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On the smoothness of L^p of a positive vector measure

L. Agud · J. M. Calabuig · E. A. Sánchez-Pérez

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Abstract We investigate natural sufficient conditions for a space $L^p(m)$ of p-integrable functions with respect to a positive vector measure to be smooth. Under some assumptions on the representation of the dual space of such a space, we prove that this is the case for instance if the Banach space where the vector measure takes its values is smooth. We give also some examples and show some applications of our results for determining norm attaining elements for operators between two spaces $L^p(m_1)$ and $L^q(m_2)$ of positive vector measures m_1 and m_2.

Keywords Banach function spaces · Norm attaining functionals · Operators

Mathematics Subject Classification (2000) Primary 46E30; Secondary 47B38 · 46B42 · 46B28

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1 Introduction

Let X be a Banach space, X^* its topological dual space and B_X the closed unit ball of X. It is said that $x^* \in X^*$ *norms* $x \in X$—or that x^* *is norming for* x—if $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$. By the Hahn–Banach theorem there always exists such a functional. A Banach space X is called *smooth* if for every $0 \neq x \in X$ the norming element for such x is unique. This element is denoted by $\Theta_X(x)$. For instance, it is known (see [1, Part 3, Ch. 1]) that the spaces $X = L^p(\mu)$—where μ is a scalar measure and $1 < p < \infty$—are smooth and, moreover, the unique norming element for a function $f \in L^p(\mu)$ is given by the formula

$$\Theta_{L^p(\mu)}(f) = \frac{sgn(f)|f|^{p-1}}{\|f\|^{p-1}}, \tag{1}$$

where the sign function $sgn(\cdot)$ is defined as usual.

The aim of this paper is to study conditions under which a space of p-integrable functions with respect to a vector measure for $1 \leq p < \infty$, is smooth. The reader may take into account that these spaces represent a broad class of Banach lattices, since each p-convex Banach lattice (with p-convexity constant equal to one) with a weak unit can be represented as such a space. In particular, we analyze the natural question of when the property is inherited from the space where the vector measure takes its values, obtaining a result in the positive (Theorem 2). We also give some examples.

A (partial) motivation of our study comes from the setting of the norm attaining operators. A bounded linear operator between Banach spaces, $T : X \to Y$, is said to be *norm attaining*—or that T attains its norm—if there exists $0 \neq x \in X$ such that $\|T(x)\|_Y = \|T\| \cdot \|x\|_X$. The following result, whose proof can be found in [6, Section 2], gives a link between smooth spaces and norm attaining operators.

Theorem (Howard and Schep). *Let $T : X \to Y$ be a linear and bounded operator between smooth Banach spaces. Given $x \in X$, the following assertions are equivalent:*

(a) *T attains its norm at x.*
(b) $T^*(\Theta_Y(T(x))) = \|T\| \cdot \Theta_X(x)$.

Actually, in the paper quoted above only the implication (a)⇒(b) is shown; however for the converse it suffices to notice that

$$\begin{aligned}\|T(x)\|_Y &= \langle T(x), \Theta_Y(T(x)) \rangle = \langle x, T^*(\Theta_Y(T(x))) \rangle \\ &= \langle x, \|T\| \Theta_X(x) \rangle = \|T\| \langle x, \Theta_X(x) \rangle = \|T\| \cdot \|x\|_X.\end{aligned}$$

This general result can be improved when X and Y are spaces of p-integrable functions with respect to a vector measure (or in a more general case where X and Y are order continuous Banach functions spaces having weak unit). This will be done in Theorem 4. The concrete formula that can be given in this case for the functional attaining the norm of a norm one element f is

$$\Theta_{L^p(m)}(f)(h) = \langle h, \Theta_{L^p(m)}(f) \rangle := \int_\Omega sgn(f)|f|^{p-1} h d\langle m, x_f^* \rangle, \quad h \in L^p(m),$$

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for a certain positive norm one element x_f^* of X^*. Moreover, if m_1 and m_2 are vector measures, this formula will provide the better expression

$$\int_{\Omega_2} \frac{sgn(T(f))|T(f)|^{q-1}}{\|T(f)\|^{q-1}} \cdot T(h)\, d\nu_2 = \alpha \int_{\Omega_1} \frac{sgn(f)|f|^{p-1}}{\|f\|^{p-1}} \cdot h\, d\nu_1, \quad h \in L^p(m_1),$$

for the equation that characterize an element f in which the norm of an operator $T : L^p(m_1) \to L^q(m_2)$ is attained, where ν_1 and ν_2 are specific scalar measures and α is a positive constant.

In order to do that, we need to give conditions that assure the smoothness of the spaces of p-integrable functions with respect to a vector measure. However it must be pointed out that not all the $L^p(m)$ spaces are smooth (see Example 1).

2 Preliminaries and notation

Let (Ω, Σ, μ) be a positive finite measure space. A Banach function space $X(\mu)$ over μ is defined to be an ideal of the space of (equivalence classes of) measurable functions $L^0(\mu)$ endowed with a complete norm that is compatible with the μ-a.e. order and such that $L^\infty(\mu) \subseteq X(\mu) \subseteq L^1(\mu)$ (see p. 28 in [7]).

Let X be a Banach space, B_X its closed unit ball and S_X its unit sphere. Let (Ω, Σ) be a measurable space and $m : \Sigma \to X$ be a (countably additive) vector measure. If $1 \leq p < \infty$ we write p' by the extended real number satisfying $1/p + 1/p' = 1$. We write $|m|$ for the variation of m. A Σ-measurable function f *is* p*-integrable with respect to* m if (i) $|f|^p$ is integrable with respect to each scalar measure $\langle m, x^* \rangle := x^* \circ m$, for each $x^* \in X^*$, and (ii) for every $A \in \Sigma$ there is an element $\int_A |f|^p dm \in X$ such that $\langle \int_A |f|^p dm, x^* \rangle = \int_A |f|^p d\langle m, x^* \rangle$, $x^* \in X^*$. The set of all (classes of m-a.e. equal) p-integrable functions is denoted by $L^p(m)$ and it defines a p-convex order continuous Banach function space with weak unit χ_Ω—in the sense of [7, Def.1.b.17]—over any Rybakov measure $\nu = |\langle m, x_0^* \rangle|$ for m (see [2, Ch.IX,2]) with the norm

$$\|f\|_{p,m} := \sup_{x^* \in B_{X^*}} \left(\int_\Omega |f|^p \, d|\langle m, x^* \rangle| \right)^{\frac{1}{p}}, \quad f \in L^p(m).$$

If only condition (i) is satisfied then the corresponding spaces (with the same m-a.e. identification and norm) are denoted by $L^p_w(m)$ and it is again a p-convex Banach function space for which $L^p(m)$ is a closed subspace. Since $L^p(m)$ is order continuous then its topological dual $L^p(m)^*$ coincides with its Köthe dual $L^p(m)'$ (cf. [8, Corollary 2.6.5]), which is defined as

$$L^p(m)' = \{h \ \Sigma\text{-measurable} : fh \in L^1(\nu) \quad \text{for all } f \in L^p(m)\}.$$

The duality is given by the formula $\langle h, f \rangle = \int_\Omega fh d\nu$. More information on $L^p(m)$ spaces can be found in [3,9] All unexplained terminology can be found in the standard references [7,8] (for Banach lattices) and [9] (for integration of scalar functions with respect to vector measures).

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Let us explain now some relevant facts regarding the integration operator $I_m : L^1(m) \to X$ associated to a vector measure m. Assume that m is positive. For a fixed positive $x_0^* \in S_{X^*}$ let $\nu = \langle m, x_0^* \rangle$ be the associated Rybakov measure. By using Radon–Nikodým derivatives the adjoint operator $I_m^* : X^* \to L^1(m)^* = L^1(m)'$ can be written as

$$I_m^*(x^*)(f) = \int_\Omega f \frac{d\langle m, x^* \rangle}{d\nu} d\nu, \quad x^* \in X^*, f \in L^1(m).$$

Consider the subspace of $L^1(m)'$ given by

$$\mathbf{R}(m) = \left\{ I_m^*(x^*) = \frac{d\langle m, x^* \rangle}{d\nu} : x^* \in B_{X^*} \right\} \subseteq B_{L^1(m)'}. \tag{2}$$

Given $1 < p < \infty$, the pointwise product space $B_{L_w^{p'}(m)} \cdot \mathbf{R}(m)$ is defined as

$$B_{L_w^{p'}(m)} \cdot \mathbf{R}(m) = \left\{ h \in L^0(\nu) : h = g \cdot I_m^*(x^*),\ g \in B_{L_w^{p'}(m)},\ x^* \in B_{X^*} \right\}.$$

Note that $B_{L_w^{p'}(m)} \cdot \mathbf{R}(m) \subseteq B_{L^p(m)'}$. Indeed for each $f \in L^p(m)$, $g \in B_{L_w^{p'}(m)}$ and $x^* \in B_{X^*}$, one has

$$\left| \left(g \cdot I_m^*(x^*) \right)(f) \right| = \left| \int_\Omega fg \frac{d\langle m, x^* \rangle}{d\nu} d\nu \right| \le \int_\Omega |fg| d\langle m, x^* \rangle \le \|f\|_{p,m} \|g\|_{p',m} \|x^*\|.$$

As we will show in the next section, the smoothness of the space $L^p(m)$ is related to the opposite containment

$$B_{L^p(m)'} \subseteq B_{L_w^{p'}(m)} \cdot \mathbf{R}(m).$$

3 Results

As we said at the end of the first section the space $L^p(m)$ is not, in the general case, smooth. This is shown in the following easy example.

Example 1 Let $1 \le p < \infty$. Let us consider the measure space $\Omega = \{1, 2\}$ with the algebra of all its subsets and the positive vector measure $m : \Sigma \to \ell_2^\infty$ defined by $m(\{i\}) := e_i$, $i = 1, 2$, where e_i is the corresponding element of the canonical basis of $\mathbb{R}^2$. Clearly $m(\Omega) = e_1 + e_2$. Since the measure is positive then (cf. [9, Lemma 3.13])

$$\|f\|_{p,m} = \left\| \int_\Omega |f|^p dm \right\|_{\ell_2^\infty}^{1/p}, \quad f \in L^p(m), \tag{3}$$

and so

$$\begin{aligned} \|(\lambda_1, \lambda_2)\|_{p,m} &= \left\| |\lambda_1|^p e_1 + |\lambda_2|^p e_2 \right\|_\infty^{1/p} = \max\{|\lambda_1|^p, |\lambda_2|^p\}^{1/p} \\ &= \|(\lambda_1, \lambda_2)\|_\infty. \end{aligned}$$

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Therefore $L^p(m)$ and ℓ_2^∞ are isometrically isomorphic. But in [1, Part 3, Ch. 1] it is shown that ℓ^∞ is not a smooth space. So, $L^p(m)$ is not a smooth space.

We are ready to state and prove the main result of this paper.

Theorem 2 *Let* $1 < p < \infty$ *and* $m : \Sigma \to X$ *be a positive vector measure satisfying*

(i) $B_{L^p(m)'} \subseteq B_{L_w^{p'}(m)} \cdot \mathbf{R}(m)$.
(ii) *X is smooth.*

Then $L^p(m)$ is smooth.

Proof Let $f \in S_{L^p(m)}$. Since m is positive then (see [9, Lemma 3.13])

$$\|f\|_{p,m}^p = \left\| \int_\Omega |f|^p dm \right\|_X .$$

Therefore, by using the Hanh–Banach Theorem we get that there is $x_f^* \in B_{X^*}$ (that we can assume $x_f^* \geq 0$) such that

$$\left\langle \int_\Omega |f|^p dm, x_f^* \right\rangle = 1. \tag{4}$$

Let us consider the function $g_f = sgn(f)|f|^{p-1}$. Clearly $g_f \in B_{L_w^{p'}(m)}$ (in fact it belongs to $S_{L_w^{p'}(m)}$) since

$$\left\| \int_\Omega |g_f|^{p'} dm \right\|_X = \left\| \int_\Omega |f|^{(p-1)p'} dm \right\|_X = \|f\|_{p,m}^p = 1.$$

Let us define now the linear map $\varphi : L^p(m) \to \mathbb{R}$ given by

$$\varphi(h) = \int_\Omega h g_f d\langle m, x_f^* \rangle, \quad h \in L^p(m). \tag{5}$$

Claim 1 $\varphi \in S_{L^p(m)'}$ and it norms f. Indeed, first note that for all $h \in B_{L^p(m)}$ one has

$$\left|\varphi(h)\right| \leq \int_\Omega |h g_f| d\langle m, x_f^* \rangle \leq \|h\|_{p,m} \|g_f\|_{p',m} \leq 1,$$

so $\varphi \in B_{L^p(m)'}$. Moreover

$$\varphi(f) = \int_\Omega f g_f d\langle m, x_f^* \rangle = \int_\Omega f sgn(f)|f|^{p-1} d\langle m, x_f^* \rangle = \left\langle \int_\Omega |f|^p dm, x_f^* \right\rangle = 1.$$

Therefore $\varphi \in S_{L^p(m)'}$ and norms f. This proves our first claim.

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Since we have to prove that $L^p(m)$ is smooth and φ norms f let us see that φ is the unique function in $S_{L^p(m)'}$ norming f. Assume then that $\psi \in S_{L^p(m)'}$ norms f. We have to see that $\psi = \varphi$. By using the hypothesis (i) we find $g \in B_{L_w^{p'}(m)}$ and $x^* \in B_{X^*}$ such that

$$\psi = g \cdot \frac{d\langle m, x^*\rangle}{d\nu}. \tag{6}$$

Define now the positive measure $\eta : \Sigma \to [0, \infty[$ given by $\eta(A) = \frac{d\langle m, x^*\rangle}{d\nu} \cdot \nu(A)$ for all $A \in \Sigma$.

Claim 2 $f \in S_{L^p(\eta)}$, $g \in S_{L^{p'}(\eta)}$ and g norms f —as a function on $L^p(\eta)$—. First it is easy to see that $\|f\|_{L^p(\eta)} \leq \|f\|_{p,m} = 1$ and $\|g\|_{L^{p'}(\eta)} \leq \|g\|_{p',m} = 1$. But, by using *Hölder's Inequality*, one has

$$1 = \int_\Omega fgd\eta \leq \|f\|_{L^p(\eta)} \|g\|_{L^{p'}(\eta)}.$$

This proves our second claim.

But bearing in mind that $L^p(\eta)$ is smooth (since $1 < p < \infty$) and $sgn(f)|f|^{p-1} = g_f$ also norms f in $L^p(\eta)$ then

$$g = sgn(f)|f|^{p-1} = g_f \text{ in } L^p(\eta). \tag{7}$$

Claim 3 x_f^* and x^* norm $x = \int_\Omega |f|^p dm \in S_X$. Indeed by Eq. (4) we have $1 = \langle x, x_f^*\rangle$. On the other hand the equality $1 = \langle x, x^*\rangle$ follows from (7) since

$$\begin{aligned} 1 = \psi(f) &= \int_\Omega fgd\langle m, x^*\rangle = \int_\Omega fgd\eta = \int_\Omega fg_f d\eta \\ &= \int_\Omega fg_f d\langle m, x^*\rangle = \int_\Omega |f|^p d\langle m, x^*\rangle = \langle x, x^*\rangle. \end{aligned}$$

So $\langle x, x_f^*\rangle = 1 = \langle x, x^*\rangle$ and our last claim follows.

By using the smoothness of X that is assumed by the hypothesis (ii) one has

$$x^* = x_f^*. \tag{8}$$

Therefore (6), (7), (8) and (5) give

$$\begin{aligned} \psi(h) &= \int_\Omega hg \frac{d\langle m, x^*\rangle}{d\nu} d\nu = \int_\Omega hgd\eta = \int_\Omega hg_f d\eta \\ &= \int_\Omega hg_f \frac{d\langle m, x^*\rangle}{d\nu} d\nu = \int_\Omega hg_f d\langle m, x_f^*\rangle = \varphi(h), \end{aligned}$$

for all $h \in L^p(m)$ and the theorem is proved. □

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As we know, for a scalar measure μ the space $L^p(\mu)$ is smooth for $1 < p < \infty$ (see the Sect. 1). However, this is not the case for the space $L^1(\mu)$. In the setting of the vector valued measures the situation is different. Indeed, if $1 < p < \infty$ the space $L^1(m_p)$ associated to the measure $m_p : \Sigma \to L^p(\mu)$ given by $m_p(A) = \chi_A$ is isometrically isomorphic to $L^p(\mu)$. This means that $L^1(m_p)$ is smooth. Actually as a consequence of Theorem 2 we can prove that in general the space $L^p(m)$ is smooth provided that $L^1(m)$ is smooth and has the Fatou property. The proof is a consequence of a result regarding products of Banach function spaces which can be found in [10, Theorem 3.7].

Theorem (Schep). *Let E and F be Banach function spaces with the Fatou property. If $E \cdot F = G$ is a product Banach function space, then $E \cdot G'$ is a product Banach function space and $E \cdot G' = F'$.*

Corollary 3.1 *Let $1 < p < \infty$ and $m : \Sigma \to X$ be a positive vector measure. If $L^1(m)$ is smooth and has the Fatou property then $L^p(m)$ is smooth.*

Proof Let is consider the (countable additive) vector measure

$$\begin{aligned} m_0 : \Sigma &\to L^1(m) \\ A &\mapsto m_0(A) = \chi_A. \end{aligned} \tag{9}$$

By the hypothesis, $Y = L^1(m)$ is smooth. Since $L^p(m_0)$ is isometrically isomorphic to $L^p(m)$, for $1 \le p < \infty$, then we only have to show that (i) in Theorem 2 holds.

First note that $\mathbf{R}(m_0) = B_{L^1(m_0)'}$. Indeed, just bearing in mind (2) we have

$$\mathbf{R}(m_0) \subseteq B_{L^1(m_0)'}.$$

But on the other hand, given $h \in B_{L^1(m_0)'}$ then

$$\langle m_0, h\rangle(A) = \langle m_0(A), h\rangle = \int_A h d\mu, \quad A \in \Sigma,$$

therefore $h = \frac{d\langle m_0, h\rangle}{d\mu} \in \mathbf{R}(m_0)$ so $\mathbf{R}(m_0) = B_{L^1(m_0)'}$.

Hence hypothesis (i) in Theorem 2 is now

$$B_{L^p(m_0)'} \subseteq B_{L_w^{p'}(m_0)} \cdot B_{L^1(m_0)'}. \tag{10}$$

Therefore let us show

$$L^p(m_0)' = L_w^{p'}(m_0) \cdot L^1(m_0)'. \tag{11}$$

It is well-known (see [9, Proposition 3.43]) that

$$L^p(m_0) \cdot L_w^{p'}(m_0) = L^1(m_0), \tag{12}$$

so taking $E = L_w^{p'}(m_0)$—which has always the Fatou property—, $F = L^p(m_0)$ and $G = L^1(m_0)$—which have the Fatou property by the hypothesis—, we can apply the

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theorem by Schep quoted above (taking into account that all the equalities are actually isometries), to obtain (11), and in consequence (10). The result follows applying our Theorem 2. □

This result can be easily adapted for the case of order continuous Banach function spaces having the Fatou property by means of the well-known representation theorems and the pth powers theory for these spaces; actually it can also be deduced from Corollary 3.1. The pth power $X(\mu)_{[p]}$ of a Banach function space $X(\mu)$ is defined as the space of functions

$$X(\mu)_{[p]} = \{f \text{ measurable} : |f|^{1/p} \in X(\mu)\},$$

It is a Banach function space with the norm

$$\|f\|_{X(\mu)_{[1/p]}} = \big\||f|^p\big\|_{X(\mu)}^{1/p}, \quad f \in X(\mu)_{[1/p]}.$$

Note that with this definition, $L^p(m) = L^1(m)_{[1/p]}$ and $L^1(m) = L^p(m)_{[p]}$. We refer the reader to [9, Sec. 2.2] for the unexplained information regarding Banach function spaces and theirs pth powers.

Corollary 3.2 *Let $X(\mu)$ be an order continuous Banach function space over a positive finite measure having the Fatou property and let $1 < p < \infty$. If $X(\mu)$ is smooth, then $X(\mu)_{[1/p]}$ is smooth.*

Let us analyze now the main requirement on the spaces that appears in our results in order to give a geometric meaning to our setting. Condition (i) in Theorem 2 can be replaced by the slightly stronger condition

$$\text{(i')} \quad B_{L^p(m)'} \subseteq B_{L^{p'}(m)} \cdot \mathbf{R}(m).$$

Clearly, condition (i') implies condition (i). However this new condition (i') can be interpreted in geometric terms. To explain this let us introduce first some terminology regarding boundaries in Banach spaces. Let X be a Banach space and K be a w^*-compact subset of X^*. A *James boundary for* K is a subset B of K such that for all $x \in X$ there is $b \in B$ such that

$$\langle x, b\rangle = \sup_{k \in K} \langle x, k\rangle.$$

It is well-known (see [5]) that if X^* is weakly compactly generated (WCG) then

$$B_{X^*} = \overline{co(B)}^{\|\cdot\|_X}, \tag{13}$$

for each James boundary B of B_{X^*}. In our setting, a result by Ferrando and Rodríguez ensures that in the case when $1 < p < \infty$ the space $L^p(m)'$ is WCG (see [4, Theorem 3.1]) so (13) applies for all James boundary B for $B_{L^p(m)'}$. Moreover Theorem 3.12 in

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[4] asserts that in the case when the vector measure m is positive then the (pointwise product) set

$$B_{L^{p'}(m)} \cdot \mathbf{R}(m)$$

is a James boundary for $B_{L^p(m)'}$. This gives directly the proof of the next lemma.

Lemma 3 *Let $m : \Sigma \to X$ be a positive vector measure and $1 < p < \infty$. The following assertions are equivalent:*

(a) *$B_{L^{p'}(m)} \cdot \mathbf{R}(m)$ is convex and closed.*
(b) *$B_{L^p(m)'} \subseteq B_{L^{p'}(m)} \cdot \mathbf{R}(m)$.*

With all these results we obtain the next

Corollary 3.3 *Let $1 < p < \infty$ and $m : \Sigma \to X$ be a positive vector measure satisfying:*

(i) *$B_{L^{p'}(m)} \cdot \mathbf{R}(m)$ is convex and closed.*
(ii) *X is smooth.*

Then $L^p(m)$ is smooth.

We finish this section with a general version of the theorem by Howard and Schep mentioned in the introduction. In order to do that we fix some notation. For $1 < p < \infty$ the space $L^p(\mu)$ for μ positive scalar measure is smooth. Hence the unique norming element for a function $f \in S_{L^p(\mu)}$ is given by the formula

$$\Theta_{L^p(\mu)}(f) = sgn(f)|f|^{p-1} = g_f.$$

In the case of the spaces $L^p(m)$ we know that $\varphi : L^p(m) \to \mathbb{R}$ given by

$$\varphi(h) = \int_\Omega h g_f d\langle m, x_f^* \rangle, \quad h \in L^p(m),$$

norms f for $x_f^* \in B_{X^*}$ satisfying

$$\left\langle \int_\Omega |f|^p dm, x_f^* \right\rangle = 1.$$

Therefore if we assume that $L^p(m)$ for $1 < p < \infty$ is smooth then φ is the unique norming element for $f \in S_{L^p(m)}$ and will be denoted by $\Theta_{L^p(m)}(f)$, i.e.,

$$\Theta_{L^p(m)}(f) = sgn(f)|f|^{p-1} \frac{d\langle m, x_f^* \rangle}{d\nu} = g_f I_m^*(x_f^*).$$

Theorem 4 *Let $m_1, m_2 : \Sigma \to X$ be positive vector measures and let $T : L^p(m_1) \to L^q(m_2)$ be a linear and bounded operator with $1 < p, q < \infty$. Given $f \in L^p(m_1)$, if $L^p(m_1)$ and $L^q(m_2)$ are smooth then the following assertions are equivalent.*

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(a) *T attains its norm at f.*
(b) *There exists $\alpha \in \mathbb{R}$ such that $T^*(\Theta_{L^q(m_2)}(Tf)) = \alpha\Theta_{L^p(m_1)}(f)$. In this case, $\alpha = \|T\|$.*
(c) *There exists $\alpha \in \mathbb{R}$ such that the positive scalar measures $\nu_1 = \langle m, x_f^* \rangle$ and $\nu_2 = \langle m, x_{T(f)}^* \rangle$ satisfy*

$$\int_{\Omega_2} \frac{sgn(T(f))|T(f)|^{q-1}}{\|T(f)\|^{q-1}} \cdot T(h)\, d\nu_2 = \alpha \int_{\Omega_1} \frac{sgn(f)|f|^{p-1}}{\|f\|^{p-1}} \cdot h\, d\nu_1,$$

for all $h \in L^p(m_1)$. In this case, $\alpha = \|T\|$.

Proof The equivalence between (a) and (b) is just the result of Howard and Schep taking $X = L^p(m_1)$ and $Y = L^q(m_2)$. So let us show now the equivalence between (b) and (c); in fact (c) is just a reformulation of (b). Consider the positive measures ν_1 and ν_2 by $\nu_1 = \langle m_1, x_f^* \rangle$ and $\nu_2 = \langle m_2, x_{T(f)}^* \rangle$. So, given $h \in L^p(m_1)$

$$\begin{aligned} T^*(\Theta_{L^q(m_2)}(T(f)))(h) &= \langle T^*(\Theta_{L^q(m_2)}(T(f))), h \rangle \\ &= \langle \Theta_{L^q(m_2)}(T(f)), T(h) \rangle \\ &= \int_{\Omega_2} \frac{sgn(T(f))\big|T(f)\big|^{q-1}}{\|T(f)\|^{q-1}} T(h) d\langle m_2, x_{T(f)}^* \rangle \\ &= \int_{\Omega_2} \frac{sgn(T(f))\big|T(f)\big|^{q-1}}{\|T(f)\|^{q-1}} T(h) d\nu_2. \end{aligned}$$

In a similar way we obtain

$$\langle \Theta_{L^p(m_1)}(f), h \rangle = \int_{\Omega_1} \frac{sgn(f)|f|^{p-1}}{\|f\|^{p-1}} hd\langle m_1, x_f^* \rangle = \int_{\Omega_1} \frac{sgn(f)|f|^{p-1}}{\|f\|^{p-1}} hd\nu_1,$$

and so the equality is proved and we finish the proof. □

4 Examples

We finish the paper with some relevant examples in which we can apply our results. In particular, we show some cases in which the inclusion

$$B_{L^p(m)'} \subseteq B_{L^{p'}(m)} \cdot \mathbf{R}(m) \tag{14}$$

holds. We start with a well-known case which comes from a canonical construction.

Example 5 Let (Ω, Σ, μ) be a positive finite measure space and $1 < p < \infty$ and consider the vector measure $m_0 : \Sigma \to L^1(\mu)$ given by $m_0(A) = \chi_A$ for each $A \in \Sigma$. Then it is well-known that $L^p(m_0)$ is isometrically isomorphic to $L^p(\mu)$. On the other

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hand, take the Rybakov measure ν associated to the function $\chi_\Omega \in L^\infty(\mu)$. Note that in such case $\nu = \langle m_0, \chi_\Omega \rangle = \mu$, since

$$\nu(A) = \langle m_0, \chi_\Omega \rangle(A) = \langle m_0(A), \chi_\Omega \rangle = \int_A \chi_\Omega d\mu = \mu(A), \quad A \in \Sigma. \tag{15}$$

For this vector measure, the relation that appears in formula (14) is $B_{L^{p'}(\mu)} \subseteq B_{L^{p'}(\mu)} \cdot \mathbf{R}(m_0)$. But this containment is trivially satisfied considering the decomposition $g = g\chi_\Omega$ for each $g \in L^{p'}(\mu)$ just taking into account that (15) gives

$$\frac{d\langle m_0, \chi_\Omega \rangle}{d\mu} = \chi_\Omega.$$

Note that in this example the space $X = L^1(\mu)$ is not smooth. However, the corresponding space $L^p(m_0)$ is a smooth space since it is isometrically isomorphic to $L^p(\mu)$ for $1 < p < \infty$. This means that conditions (i) and (ii) in Theorem 2 do not characterize the smoothness of the corresponding space $L^p(m)$.

We present now a generalization of Example 5. Note that its proof follows the lines of the proof of Corollary 3.1.

Example 6 Let us consider bounded linear map $T : X(\mu) \to Y(\mu)$ where

(a) $X(\mu)$ is an order continuous Banach function space having weak unit.
(b) $Y(\mu)$ is an order continuous Banach function space having weak unit and satisfying the Fatou property.

Take now the (countable additive) vector measure $m_T : \Sigma \to Y(\mu)$ given by $m_T(A) = T(\chi_A)$, $A \in \Sigma$, and suppose that the spaces $X(\mu)$ and $Y(\mu)$ are Banach function spaces over a Rybakov measure for m_T (for instance, if μ is a Rybakov measure for m_T). Assume finally that

(c) the integration operator associated to the vector measure m_T, that is, the operator $I_{m_T} : L^1(m_T) \to Y(\mu)$ defined by $I_{m_T}(f) = \int_\Omega f dm_T = T(f)$, $f \in L^1(m_T)$ is an isometry.

With these assumptions we have:

- $L^1(m_T) = Y(\mu)$ and then $L^2(m_T) = Y(\mu)_{[\frac{1}{2}]}$.
- $\mathbf{R}(m_T) = B_{Y(\mu)'}$ isometrically. Indeed, since I_{m_T} is an isometry then $I^*_{m_T}$

also is and then $\mathbf{R}(m_T) = I^*_{m_T}(B_{Y(\mu)'}) = B_{Y(\mu)'}$. Moreover if $y'_0 \in Y(\mu)'$ then

$$\|y'_0\|_{Y(\mu)'} = \|I^*_{m_T}(y'_0)\|_{L^1(m_T)'} = \left\| \frac{d\langle m_T, y'_0 \rangle}{d\mu} \right\|_{L^1(m_T)'},$$

so the equality is an isometry. Let us show that

$$B_{L^2(m_T)'} \subseteq B_{L^2(m_T)} \cdot \mathbf{R}(m_T),$$

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that is

$$B_{L^2(m_T)'} \subseteq B_{L^2(m_T)} \cdot B_{L^1(m_T)'}.$$

First note that since $Y(\mu)$ has the Fatou property then $L^2(m_T) = Y(\mu)_{[\frac{1}{2}]}$ also has. This means that $L^2(m_T) = L^2_w(m_T)$ and then using [9, Proposition 3.43] one has the isometry

$$L^2(m_T) \cdot L^2(m_T) = L^1(m_T).$$

On the other hand taking into account again that $L^2(m_T)$ has the Fatou property, using [10, Theorem 3.7] with $E = L^2(m_T) = F$ and $L^2(m_T) \cdot L^2(m_T) = L^1(m_T)$ we obtain

$$L^2(m_T) \cdot L^1(m_T)' = L^2(m_T)',$$

and then

$$B_{L^2(m_T)'} = B_{L^2(m_T)} \cdot B_{L^1(m_T)'}.$$

It is easy to see that the previous example apply if, for instance, one considers as T the inclusion map between classical Lebesgue spaces as $i : L^2[0,1] \to L^1[0,1]$.

We finish this paper by giving an example where we use our results in order to obtain the smoothness of the space

$$X = \bigoplus_4 L^2(\mu_{|A_i}),$$

where μ is the Lebesgue measure on $[0,1]$ and $(A_i)_{i\geq 1}$ is a disjoint family of measurable subsets of $[0,1]$.

Example 7 Let us consider μ the Lebesgue measure on $[0,1]$ and take $(A_i)_{i\geq 1}$ a disjoint family of measurable subsets of $[0,1]$. Define the positive ℓ_2-valued vector measure $m : \Sigma \to \ell_2$ given by

$$m(A) = \sum_{i\geq 1} \mu(A \cap A_i) e_i, \quad A \in \Sigma,$$

where $(e_i)_{i\geq 1}$ is the usual canonical basis of ℓ_2. Take now the Rybakov measure for m associated to $x_0^* = (2^{-i/2})_{i\geq 1} \in \ell_2$. It is easy to see that

$$L^1(m) = \left\{ f \in L^0(\nu) : \sum_{i\geq 1} \left(\int_{A_i} |f| d\mu \right)^2 < \infty \right\} = \bigoplus_2 L^1(\mu_{|A_i}),$$

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and, consequently

$$L^2(m) = \left\{ f \in L^0(\nu) : \sum_{i\geq 1} \left(\int_{A_i} |f|^2 d\mu \right)^2 < \infty \right\} = \bigoplus_4 L^2(\mu_{|A_i}).$$

Therefore

$$L^2(m)' = \left(\bigoplus_4 L^2(\mu_{|A_i}) \right)' = \bigoplus_{4/3} L^2(\mu_{|A_i}).$$

In order to have (14) we have to show that $B_{\bigoplus_{4/3} L^2(\mu_{|A_i})} \subseteq B_{L^2(m)} \cdot \mathbf{R}(m)$ so take $\sum_{i\geq 1} g\chi_{A_i} \in \bigoplus_{4/3} L^2(\mu_{|A_i})$ such that $\sum_{i\geq 1} \left(\int_{A_i} |g|^2 d\mu \right)^{2/3} \leq 1$. Let us take the sequence $(\alpha_i)_{i\geq 1}$ defined by

$$\alpha_i = \frac{1}{2^{i/2}} \left(\int_{A_i} |g|^2 d\mu \right)^{\frac{1}{3}},$$

and consider the decomposition

$$\sum_{i\geq 1} g\chi_{A_i} = \left(\sum_{i\geq 1} \frac{g}{2^{i/2}\alpha_i} \chi_{A_i} \right) \cdot \left(\sum_{i\geq 1} 2^{i/2}\alpha_i \chi_{A_i} \right).$$

- First of all note that $(\alpha_i)_{i\geq 1} \in B_{\ell_2}$ since

$$\sum_{i\geq 1} |\alpha_i|^2 = \sum_{i\geq 1} \frac{1}{2^i} \left(\int_{A_i} |g|^2 d\mu \right)^{\frac{2}{3}} \leq \sum_{i\geq 1} \left(\int_{A_i} |g|^2 d\mu \right)^{\frac{2}{3}} \leq 1.$$

- On the other hand $\left(\sum_{i\geq 1} g/(2^{i/2}\alpha_i)\chi_{A_i} \right) \in B_{L^2(m)}$ since

$$\sum_{i\geq 1} \left(\int_{A_i} \left|\frac{g}{2^{i/2}\alpha_i}\right|^2 d\mu \right)^2 = \sum_{i\geq 1} \frac{1}{(2^{i/2}\alpha_i)^4} \left(\int_{A_i} |g|^2 d\mu \right)^2$$
$$= \sum_{i\geq 1} \left(\int_{A_i} |g|^2 d\mu \right)^{\frac{2}{3}} \leq 1.$$

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- Finally $\sum_{i\geq 1} 2^{i/2}\alpha_i \chi_{A_i} \in \mathbf{R}(m)$. Indeed, if we write $x^* = (\alpha_i)_{i\geq 1} \in B_{\ell_2}$, then

$$I_m^*(x^*) = \frac{d\langle m, x^*\rangle}{d\nu} = \frac{d\langle m, x^*\rangle}{d\mu}\frac{d\mu}{d\nu} = \left(\sum_{i\geq 1}\alpha_i \chi_{A_i}\right)\left(\sum_{i\geq 1} 2^{i/2}\chi_{A_i}\right)$$

$$= \sum_{i\geq 1} 2^{i/2}\alpha_i \chi_{A_i}.$$

Since the space $X = \ell_2$ is smooth then as a direct consequence of our results one obtains the following

Corollary 8 *Let μ be the Lebesgue measure on* $[0, 1]$ *and take* $(A_i)_{i\geq 1}$ *a disjoint family of measurable sets of* $[0, 1]$*. Then the space*

$$X = \bigoplus_4 L^2(\mu_{|A_i}),$$

is smooth.

Clearly with the corresponding easy modifications one can get that if μ is the Lebesgue measure on $[0, 1]$ and $(A_i)_{i\geq 1}$ is a disjoint family of measurable subsets of $[0, 1]$ then the space

$$X = \bigoplus_p L^q(\mu_{|A_i}),$$

is smooth for adequate $1 < p, q < \infty$.

Open problems. All the proofs of the results in this paper depend strongly of the condition given by the equation

$$B_{L^p(m)'} \subseteq B_{L_w^{p'}(m)} \cdot \mathbf{R}(m).$$

However we do not know the answers to the following general questions without this requirement. Let $1 < p < \infty$.

(Q1) *If X is smooth, is $L^p(m)$ also a smooth space?*
(Q2) *If $L^1(m)$ is smooth, is $L^p(m)$ also a smooth space?*

If the Fatou property is required for $L^1(m)$—equivalently, if $L^1(m) = L_w^1(m)$—, Corollary 3.1 gives the answer. This happens for example if X is reflexive. But the general result is unknown.

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