*\mathbf{C}^{-1}\mathbf{W}_2\mathbf{W}_2^*\mathbf{S} \\ &= \mathbf{C}^{-1}\mathbf{S}. \end{aligned}$$

□

In next results we show that if $\mathbf{A}$ is group invertible, then $\mathbf{A}\{1,5\}$ is a linear manifold passing through $\mathbf{A}^\#$. Let us recall that a linear manifold is a subset of a vector space $\mathcal{V}$ of the form $\mathbf{v} + \mathcal{S}$, where $\mathcal{S}$ is a linear subspace of $\mathcal{V}$ and $\mathbf{v} \in V$.

Corollary 14. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$ be group invertible.*

(i) *If $\mathbf{A}$ is of the form (1), then $\mathbf{A}\{1,5\} = \{\mathbf{A}^\# + \mathbf{U}(\mathbf{0} \mid \mathbf{\Delta})\mathbf{U}^* : \mathbf{\Delta} \in \mathbb{C}_{n,n-r}, (\mathbf{K} \mid \mathbf{L})\mathbf{\Delta} = \mathbf{0}\}$.*

(ii) *If $\mathbf{A}$ is of the form (4), then $\mathbf{A}\{1,5\} = \{\mathbf{A}^\# + \mathbf{V}(\mathbf{0} \mid \mathbf{\Lambda})\mathbf{V}^* : \mathbf{\Lambda} \in \mathbb{C}_{n,n-r}, (\mathbf{C} \mid \mathbf{S})\mathbf{\Lambda} = \mathbf{0}\}$.*

Proof. (i): Pick any $\mathbf{B}_2 \in \mathbb{C}_{r,n-r}$ and $\mathbf{B}_4 \in \mathbb{C}_{n-r,n-r}$ such that $\mathbf{\Sigma}(\mathbf{K}\mathbf{B}_2 + \mathbf{L}\mathbf{B}_4) = \mathbf{K}^{-1}\mathbf{L}$ and define $\mathbf{A}^-$ as in (31). We have

$$\mathbf{A}^- = \mathbf{U} \begin{pmatrix} \mathbf{K}^{-1}\mathbf{\Sigma}^{-1} & \mathbf{K}^{-1}\mathbf{\Sigma}^{-1}\mathbf{K}^{-1}\mathbf{L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^* + \mathbf{U} \begin{pmatrix} \mathbf{0} & \mathbf{B}_2 - \mathbf{K}^{-1}\mathbf{\Sigma}^{-1}\mathbf{K}^{-1}\mathbf{L} \\ \mathbf{0} & \mathbf{B}_4 \end{pmatrix} \mathbf{U}^*.$$

Thus, if we define $\mathbf{\Delta} = \binom{\mathbf{B}_2 - \mathbf{K}^{-1}\mathbf{\Sigma}^{-1}\mathbf{K}^{-1}\mathbf{L}}{\mathbf{B}_4}$, then it is sufficient to prove $(\mathbf{K} \mid \mathbf{L})\mathbf{\Delta} = \mathbf{0}$, and this is easy to check in view of $\mathbf{\Sigma}(\mathbf{K}\mathbf{B}_2 + \mathbf{L}\mathbf{B}_4) = \mathbf{K}^{-1}\mathbf{L}$ and the definition of $\mathbf{\Delta}$.

To prove (ii), pick $\mathbf{\Delta} \in \mathbb{C}_{n,n-r}$ such that $(\mathbf{K} \mid \mathbf{L})\mathbf{\Delta} = \mathbf{0}$. We can write $\mathbf{\Delta} = \binom{\mathbf{\Delta}_1}{\mathbf{\Delta}_2}$, where $\mathbf{\Delta}_1 \in \mathbb{C}_{r,n-r}$ and $\mathbf{\Delta}_2 \in \mathbb{C}_{n-r,n-r}$. Define $\mathbf{\Lambda} = \binom{\mathbf{W}_1\mathbf{\Delta}_1}{\mathbf{\Delta}_2} \in \mathbb{C}_{n,n-r}$. Now

$$\mathbf{0} = (\mathbf{K} \mid \mathbf{L})\mathbf{\Delta} = \mathbf{K}\mathbf{\Delta}_1 + \mathbf{L}\mathbf{\Delta}_2 = \mathbf{W}_2^*\mathbf{C}\mathbf{W}_1\mathbf{\Delta}_1 + \mathbf{W}_2^*\mathbf{S}\mathbf{\Delta}_2 = \mathbf{W}_2^*(\mathbf{C}\mathbf{W}_1\mathbf{\Delta}_1 + \mathbf{S}\mathbf{\Delta}_2),$$

which yields $\mathbf{0} = (\mathbf{C} \mid \mathbf{S})\mathbf{\Lambda}$. It remains to check $\mathbf{U}(\mathbf{0} \mid \mathbf{\Delta})\mathbf{U}^* = \mathbf{V}(\mathbf{0} \mid \mathbf{\Lambda})\mathbf{V}^*$:

$$\begin{aligned} \mathbf{U}(\mathbf{0} \mid \mathbf{\Delta})\mathbf{U}^* &= \mathbf{V}\begin{pmatrix} \mathbf{W}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathbf{\Delta}_1 \\ \mathbf{0} & \mathbf{\Delta}_2 \end{pmatrix}\begin{pmatrix} \mathbf{W}_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{V}^* \\ &= \mathbf{V}\begin{pmatrix} \mathbf{0} & \mathbf{W}_1\mathbf{\Delta}_1 \\ \mathbf{0} & \mathbf{\Delta}_2 \end{pmatrix}\mathbf{V}^* = \mathbf{V}(\mathbf{0} \mid \mathbf{\Lambda})\mathbf{V}^*. \end{aligned}$$

□

Now we give an explicit representation of $\mathbf{A}\{1,5\}$ without using representations (1) and (4).

Corollary 15. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be group invertible. Then*

(i) $\mathbf{A}\{1,5\} = \{\mathbf{A}^\# + \mathbf{B} : \mathbf{B} \in \mathbb{C}_{n,n}, \mathbf{AB} = \mathbf{BA} = \mathbf{0}\}$.

(ii) *If* $\mathbf{A}^- \in \mathbf{A}\{1,5\}$*, then* $\mathbf{A}\{1,5\} = \{\mathbf{A}^- + \mathbf{B} : \mathbf{B} \in \mathbb{C}_{n,n}, \mathbf{AB} = \mathbf{BA} = \mathbf{0}\}$.

Proof. (i) follows from Corollary 14. (ii) follows from (i) of this corollary. □

Acknowledgments

The authors wish to thank Prof. Oskar Maria Baksalary for the comments that improved the manuscript. They are also grateful to three anonymous referees for remarking that some proofs can be simplified. The first author is supported by Work funded by Vicerrectorado de Investigación U.P.V. PAID 06-2010-2285. The second author is supported by the NSFC grant (11061005, 61165015) the Ministry of Education Science and Technology Key Project under grant 210164 and Grants (HCIC201103) of Guangxi Key Laborarory of Hybrid Computational and IC Design Analysis Open Fund and Key issues for Department of Education of Guangxi (201202ZD031).

References

[1] J. Benítez, A new decomposition for square matrices, Electron. J. Linear Algebra, 20 (2010) 207-225.

[2] F. Zhang, Matrix Theory, Springer, New York, 1999.

[3] B. Zheng, L. Ye, D.S. Cvetković-Ilić, Generalized inverses of a normal matrix, Appl. Math. Comput., 206 (2008) 788-795.

[4] R.E. Hartwig, K. Spindelböck, Matrices for which A^* and $A^\dagger$ commute, Linear Multilinear Algebra, 14 (1984) 241-256.

[5] O.M. Baksalary, G.P.H. Styan, G. Trenkler, On a matrix decomposition of Hartwig and Spindelböck, Linear Algebra Appl., 430 (2009) 2798-2812.

[6] A. Ben-Israel, T.N.E. Greville, Generalized Inverses, Springer-Verlag, New York, 2003.

[7] S.L. Campbell, C.D. Meyer, Generalized Inverses of Linear Transformations, Pitman, London, 1979.