CLASSICAL OPERATORS ON THE HÖRMANDER ALGEBRAS

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Abstract. We study the integration operator, the differentiation operator and more general differential operators on radial Fréchet or (LB) Hörmander algebras of entire functions. We analyze when these operators are power bounded, hypercyclic and (uniformly) mean ergodic.

1. Introduction and notation

The aim of this paper is to investigate the dynamics of the integration operator $Jf(z) = \int_0^z f(\zeta)d\zeta$, the differentiation operator $Df(z) = f'(z)$ and differential operators $\phi(D) = \sum_{n=0}^{\infty} a_n D^n$, with $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ an entire function of exponential type, on radial Hörmander algebras of entire functions. The continuity of the differentiation and the integration operators between weighted Banach spaces of holomorphic functions has been studied by Harutyunyan and Lusky [19]. Here we continue our research in [13], [14], [15], [5] and [4].

In what follows $\mathcal{H}(\mathbb{C})$ denotes the space of all entire functions endowed with the compact open topology τ_{co}. It is easy to see that the operators D and J are continuous on $\mathcal{H}(\mathbb{C})$.

Throughout the paper, a *weight* v on $\mathbb{C}$ is a strictly positive continuous function on $\mathbb{C}$ which is radial, i.e., $v(z) = v(|z|),\ z \in \mathbb{C}$, $v(r)$ is non-increasing on $[0, \infty[$ and rapidly decreasing, that is, $\lim_{r\to\infty} r^m v(r) = 0$ for each $m \in \mathbb{N}$.

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For such a weight, the *weighted Banach spaces of entire functions* are defined by

$$H_v := \{f \in \mathcal{H}(\mathbb{C}) \mid ||f||_v := \sup_{z\in\mathbb{C}} v(|z|)|f(z)| < +\infty\},$$
$$H_v^0 := \{f \in \mathcal{H}(\mathbb{C}) \mid \lim_{|z|\to\infty} v(|z|)|f(z)| = 0\},$$

endowed with the sup norm $\|\cdot\|_v$. Clearly H_v^0 is a closed subspace of H_v which contains the polynomials. Both are Banach spaces and the closed unit ball of H_v is τ_{co}-compact. The polynomials are contained and dense in H_v^0 but the monomials are not in general a Schauder basis [20]. Clearly, changing the value of v on a compact interval does not change the spaces and gives an equivalent norm. By [11, Ex 2.2] the bidual of H_v^0 is isometrically isomorphic to H_v.

Let $V = (v_n)_n$ be a decreasing sequence of weights on $\mathbb{C}$. Then the weighted inductive limits of spaces of entire functions are defined by

$$VH := \mathrm{ind_n}\ H_{v_n},$$

$$V_0H := \mathrm{ind_n}\ H_{v_n}^0,$$

that is, VH is the increasing union of the Banach spaces H_{v_n} with the strongest locally convex topology for which all the injections $H_{v_n} \to VH$ become continuous, $n \in \mathbb{N}$, and similarly for V_0H.

Given an increasing sequence of weights $W = (w_n)_n$ on $\mathbb{C}$, the weighted projective limits of spaces of entire functions are defined by

$$HW := \mathrm{proj_n} H_{w_n},$$

$$H_0W := \mathrm{proj_n} H_{w_n}^0,$$

that is, HW is the decreasing intersection of the Banach spaces H_{w_n} whose topology is defined by the sequence of norms $||\cdot||_{w_n}$ and similarly for H_0W. Both are Fréchet spaces. We refer the reader to [9], [10] and [27].

We use here and in the rest of the paper Landau's notation of little o-growth and capital O-growth.

A function $p : \mathbb{C} \to [0,\infty[$ is called a *growth condition* if it is continuous, radial, increases with $|z|$ and satisfies

(α) $\log(1 + |z|^2) = o(p(|z|))$ as $|z| \to \infty$,
(β) $p(2|z|) = O(p(|z|))$ as $|z| \to \infty$.

Given a growth condition p, consider the weight $v(z) = e^{-p(|z|)}$, $z \in \mathbb{C}$, and the decreasing sequence of weights $V = (v_n)_n$, $v_n = v^n$. We define the following weighted spaces of entire functions (see e.g. [7], [6]):

$$A_p := \{f \in \mathcal{H}(\mathbb{C})| \text{ there is } A > 0 : \sup_{z\in\mathbb{C}} |f(z)| \exp(-Ap(z)) < \infty\},$$

that is, $A_p = VH$, endowed with the inductive limit topology, for which it is a (DFN)-algebra (cf. [23]). If we consider the increasing sequence of weights $W = (w_n)_n$, $w_n = v^{1/n}$, we define

$$A_p^0 := \{f \in \mathcal{H}(\mathbb{C})| \text{ for all } \varepsilon > 0 : \sup_{z\in\mathbb{C}} |f(z)| \exp(-\varepsilon p(z)) < \infty\},$$

that is, $A_p^0 = HW$, endowed with the projective limit topology, for which it is a nuclear Fréchet algebra (cf. [24]).

Clearly $A_p^0 \subset A_p$. Condition (α) implies that for each $a > 0$ the weight $v_a(z) := e^{-ap(|z|)}$ is rapidly decreasing, consequently, the polynomials are contained and dense in $H_{v_a}^0$, and that for $a < b$ the inclusion $H_{v_a} \subset H_{v_b}^0$ is compact. Therefore the polynomials are dense in A_p and in A_p^0. Condition (β) implies that both spaces are stable under differentiation. By the closed graph theorem, the differentiation operator D is continuous on A_p and on A_p^0.

Weighted algebras of entire functions of this type, usually known as Hörmander algebras, have been considered since the work of Berenstein and Taylor [7] by many authors; see e.g. [6] and the references therein. Braun, Meise and Taylor studied in [16], [23] and [24] the structure of (complemented) ideals in these algebras. The growth condition defining the Hörmander algebras is usually required to be subharmonic, mainly to use Hörmander's L^2-estimates on the solution of the $\overline{\partial}$-equation. For our purposes we do not need, and therefore we do not assume, this condition.

As an example, when $p(z) = |z|^a$, then A_p consists of all entire functions of order a and finite type or order less than a, and A_p^0 is the space of all entire functions of order at most a and type 0. For $a = 1$, A_p is the space of all entire functions of exponential type, and A_p^0 is the space of entire functions of infraexponential type.

Our notation for locally convex spaces and functional analysis is standard [25]. For a locally convex space E, $\mathrm{cs}(E)$ denotes a system of continuous seminorms determining the topology of E and the space of all continuous linear operators on E is denoted by $L(E)$. A continuous linear operator T from a locally convex space E into itself is called *hypercyclic* if there is a vector x (which is called a hypercyclic vector) in E such that its orbit $(x, Tx, T^2x, \dots)$ is dense in E. Every hypercyclic operator T on E is *topologically transitive* in the sense of dynamical systems, that is, for every pair of non-empty open subsets U and V of E there is n such that $T^n(U)$ meets V. The operator T is *topologically mixing* if for every pair of non-empty open subsets U and V of E there is n_0 such that for each $n \geq n_0$, $T^n(U)$ meets V. T is *chaotic* if it is topologically transitive and has a dense set of periodic points.

The basic references for dynamics of linear and continuous operators are [3] and [18].

An operator $T \in L(E)$ is said to be *power bounded* if $\{T^m : m \in \mathbb{N}\}$ is an equicontinuous set of $L(E)$. T is called *mean ergodic* if the limits of the Cesàro means of the iterates of T

$$Px := \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} T^m x,\ x \in E,$$

exist in E. The spaces we are considering are barrelled, therefore an operator $T \in L(E)$ is power bounded if and only if for each $x \in E$ its orbit $(x, Tx, T^2x, \dots)$ is bounded in E.

Let us recall that the topology τ_b of uniform convergence on bounded sets is defined in $L(E)$ via the seminorms

$$q_B(S) := \sup_{x\in B} q(Sx),\ S \in L(E),$$

for each bounded set B of E and each $q \in \mathrm{cs}(E)$. If the sequence of the Cesàro means of the iterates of T converges in τ_b, the operator T is called *uniformly mean ergodic.* Mean ergodic operators in Fréchet spaces and barrelled locally convex spaces have been considered in [1]. According to [2, Proposition 2.4], A_p and A_p^0 are uniformly mean ergodic, that is, each power bounded operator is automatically uniformly mean ergodic.

By [5] the Hardy operator $(Hf)(z) = \frac{1}{z}\int_0^z f(\zeta)d\zeta$ is continuous and power bounded on H_v^0 for every weight v, hence it is power bounded and uniformly mean ergodic on A_p and on A_p^0. Thus, we restrict our attention to the integration operator, the differentiation operator and more general differential operators.

2. The differentiation operator

In this section we consider the action of the differentiation operator $Df(z) := f'(z)$ on the Hörmander algebras A_p and A_p^0.

Lemma 2.1. *Let u and v be two weights on $\mathbb{C}$ such that there are $0 < \alpha < 1$ and $C > 0$ such that $u(r)e^{\alpha r} \leq Cv(R)e^{\alpha R}$ for each $0 \leq r \leq R$. Then for each $n \in \mathbb{N}$, the operator*

$$D^n : H_v \to H_u$$

is continuous. Moreover, for each $f \in H_v$ the sequence $(D^n(f))_n$ converges to 0 in H_u.

Proof. Fix $f \in H_v$ and $n \in \mathbb{N}$. Given $z \in \mathbb{C}$ and $\varepsilon > 0$, by the Cauchy integral formula we have, since v is decreasing,

$$|f^{(n)}(z)| \leq \frac{n!}{\varepsilon^n}\sup_{|z-\omega|=\varepsilon}|f(\omega)| \leq \frac{n!}{\varepsilon^n}||f||_v \frac{1}{v(|z|+\varepsilon)}.$$

Thus

$$u(|z|)|f^{(n)}(z)| \leq \frac{n!}{\varepsilon^n}||f||_v \frac{u(|z|)}{v(|z|+\varepsilon)} \leq C||f||_v \frac{n!}{\varepsilon^n}e^{\alpha\varepsilon}.$$

This implies, by the Stirling formula,

$$||f^{(n)}||_u \leq C||f||_v n! \inf_{\varepsilon>0}(\tfrac{e^{\alpha\varepsilon}}{\varepsilon^n}) =$$

$$= C||f||_v n! \tfrac{\alpha^n e^n}{n^n} \leq C'||f||_v \alpha^n \sqrt{2\pi n},$$

which converges to 0 as $n \to \infty$, since $0 < \alpha < 1$. □

Corollary 2.2. *If v is a weight such that for some $0 < \alpha < 1$ the function $v(r)e^{\alpha r}$ is increasing, then $D : H_v \to H_v$ is continuous and the orbit of each $f \in H_v$ converges to 0 in norm.*

Corollary 2.3. *(a) Let $(v_k)_k$ be a decreasing sequence of weights such that for every m there are $k \geq m$, $C > 0$ and $0 < \alpha < 1$ such that $v_k(r)e^{\alpha r} \leq Cv_m(R)e^{\alpha R}$ if $0 \leq r \leq R$. Then $D : VH \to VH$ is continuous and $(D^n(f))_n$ converges to 0 in VH for each $f \in VH$.*

(b) Let $(w_k)_k$ be an increasing sequence of weights such that for each m there are $k \geq m$, $C > 0$ and $0 < \alpha < 1$ such that $w_m(r)e^{\alpha r} \leq Cw_k(R)e^{\alpha R}$ if $0 \leq r \leq R$. Then the differentiation operator $D : HW \to HW$ is continuous and $(D^n(f))_n$ converges to 0 in HW for each $f \in HW$.

Proof. It follows easily from Lemma 2.1. □

Lemma 2.4. *Let E be a locally convex space of entire functions continuously included in $\mathcal{H}(\mathbb{C})$ and assume that there is $a > 1$ such that $e^{az} \in E$. If $D : E \to E$ is continuous, then it is not mean ergodic.*

Proof. Set $e_a(z) := e^{az}$, $z \in \mathbb{C}$. If D is continuous on E and mean ergodic, then $\frac{1}{n}\sum_{m=1}^{n} D^m e_a$ converges in E. Since E is continuously included $\mathcal{H}(\mathbb{C})$, the sequence

$$(\frac{1}{n}\sum_{m=1}^{n} D^m e_a(0))_n = (\frac{1}{n}\sum_{m=1}^{n} a^m)_n$$

converges. This is impossible for $a > 1$. □

Theorem 2.5. (i) *If $r = O(p(r))$ as $r \to \infty$, then D is not mean ergodic on A_p.*

(ii) *If $r = o(p(r))$ as $r \to \infty$, then D is not mean ergodic on A_p^0.*

(iii) *If $p(r) = o(r)$ as $r \to \infty$, then D is power bounded, hence uniformly mean ergodic and not hypercyclic on A_p and on A_p^0.*

Proof. Statements (i) and (ii) follow from Lemma 2.4.

We conclude (iii) from Corollary 2.3. We give the details for A_p applying Corollary 2.3(a). The proof for A_p^0 is similar. Fix $m \in \mathbb{N}$. Apply condition (β) for the growth condition p to find $A > 0$ such that $p(r+s) \leq A(1+p(r)+p(s))$ for each r, $s > 0$. As $p(r) = o(r)$, there exists $C > 0$ such that $Amp(s) < C + \frac{1}{2}s$ for each $s \geq 0$. If $k > Am$ and $0 \leq r \leq R$ we have, for $s = R - r$,

$$mp(R) \leq Am(1 + p(r)) + C + \frac{R-r}{2} \leq C_1 + kp(r) + \frac{R}{2} - \frac{r}{2}.$$

This yields, for $0 \leq r \leq R$,

$$e^{-kp(r)}e^{r/2} \leq e^{C_1}e^{-mp(R)}e^{R/2}.$$

This completes the proof. □

Proposition 2.6. (i) *If $p(r) = o(r - \frac{1}{2}\log(r))$ as $r \to \infty$, then D is not hypercyclic on A_p.*

(ii) *If $r = O(p(r))$ as $r \to \infty$, then D is topologically mixing and has a dense set of periodic points on A_p.*

(iii) *If $r = o(p(r))$ as $r \to \infty$, then D is topologically mixing and has a dense set of periodic points on A_p^0.*

Proof. Statements (i) and (ii) were proved in [13]. In fact, (ii) and (iii) follow from Bonet [14], since the weighted Banach space

$$H_\alpha^0 := \{f \in \mathcal{H}(\mathbb{C}) | \sup_{z\in\mathbb{C}} |f(z)|e^{-\alpha|z|} < \infty\},$$

$\alpha > 1$, is continuously and densely included in A_p if $r = O(p(r))$ $(r \to \infty)$ and in A_p^0 if $r = o(p(r))$ $(r \to \infty)$. The conclusions now follow by the comparison principle, e.g. [13, Lemma 3]. □

Corollary 2.7. *Let $p_a(r) = r^a$, $a > 0$:*

(i) *If $a > 1$, then D is topologically mixing, chaotic and not mean ergodic on A_{p_a} and on $A_{p_a}^0$.*

(ii) *If $a < 1$, then D is power bounded, hence uniformly mean ergodic on A_{p_a} and $A^0_{p_a}$.*

(iii) *If $a = 1$, then D is topologically mixing, chaotic and not mean ergodic on A_{p_1}, and it is power bounded on $A^0_{p_1}$.*

Proof. (i) follows from Theorem 2.5 (i) and (ii) and Proposition 2.6 (ii) and (iii). (ii) is a consequence of Theorem 2.5 (iii). Since statement (iii) for A_{p_1} follows easily from Theorem 2.5 (i) and Proposition 2.6 (ii), only the statement for $A^0_{p_1}$ ($p(r) = r$) needs a proof. In this case, $A^0_{p_1}$ is the intersection of the spaces $H^0_{v_n}$ for $v_n(r) = e^{-\frac{r}{n+1}}$ and the differentiation operator D is power bounded on each $H^0_{v_n}(\mathbb{C})$ by [5, Theorem 1.1.(1)]. □

It is possible to extend some of these results to the weighted Fréchet space H_0W. See also [14, Theorems 2.3, 2.4].

Proposition 2.8. *Assume that the differentiation operator $D : H_0W \to H_0W$ is continuous. Then the following are equivalent:*

(i) *D satisfies the hypercyclicity criterion (see* [18, Theorem 3.12]*).*

(ii) *D is hypercyclic.*

(iii) *there exists a sequence $(k_s)_s$ such that $\lim_{s\to\infty} \frac{\|z^{k_s}\|_n}{k_s!} = 0$ for every $n \in \mathbb{N}$.*

Proof. (i)⇒(ii) is clear.

By the Cauchy inequalities, for each $n \in \mathbb{N}$,

$$w_n(r)\frac{|f^{(k)}(0)|}{k!}r^k = w_n(r)\left|\frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)}{z^{k+1}}dz\right| r^k \le w_n(r)\max_{|z|=r}|f(z)| \le \|f\|_n, \tag{1}$$

Hence, $\frac{|f^k(0)|}{k!}\|z^k\|_n \le \|f\|_n$ for every $f \in H_{v_n}$ and each k. Therefore, the monomials are an OP-basis of H_0W (see [17] for the definition) and the equivalence (ii)⇔(iii) is a special case of [17, Theorem 7] since the differentiation operator D becomes the weighted backward shift B_w, with weights $w_n = n$, $n \in \mathbb{N}$.

Let us see (iii)⇒(i). Take Y as the set of all polynomials, which is dense in H_0W. Define $S_j := S^j$ on Y, $j \in \mathbb{N}$, with S the integration map defined on the monomials by $S(z^k) = z^{k+1}/(k+1)$. Since $D \circ S(q) = q$ for each polynomial q, and for each q of degree less or equal to M, $D^k q = 0$ for $k \ge M+1$, it only remains to show that there exists some sequence $(N_j)_j$ such that $\lim_{j\to\infty} S^{N_j} q = 0$ in H_0W for each polynomial q.

Since D is continuous, given $m \in \mathbb{N}$ and $j \in \mathbb{N}$ there exist $n_{m,j} \in \mathbb{N}$ and $C_{m,j} > 1$ such that $\|D^l f\|_m \le C_{m,j}\|f\|_{n_{m,j}}$ for every $l \le j$ and each $f \in H_0W$.

Given $j \in \mathbb{N}$, consider the weight $w_{n_{j,j}}$. Since $\lim_{s\to\infty} \frac{\|z^{k_s}\|_{n_{j,j}}}{k_s!} = 0$, set $k_{s_0} = 0$ and find k_{s_j} such that $\frac{\|z^{k_{s_j}}\|_{n_{j,j}}}{k_{s_j}!} \le \frac{1}{jC_{j,j}}$ and $k_{s_{j+1}} > k_{s_j} + j + 2$. Consider $N_j := k_{s_j} - j - 1$, $j \in \mathbb{N}$. Notice that $N_{j+1} > N_j + j + 1$. Since

$$S^{N_j}(z^k) = \frac{k!}{(N_j+k)!} z^{N_j+k}$$

and

$$D^{j+1-k}(z^{N_j+j+1}) = \frac{(N_j+j+1)!}{(N_j+k)!} z^{N_j+k}$$

for $k \leq j+1$, $k \in \mathbb{N}_0$, given m, for $j \geq m$ we get

$$\|S^{N_j}(z^k)\|_m \leq \frac{k!}{(N_j+k)!}\|z^{N_j+k}\|_j = \frac{k!}{(N_j+j+1)!}\|D^{j+1-k}(z^{N_j+j+1})\|_j \leq$$

$$\leq k!C_{j,j}\frac{\|z^{N_j+j+1}\|_{n_{j,j}}}{(N_j+j+1)!} \leq \frac{k!}{j}$$

Thus, for every $k \in \mathbb{N}$ and each $m \in \mathbb{N}$, we get $\lim_{j\to\infty}\|S^{N_j}(z^k)\|_m = 0$. Therefore, S^{N_j} tends to zero on the polynomials, and the hypercyclicity criterion is satisfied. □

Proposition 2.9. *Assume that the differentiation operator $D : H_0W \to H_0W$ is continuous. The following are equivalent:*

(i) *D is topologically mixing.*
(ii) $\lim_{k\to\infty}\frac{\|z^k\|_n}{k!} = 0$ *for every $n \in \mathbb{N}$.*

Proof. Suppose that for some $n \in \mathbb{N}$ the sequence $(\frac{\|z^k\|_n}{k!})_n$ does not converge to zero. Then we find $M > 0$ and a subsequence $(k_s)_s$ such that $\frac{\|z^{k_s}\|_n}{k_s!} > M$. Proceeding analogously to the proof of [14, Lemma 2.2] in the case of Fréchet spaces, we get that the sequence $\{(D^t)^{k_s}(\delta_0) : s \in \mathbb{N}\}$ is unbounded in $(H_0W)'$, hence there is $f \in H_0W$ such that $\{f^{(k_s)}(0) : s \in \mathbb{N}\}$ is unbounded, and therefore there is a subsequence that we still denote by $(k_s)_s$ such that $\lim_{s\to\infty}|f^{(k_s)}(0)| = \infty$. Now, as in (1),

$$w_n(r)|f^{(k)}(0)|\frac{r^k}{k!} \leq w_n(r)\max_{|z|=r}|f(z)| \leq \|f\|_n.$$

Then, for $n, s \in \mathbb{N}$ and $r > 0$,

$$w_n(r)\frac{r^{k_s}}{k_s!} \leq \frac{\|f\|_n}{|f^{(k_s)}(0)|},$$

which implies

$$M < \frac{\|z^{k_s}\|_n}{k_s!} \underset{s\to\infty}{\to} 0,$$

a contradiction.

(ii)⇒(i). It is enough to show that D satisfies the assumptions of the criterion of Kitai-Gethner-Shapiro [18, Theorem 3.4]. As in the proof of Proposition 2.8, we take Y the set of all polynomials and denote by S the operator of integration in the set of polynomials. Clearly $(D^j)_j$ tends pointwise to 0 in the set of polynomials and $D \circ S$ coincides with the identity on this set. So, it only remains to prove that $(S^j(g))_j$ converges to 0 in H_0W for all polynomial g. Since $S^j(z^k) = k!z^{k+j}/(k+j)!$ for each $k \in \mathbb{N}$, it is enough to show that $(\|\frac{z^k}{k!}\|_n)_k$ converges to 0 for every $n \in \mathbb{N}$, which holds by condition (ii). □

Proposition 2.10. *Assume that the differentiation operator $D : H_0W \to H_0W$ is continuous. The following conditions are equivalent:*

(i) *D is chaotic.*
(ii) *D has a periodic point different from 0.*
(iii) $\lim_{r\to\infty} w_n(r)e^r = 0$ *for every $n \in \mathbb{N}$.*

Proof. Clearly (i) implies (ii). Let us see (ii)⇒(iii). By hypothesis, there exists a function $0 \neq f \in H_0W$ such that, for some $n \in \mathbb{N}$, $D^n f = f$. Using the trivial decomposition $D^n - I = (D - \theta_1 I) \dots (D - \theta_n I)$, $\theta_j^n = 1$, $j = 1, \dots, n$, we conclude that there is $\theta \in \mathbb{C}$, $|\theta| = 1$, and $g \in H_0W$, $g \neq 0$, such that $(D - \theta I)g = 0$. This yields $e^{\theta z} \in H_0W$, and thus, (iii) is satisfied.
(iii)⇒(i) Denote by P the linear span of the functions $e^{\theta z}$, $\theta \in \mathbb{C}$, $\theta^n = 1$ for some $n \in \mathbb{N}$. Obviously, P is formed by periodic points and, by the proof of [15, Theorem 2.3], it is dense in $H^0_{w_n}$ for every $n \in \mathbb{N}$, and thus, on H_0W. On the other hand, as $w_n(r) = o(e^{-r})$ for every $n \in \mathbb{N}$, by the proof of [14, Corollary 2.5.(2)], $\lim_{k\to\infty} \frac{\|z^k\|_n}{k!} = 0$ for every $n \in \mathbb{N}$. Therefore, D is topologically mixing by Proposition 2.9, and thus, chaotic. □

To close this section we describe the spectrum of the differentiation operator on the Hörmander algebras.

Proposition 2.11. (i) *If $r = O(p(r))$ (resp. $r = o(p(r))$) as $r \to \infty$, then the spectrum of D in A_p (resp. in A_p^0) is $\mathbb{C}$.*
(ii) *If $p(r) = o(r)$ as $r \to \infty$, then the spectrum of D in A_p and in A_p^0 reduces to $\{0\}$.*

Proof. (i) follows from the fact that $e^{az} \in A_p$ (resp. $e^{az} \in A_p^0$) for every $a \in \mathbb{C}$, hence each complex number is an eigenvalue of the operator.

(ii) In the proof of Theorem 2.5 (iii), we get that for each $0 < \alpha$ and each m there exist $k > m$ and $C > 0$ such that $-kp(r) + \alpha r < C - mp(R) + \alpha R$. Hence, given $b \in \mathbb{C}$, $\varepsilon > 0$, and $0 < \beta < 1$, take $\alpha = \frac{\beta}{|b|}$. Proceeding as in the proof of Lemma 2.1, with the Cauchy formulas integrating in $|z - w| = |b|\varepsilon$, we get

$$e^{-kp(|z|)}|b^n f^{(n)}(z)| \leq C'||f||_m n! \inf_{\varepsilon>0}\left(\frac{e^{\beta\varepsilon}}{\varepsilon^n}\right) \leq C'||f||_m \beta^n \sqrt{2\pi n}. \tag{2}$$

If we consider β^2 instead of β in (2), there exists $k_1 \in \mathbb{N}$ such that

$$e^{-k_1 p(|z|)}|b^n f^{(n)}(z)| \leq C'||f||_m (\beta^2)^n \sqrt{2\pi n} \leq C''\beta^n ||f||_m.$$

Then, for each $b \in \mathbb{C}$ and every $f \in A_p$ the series

$$\sum_{n=0}^{\infty} b^n D^n f$$

converges in A_p and therefore it defines a continuous operator in this space. This immediately implies that $aI - D$ is invertible in A_p whenever $a \neq 0$. The proof for A_p^0 follows the same steps. □

3. Differential operators

This section is devoted to study the dynamics of differential operators $\phi(D)$ on the Hörmander algebras A_p and A_p^0 whenever ϕ is an entire function of exponential type.

An entire function ϕ is said to be of *exponential type* if there are constants $C, R > 0$ such that

$$|\phi(z)| \leq Ce^{R|z|} \text{ for all } z \in \mathbb{C}.$$

It is well-known that an entire function $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ is of exponential type if and only if there are $C, B > 0$ such that, for $n \geq 0$, $|a_n| \leq C\frac{B^n}{n!}$ (see e.g. [18, Lemma 4.18]). For every entire function of exponential type $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ the differential operator $\phi(D)f = \sum_{n=0}^{\infty} a_n D^n f$ is well defined and continuous on $\mathcal{H}(\mathbb{C})$. Godefroy and Shapiro proved that if $T : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$, $T \neq \lambda I$, commutes with D, that is, $TD = DT$, it can be expressed as a differential operator $\phi(D)$ for an entire function ϕ of exponential type [22]. Moreover, they proved that T is chaotic. See [18, Section 4.2] for more references and background on this topic.

To start, let us observe that given an operator T on a Banach space X and $\phi(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{C})$, the expression $\phi(T) = \sum_{n=0}^{\infty} a_n T^n$ defines also an operator on X. In fact, $\|\phi(T)\| \leq \sum_{n=0}^{\infty} |a_n| \|T\|^n < \infty$. In particular, if the differentiation operator D is continuous on the weighted Banach space of entire functions H_v^0, the differential operator

$$\phi(D)f = \sum_{n=0}^{\infty} a_n f^{(n)}$$

is continuous on this space.

In [8] Bernal and Bonilla show that if $T : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$, $T \neq \lambda I$, commutes with D and $T = \phi(D)$ is its representation with an entire function ϕ of exponential type, then for any $\tau > \min\{|z| : \ |\phi(z)| = 1\}$ there is a T-hypercyclic entire function with $|f(z)| \leq Me^{\tau|z|}$, $z \in \mathbb{C}$. This seems to be the best growth result known for general operators T commuting with D. For individual operators, much better results are available. The next theorem is an adaptation of the results given in [8] for weighted Banach spaces of entire functions.

Theorem 3.1. *Let ϕ be an entire function and $\alpha > \min\{|z| : |\phi(z)| = 1\}$. Suppose v is a weight such that $\lim_{r\to\infty} v(r)e^{\alpha r} = 0$ and $D : H_v^0 \to H_v^0$ is continuous. Then, the operator $\phi(D) : H_v^0 \to H_v^0$ is topologically mixing. Moreover, $\phi(D)$ is chaotic and not mean ergodic.*

Proof. We prove that $\phi(D)$ satisfies the Hypercyclicity Criterion for the entire sequence of positive integers. Consider the sets

$$V := \operatorname{span}\{e_a : |a| < \alpha, |\phi(a)| < 1\},$$

$$W := \operatorname{span}\{e_a : |a| < \alpha, |\phi(a)| > 1\},$$

where $e_a(z) := e^{az}$, $a \in \mathbb{C}$, $z \in \mathbb{C}$. Since ϕ is non constant and open, $\phi(\alpha\mathbb{D})$ is an open set, which together with $\phi(\alpha\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ implies that $\alpha\mathbb{D} \cap \phi^{-1}(\mathbb{D})$ and $\alpha\mathbb{D} \cap \phi^{-1}(\mathbb{C}\setminus\overline{\mathbb{D}})$ are non-empty open sets of $\mathbb{C}$. Thus, they have an accumulation point in $\alpha\mathbb{D}$. By hypothesis, $e_a \in H_v^0$ for all $a \in \alpha\mathbb{D}$, and by the proof of [15, Theorem 2.3] (see also [4, Lemma 5.4]), V and W are dense subsets of H_v^0. Furthermore, $\phi(D)e_a = \phi(a)e_a$ for all $n \in \mathbb{N}$, then, $\phi(D)^n e_a = \phi(a)^n e_a$ for each $n \in \mathbb{N}$, $a \in \mathbb{C}$. Hence, by linearity, $\phi(D)^n f \to 0$ as n tends to infinity for all $f \in V$. Let $a \in \mathbb{C}$ with $|a| < \alpha$ and $|\phi(a)| > 1$. Define $S : W \to H_v^0$ as $S(e_a) = \frac{1}{\phi(a)} e_a \in W$ and extend it by linearity to W. Then, $S^n(e_a) = \frac{1}{\phi(a)^n} e_a \to 0$ if n tends to infinity, so $S^n f$ converges to 0 for every $f \in W$. As $\phi(D)^n S^n f = f$ for all $f \in W$, the Hypercyclicity Criterion holds.

Consider now the set

$$P := \operatorname{span}\{e_a : |a| < \alpha,\ \phi(a)^n = 1,\ n \in \mathbb{N}\} \subseteq H_v^0.$$

As $\phi(D)^n e_a = \phi(a)^n e_a$, P is formed by periodic points. There exists $z_0 \in \alpha\mathbb{D}$ such that $|\phi(z_0)| = 1$. Take an open set U containing z_0 and such that $\overline{U} \subseteq \alpha\mathbb{D}$ As $\phi(U)$ is open, $\phi(U) \cap \mathbb{T}$ contains a dense set formed by roots of the unity. Then, the preimages by ϕ of this set contain a sequence in $\overline{U}$, and thus, the set $\{a \in \mathbb{C} : |a| < \alpha,\ \phi(a)^n = 1,\ n \in \mathbb{N}\}$ has an accumulation point in $\alpha\mathbb{D}$. Therefore, again by the proof of [15, Theorem 2.3], [4, Lemma 5.4], P is a dense set of periodic points, and hence, $\phi(D)$ is chaotic. Finally, given $a \in \mathbb{C}$, $|a| < \alpha$ and $|\phi(a)| > 1$, the sequence $\{\frac{\phi(D)^n(e_a)}{n}\}_n$ does not tend to zero. Therefore, $\phi(D)$ cannot be mean ergodic. □

Our next aim is to check that for each entire function of exponential type $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$, the differential operator $\phi(D)f = \sum_{n=0}^{\infty} a_n D^n f$ is continuous on the Hörmander algebras.

Proposition 3.2. *If $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of exponential type, then the differential operator $\phi(D)$ acts continuously on the Hörmander algebras A_p and A_p^0 for every growth condition p.*

Proof. If ϕ is an entire function of exponential type, then $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$, with $|a_n| \leq CB^n/n!$ for some constants C, $B > 0$. For a growth condition p, and $a > 0$, consider $v_a(r) := e^{-ap(r)}$. By condition (β) in the definition of growth condition, there exists $A > 0$ such that

$$\sup\{p(w) : |w - z| \leq (1 + B)\} \leq Ap(r) + A.$$

Given $f \in H_{v_a}$, by the Cauchy integral formulas

$$|f^{(n)}(z)| \leq \frac{n!}{(1+B)^n} ||f||_{v_a} e^{aAp(|z|)+aA}.$$

Therefore,

$$\sum_{n=0}^{\infty} |a_n||f^{(n)}(z)| \leq C \sum_{n=0}^{\infty} (\frac{B}{1+B})^n ||f||_{v_a} e^{aAp(|z|)+aA}.$$

Hence, if $b > aA$,

$$e^{-bp(|z|)}|\phi(D)(f)(z)| \leq Ce^{aA} \sum_{n=0}^{\infty} (\frac{B}{1+B})^n ||f||_{v_a},$$

which shows that $\phi(D)$ acts continuously on the Hörmander algebras A_p and A_p^0. □

Proposition 3.3. *Let ϕ be a non-constant entire function of exponential type.*

(i) *If $r = O(p(r))$ as $r \to \infty$, then $\phi(D)$ is chaotic, topologically mixing and not mean ergodic on A_p.*

(ii) *If $r = o(p(r))$ as $r \to \infty$, then $\phi(D)$ is chaotic, topologically mixing and not mean ergodic on A_p^0.*

Proof. The weighted Banach space

$$H_\beta^0 := \{f \in \mathcal{H}(\mathbb{C}) | \sup_{z\in\mathbb{C}} |f(z)| e^{-\beta|z|} < \infty\}$$

is continuously and densely included in A_p if $r = O(p(r))$ $(r \to \infty)$ and in A_p^0 if $r = o(p(r))$ $(r \to \infty)$ for every $\beta > 0$. If we consider $\alpha > \min\{|z| : |\phi(z)| = 1\}$, by Theorem 3.1, $\phi(D)$ is chaotic and topologically mixing on H_β^0 for every $\beta > \alpha$. The conclusion now follows by the comparison principle (e.g. [13, Lemma 3]). For the assertion about non mean ergodicity, it suffices to argue as in Theorem 3.1. □

Remark 3.4. By Propositions 2.6 and 3.3, for $r = O(p(r))$ $(r = o(p(r)))$, the behavior of the differential operator $\phi(D)$ on A_p (A_p^0) is the same as the one of the differentiation operator D.

For $p(r) = o(r)$, the behaviour of $\phi(D)$ is not determined. Whereas D is always power bounded, $\phi(D)$ need not be mean ergodic. Indeed, if $|\phi(0)| > 1$, it is enough to consider the orbit of a non-zero constant function, since $\phi^n(D)(c) = \phi(0)^n c$ for every $c \in \mathbb{C}$. We include some results about concrete differential operators.

Proposition 3.5. *Assume that* $p(r) = o(r)$.

(i) *If* $|a| < 1$ *and* $b \in \mathbb{C}$, *the operator* $aI + bD$ *is power bounded, hence uniformly mean ergodic on* A_p *and* A_p^0.

(ii) *If* $|a| = 1$, *and* $b \neq 0$, *the operator* $aI + bD$ *is hypercyclic on* A_p *and* A_p^0.

(iii) *If* $|a| > 1$ *and* $b \in \mathbb{C}$, *the operator* $aI + bD$ *is hypercyclic neither on* A_p *nor on* A_p^0.

Proof. (i) As in the proof of Proposition 2.11, given m, $b \in \mathbb{C}$ and $0 < \beta < 1$ we find k_1 and $C > 0$ such that

$$e^{-k_1 p(|z|)} |b^n f^{(n)}(z)| \leq C\beta^n ||f||_m.$$

Then, for $a \in \mathbb{C}$ with $|a| < 1$, choosing $\beta > 0$ with $|a| + \beta < 1$, we have for each N,

$$e^{-k_1 p(|z|)} |(aI + bD)^N(f)| \leq \sum_{n=0}^N \binom{N}{n} |a^n| e^{-k_1 p(|z|)} |b^{N-n} f^{(N-n)}(z)|$$

$$\leq C' ||f||_m \sum_{n=0}^N \binom{N}{n} |a|^n \beta^{N-n} \leq C' ||f||_m (|a| + \beta)^N,$$

from where we conclude.

(ii) Since the operator cD satisfies the hypothesis of [21, Theorem 5.2] for every $c \in \mathbb{C}$, $c \neq 0$, we get that, on A_p^0, the operator $I + cD$ satisfies the Hypercyclicity Criterion. Therefore, by [18, Theorem 6.7], $aI + bD = a(I + b/aD)$ is hypercyclic for every $a \in \mathbb{C}$ with $|a| = 1$. Since A_p^0 is densely and continuously included in A_p, $aI + bD$ is hypercyclic on A_p by the comparison principle.

(iii) As A_p^0 is densely included in A_p it suffices to show that $aI + bD$ is not hypercyclic on A_p. As in the proof of Proposition 3.3, set $H_\beta^0 := \{f \in \mathcal{H}(\mathbb{C}) | \sup_{z\in\mathbb{C}} |f(z)| e^{-\beta|z|} < \infty\}$. The assumptions on the growth condition p imply that A_p is continuously and densely included in H_β^0 for all $\beta > 0$. By [5, Proposition 3.8] and the spectral mapping theorem, the spectrum of $aI + bD$ on H_β^0 is $\overline{D(a, |b|\beta)}$. Therefore, if β is small enough and $|a| > 1$, the spectrum of the operator $aI + bD$ on H^0_β does not meet the unit circle. Hence, by Kitai's criterion [18, Proposition 5.3], $aI + bD$ is not hypercyclic on H^0_β, so it cannot be hypercyclic on A_p. $\square$

To finish, we consider the translation operator $T_{z_0}f(z) = f(z + z_0)$, $z_0 \neq 0$. By condition (β) in the definition of growth condition, T_{z_0} is continuous on the Hörmander algebras. Moreover T_{z_0} is a differential operator, since the Taylor series of f centred at $z \in \mathbb{C}$ gives

$$f(z + z_0) = \sum_{k=0}^{\infty} \frac{z_0^k}{k!} f^{(k)}(z).$$

Hence, $T_{z_0} = \phi(D)$, for $\phi(z) = e^{z_0 z}$. The theorem of Duyos-Ruiz [18, Theorem 4.23] says that there are hypercyclic functions for the translation operator of arbitrarily slow transcendental growth.

In the case of weighted Banach spaces H^0_v we get the following result:

Proposition 3.6. *Suppose v is a weight such that there exists $\alpha > 0$ such that $\lim_{r\to\infty} v(r)e^{\alpha r} = 0$. If the differentiation operator $D : H^0_v \to H^0_v$ is continuous, then the translation operator $T_{z_0} : H^0_v \to H^0_v$, $T_{z_0}f(z) = f(z+z_0)$, $z \in \mathbb{C}$, $z_0 \neq 0$, is continuous, topologically mixing and chaotic.*

Proof. As $\min\{|z| : \ |\phi(z)| = 1\} = 0$, it suffices to apply Theorem 3.1. $\square$

Proceeding as in the proof of Duyos-Ruiz theorem (see [18, Theorem 4.23 and Exercise 8.1.2]), in the case of the Hörmander algebras we get the following:

Proposition 3.7. *The translation operator T_{z_0}, $z_0 \neq 0$, is topologically mixing on A_p and A^0_p for every growth condition p.*

Proof. Given $b > 0$, take $v_b(r) = e^{-bp(r)}$. Since the weights are rapidly decreasing, there exists some $M_{n,b} > 0$ such that $\sup_{r\geq 0} r^n e^{-bp(r)} \leq M_{n,b}$. Take $\sigma_{n,b} := \frac{1}{M_{n,b}n!}$. Since the sequence

$$(n + 1)\frac{\sigma_{n+1,b}}{\sigma_{n,b}} = \frac{M_{n,b}}{M_{n+1,b}} \leq 1,$$

by [18, Exercise 8.1.2], T_{z_0} is topologically mixing on the Banach space

$$E(\sigma_b) := \{f \in \mathcal{H}(\mathbb{C}) : \ f(z) = \sum_{n=0}^{\infty} a_n z^n, \|f\|_\sigma = \sum_{n=0}^{\infty} \sigma_{n,b}^{-1}|a_n| < \infty\}.$$

Observe that $\sup_{z\in\mathbb{C}} e^{-bp(|z|)}|f(z)| \leq \|f\|_{\sigma_b}$ for every $f \in E(\sigma_b)$. So, we get $E(\sigma_b) \hookrightarrow H^0_{v_b}$ continuously. Since the monomials are a Schauder basis of $E(\sigma_b)$, the space is densely and continuously included in $H^0_{v_b}$. Hence, by the comparison principle (e.g. [13, Lemma 3]), T_{z_0} is topologically mixing on A_p. Since $\bigcap_{m\in\mathbb{N}} E(\sigma_{1/m})$ endowed with the projective limit topology is a reduced projective limit, T_{z_0} is topologically mixing on $\bigcap_{m\in\mathbb{N}} E(\sigma_{1/m})$ by [18, Corollary 12.19]. It is continuously and densely included in A^0_p, so again the comparison principle yields the conclusion. $\square$

4. The integration operator

In this section we consider the action of the integration operator

$$Jf(z) := \int_0^z f(\zeta)d\zeta$$

on the Hörmander algebras A_p and A_p^0. We start with the following observation:

Lemma 4.1. (i) *Let u and v be two weights and assume that there is $C>0$ such that $ru(r)\le Cv(r)$. Then, $J: H_v\to H_u$ is continuous.*
(ii) *The integration operators $J: A_p\to A_p$ and $J: A_p^0\to A_p^0$ are well defined and continuous.*

Proof. (i) Since

$$u(|z|)|J(z)|\le |z|u(|z|)\int_0^1|f(tz)|dt\le \\ Cv(|z|)\int_0^1|f(tz)|dt\le C\int_0^1 v(t|z|)|f(tz)|dt\le C||f||_v$$

for every $f\in H_v$, the operator J is continuous. (ii) It is enough to observe that condition (α) of the weight p implies that $re^{-ap(r)}/e^{-bp(r)}$ is bounded whenever $a>b$ and apply part (i). $\square$

It is worth mentioning that J is not continuous on H_v for $v(r)=e^{-\alpha r^a}$, $a<1$, $\alpha>0$. In fact, an easy computation gives that

$$||z^j||_v = (\frac{j}{e\alpha a})^{\frac{j}{a}},$$

hence,

$$\frac{\|J(z^j)\|_v}{\|z^j\|_v}=\frac{\|z^{j+1}\|_v}{(j+1)\|z^j\|_v}=(\frac{1}{e\alpha a})^{\frac{1}{a}}(\frac{j+1}{j})^{\frac{j}{a}}(j+1)^{\frac{1}{a}-1}$$

As $a<1$, we get that J is not continuous, since the right hand side diverges to infinity.

Proposition 4.2. *Let u, v be two weights and assume that for some $\alpha\ge 1$ and $C>0$, $u(R)e^{\alpha R}\le Cv(r)e^{\alpha r}$ for all $0\le r\le R$. Then, for every $n\in\mathbb{N}$ the operator*

$$J^n: H_v\to H_u$$

is continuous and for each $f\in H_v$ the sequence $(J^nf)_n$ is bounded in H_u, and in case $\alpha>1$, it converges to 0.

Proof. Observe that

$$|J^nf(z)|\le\int_0^1|z||J^{n-1}f(t_1z)|dt_1=\int_0^1|z|\int_0^1 t_1|z||J^{n-2}f(t_2t_1z)|dt_2dt_1\le$$

$$\le\dots\le\int_0^1|z|\int_0^1 t_1|z|\int_0^1 t_2t_1|z|\int_0^1\dots\int_0^1 t_{n-1}\dots t_1|z||f(t_n\dots t_1z)|dt_n\dots dt_1.$$

By hypothesis, there exist $\alpha\ge1$ and $C>0$ such that

$$u(|z|)\le Cv(t_1\dots t_nz)e^{\alpha t_1\dots t_n|z|}e^{-\alpha|z|}=$$

$$Cv(t_1\dots t_nz)e^{\alpha|z|(t_1-1)}e^{\alpha t_1|z|(t_2-1)}e^{\alpha t_1t_2|z|(t_3-1)}\dots e^{\alpha|z|t_1\dots t_{n-1}(t_n-1)}.$$

Then,

$u(|z|)|J^n f(z)| \leq$

$$C\|f\|_v \int_0^1 |z|e^{\alpha|z|(t_1-1)} \int_0^1 \dots \int_0^1 t_{n-1}\dots t_1|z|e^{\alpha|z|t_1\dots t_{n-1}(t_n-1)}dt_n\dots dt_1$$

which yields

$$||J^n f||_u \leq ||f||_v \frac{C}{\alpha^n}.$$

□

Observe that the statement of Proposition 4.2 for $u = v$ follows from [5, Theorem 3.5].

Proposition 4.3. *The operator J is power bounded on $\mathcal{H}(\mathbb{C})$, therefore it is not hypercyclic, and it has no periodic points apart from 0. Consequently, J is not hypercyclic on the Hörmander algebras A_p nor A_p^0.*

Proof. To see that J is power bounded on $\mathcal{H}(\mathbb{C})$ it suffices to show that for every entire function f, the sequence $(J^n(f))_n$ converges to zero in the compact open topology. This fact is well known (see e.g. [12]), but we include the proof here for the sake of completeness. If $f(z) = \sum_{m=0}^{\infty} a_m z^m$ we know that for each $R > 0$ there is C_R such that

$$|a_m| \leq \frac{C_R}{R^m}.$$

Given $r > 0$, take $R > r$, then, for $|z| \leq r$,

$$|J^n(f)(z)| \leq \frac{r^n}{n!}\sum_{m=0}^{\infty}|a_m|r^m \leq C_R\frac{r^n}{n!}\sum_{m=0}^{\infty}(\frac{r}{R})^m,$$

from where $(J^n(f))_n$ converges to zero uniformly in $\{z : |z| \leq r\}$. The fact that J has no periodic points except zero was shown in [5]. As A_p and A_p^0 are densely and continuously included in $\mathcal{H}(\mathbb{C})$, by the comparison principle, e.g. [13, Lemma 3], the non-hypercyclicity of J on these spaces follows. □

Theorem 4.4. (i) *The operator of integration is power bounded and hence uniformly mean ergodic on A_p, provided that $r = O(p(r))$ as $r \to \infty$.*

(ii) *If $p(r) = o(r)$ as $r \to \infty$, then J is not mean ergodic on A_p.*

(iii) *J is power bounded and hence uniformly mean ergodic on A_p^0 provided that $r = o(p(r))$ as $r \to \infty$.*

(iv) *If $p(r) = O(r)$ as $r \to \infty$, then J is not mean ergodic on A_p^0.*

Proof. (i) As $r = O(p(r))$, we may assume without loss of generality that $2r \leq p(r) + c$ for some $c > 0$ and all $r > 0$. Indeed, if $r \leq A(p(r) + 1)$ for some $A > 0$ and every $r \geq 0$, we can consider the growth condition $q(r) = Ap(r)$, since $A_p = A_q$. Put $v_m(r) = e^{-mp(r)}$. Then, for all $0 \leq r \leq R$ we have

$$-(m+1)p(R) + mp(r) = -p(R) + m(p(r) - p(R)) \leq c - 2R \leq c + 2(r - R),$$

that is,

$$v_{m+1}(R)e^{2R} \leq Cv_m(r)e^{2r},$$

hence, if $\sup_{z\in\mathbb{C}}|f(z)|v_m(z) < \infty$, the sequence $(J^n f)_n$ converges to 0 in the next step of the inductive limit by Proposition 4.2.

(ii) If J is mean ergodic, then for each $f \in A_p$ the sequence $(\frac{J^n f}{n})_n$ tends to zero in A_p. For the weights $v_a(r) = e^{-ap(r)}$ the inclusions $H_{v_a} \subset H_{v_b}$ $(a < b)$ are compact, thus by [26] each convergent sequence in A_p is contained and convergent in some H_{v_m}. So, there is m such that $(||\frac{J^n f}{n}||_m)_n$ converges to 0. In particular, for $f \equiv 1$ this means that $(\frac{||z^n||_m}{n!n})_n$ converges to zero. But since $p(r) = o(r)$, we have that for some constant $C > 0$, $mp(r) \leq \frac{r}{2} + C$ for all r, hence

$$\frac{||z^n||_m}{n!n} \geq \frac{1}{e^C} \sup\{\frac{r^n}{n!n} e^{-r/2} : r \geq 0\}$$

and the right hand side diverges by the Stirling formula. The proof of (iv) is analogous.

(iii) Since $r = o(p(r))$, for each m there is $c_m > 0$ such that for all r, $4mr \leq c_m + p(r)$. Then, as before, for $0 \leq r \leq R$ we have

$$-\frac{1}{m}p(R) + \frac{1}{2m}p(r) \leq -\frac{1}{2m}p(R) \leq c_m - 2R \leq c_m + 2(r - R),$$

that is,

$$e^{-\frac{p(R)}{m}} e^{2R} \leq e^{c_m} e^{-\frac{p(r)}{2m}} e^{2r},$$

which by the proof of Proposition 4.2 implies that there exists some $C_m > 0$ such that, for all n,

$$\|J^n f\|_{w_m} \leq \frac{C_m}{2^n} \|f\|_{w_{2m}}.$$

$\square$

Remark 4.5. The function $p(0) = 0$, $p(r) = 2^n n$ for $2^n \leq r \leq 2^n + 2^{n-1}$, and linear and continuous in $[2^n + 2^{n-1}, 2^{n+1}]$, is increasing, $p(2r) = O(p(r))$ and $r = o(p(r))$ as $r \to \infty$. Hence J is power bounded on A_p as well as on A_p^0 by Theorem 4.4. However, $e^{-p(r)} e^{\alpha r}$ is not decreasing for each $\alpha > 0$, hence [5, Theorem 3.5] cannot be applied to the operator J on weighted Banach spaces of entire functions defined by weights of the form $e^{-ap(|z|)}$, $a > 0$. Observe that the function p is not convex. However, the subharmonicity of p is not assumed in this note to study the behaviour of the differentiation or the integration operators on the algebras A_p and A_p^0.

Corollary 4.6. *Let $p_a(r) = r^a$, then:*

(i) *J is power bounded on A_{p_a} for $a \geq 1$, and it is not mean ergodic for $a < 1$.*

(ii) *J is power bounded on $A_{p_a}^0$ for $a > 1$ and it is not mean ergodic for $a \leq 1$.*

Proposition 4.7. (i) *If $p(r) = o(r)$ (resp. $p(r) = O(r)$) as $r \to \infty$, the spectrum of J in A_p (resp. in A_p^0) is $\mathbb{C}$.*

(ii) *If $r = O(p(r))$ (resp. $r = o(p(r))$) as $r \to \infty$, the spectrum of J in A_p (resp. in A_p^0) reduces to $\{0\}$.*

Proof. (i) Follows from the fact that for each $\lambda \in \mathbb{C}$ the equation $Jf - \lambda f = 1$ has no solution in the space.

(ii) As in the proof of Theorem 4.4 (iii), for each $\alpha > 0$ and m we find $C > 0$ and n such that for all $0 \leq r \leq R$

$$e^{-\frac{p(R)}{m}} e^{\alpha R} \leq C e^{-\frac{p(r)}{n}} e^{\alpha r}.$$

Then by the proof of Proposition 4.2

$$||J^k f||_{w_m} \leq \frac{C}{\alpha^k} ||f||_{w_n}.$$

This again implies that for each $f \in A_p^0$ and each $b \in \mathbb{C}$,

$$\sum_{k=0}^{\infty} b^k J^k f$$

converges in A_p^0, hence we conclude as in Proposition 2.11. The proof for A_p is similar. □

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