APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 17, no. 1(2016), 57-70
doi:10.4995/agt.2016.4401

Two classes of metric spaces

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Abstract

The class of metric spaces (X, d) known as small-determined spaces, introduced by Garrido and Jaramillo, are properly defined by means of some type of real-valued Lipschitz functions on X. On the other hand, B-simple metric spaces introduced by Hejcman are defined in terms of some kind of bornologies of bounded subsets of X. In this note we present a common framework where both classes of metric spaces can be studied which allows us to see not only the relationships between them but also to obtain new internal characterizations of these metric properties.

2010 MSC: *Primary 54E35; Secondary 46A17; 54E40.*

Keywords: *Metric spaces; real-valued uniformly continuous functions; real-valued Lipschitz functions; bornologies; Bourbaki-boundedness; countable uniform partitions; small-determined spaces; B-simple spaces.*

1. Introduction and preliminaries

We are concerned here with two classes of metric spaces, namely *small-determined* spaces and *B-simple* spaces, which appear in separate frameworks into the general context of metric spaces. More precisely, one of them is related with the approximation and the extension of real-valued uniformly continuous functions (see [2]), whereas the other one is closer to boundedness and uniform

*Partially supported by MINECO Project MTM2012-34341 (Spain).

Received 30 November 2015 – Accepted 10 February 2016

bornologies (see [6]). More recently, in a paper devoted to the study of some special realcompactifications of metric spaces ([4]), we have noticed that these worlds apparently far away can be connected. Our main purpose in this note is to present a common framework where these classes of metric spaces can be studied. This setting will be simply of those "spaces in which all uniform partitions are countable". Spaces having this property are for instance every connected metric space, or more generally every uniformly connected, and also every separable metric space. We will see that on such spaces we can define a countable family of metrics uniformly equivalent to the original one that will be the key to obtaining the main results.

1.1. **Small-determined spaces.** The class of small determined metric spaces, introduced in [2], is properly defined in terms of the so-called Lipschitz in the small functions. Recall that a function between metric spaces $f:(X,d)\to(Y,d')$ is said to be *Lipschitz in the small* if there exist $\delta>0$ and $K>0$ such that $d'(f(x),f(y))\leq K\cdot d(x,y)$ whenever $d(x,y)<\delta$.

Definition 1.1 ([2])**.** A metric space (X,d) is said to be small-determined whenever every real-valued Lipschitz in the small function is Lipschitz.

If we denote by $Lip_d(X)$ the set of all real-valued Lipschitz functions and by $LS_d(X)$ the set of all real-valued Lipschitz in the small functions on the metric space (X,d) then, clearly $Lip_d(X)\subset LS_d(X)$. To see that the reverse inclusion is not true, it is enough to consider the space of natural numbers $\mathbb{N}$ endowed with the usual metric and the function $f:\mathbb{N}\to\mathbb{R}$ defined by $f(n)=n^2$, since f is Lipschitz in the small but not Lipschitz. Hence $\mathbb{N}$ is not a small-determined space.

Nevertheless, as we can see in [2], small-determined metric spaces form a huge class of metric spaces containing for instance all the normed spaces and more generally all the length spaces. Moreover these spaces have good properties of approximation and extension of real-valued functions. More precisely, they can be characterized by the fact that every real-valued uniformly continuous function can be uniformly approximated by Lipschitz functions, and also characterized in terms of the notion of U-embedding. Recall that a subspace X of a metric space (Y,d) is said to be *U-embedded* in Y if every real-valued uniformly continuous function defined on X admits a uniformly continuous extension to the whole space Y.

Next result, taken from [2], summarizes all the above comments.

Theorem 1.2 ([2])**.** *For a metric space (X,d) the following statements are equivalent:*

(1) *X is small-determined.*
(2) *Every real uniformly continuous function can be uniformly approximated by Lipschitz functions.*
(3) *X is U-embedded in every metric space bi-Lipschitz containing it.*
(4) *X is U-embedded in every metric space isometrically containing it.*

(5) *X is U-embedded in some normed space isometrically containing it.*
(6) *X is U-embedded in some normed space bi-Lipschitz containing it.*

Metric spaces fulfilling above condition (4), were studied by Ramer in [9] and by Levi and Rice in [8]. Their results as well as the characterizations of small-determined contained in [2] are somehow similar. And, in fact, they can be considered as results given in an external way, since they need to embed the initial metric space into a family of different spaces constructed for each $\varepsilon > 0$. Here, we will see that an internal characterization can be done (Theorem 3.1).

1.2. **B-simple spaces.** In order to introduce the second class of metric spaces we are interested in, we need to recall the notion of bornology on a set. So, a family $\mathbf{B}$ of subsets of a non-empty set X is said to be a *bornology* in X when it satisfies the following conditions:

- For every $x \in X$ the set $\{x\} \in \mathbf{B}$.
- If $B \in \mathbf{B}$ and $A \subset B$ then $A \in \mathbf{B}$.
- If $A, B \in \mathbf{B}$ then $A \cup B \in \mathbf{B}$.

Clearly the smallest bornology is the family $\mathcal{P}_{finite}(X)$ of finite subsets of X, while the biggest one is just $\mathcal{P}(X)$ the power set of X. One of the most interesting examples of bornology is obtained when we consider a metric space (X, d) and $\mathbf{B} = \mathbf{B}_d(X)$ is the family of all d-bounded subsets. We say that $B \subset X$ is d-bounded when it has finite d-diameter, i.e., $\operatorname{diam}_d(B) = \sup\{d(x, y) : x, y \in B\} < \infty$.

Another interesting bornology in a metric space is the formed by the so-called Bourbaki-bounded subsets. A subset B in the metric space (X, d) is called *Bourbaki-bounded* in X if for every $\varepsilon > 0$ there exist some points $x_1, ..., x_n$ and some $m \in \mathbb{N}$ such that,

$$B \subset B_d^m(x_1, \varepsilon) \cup \cdots \cup B_d^m(x_n, \varepsilon)$$

where $B_d^m(x, \varepsilon)$ denotes the set of points $y \in X$ that can be joined to x by means of an ε-chain in X of length m. Recall that an *ε-chain* in X of length m from x to y, is any collection of points $u_0, \ldots, u_m \in X$ such that, $u_0 = x$, $u_m = y$ and $d(u_i, u_{i-1}) < \varepsilon$, for $i = 1, \ldots, m$.

Note that if in the above definition, we have always $m = 1$ for every $\varepsilon > 0$ then we get the notion of totally boundedness, since $B_d^1(x, \varepsilon)$ is just $B_d(x, \varepsilon)$ the open ball centered in x and radius ε. Bourbaki-bounded subsets in metric spaces were firstly considered by Atsuji under the name of *finitely-chainable* subsets ([1]). In the general context of uniform spaces, they were introduced by Hejcman in [5] where they are called *uniformly bounded* subsets.

It is easy to check that Bourbaki-boundedness is preserved by uniformly continuous functions, and therefore uniformly equivalent metrics define the same Bourbaki-bounded subsets. Moreover, if we denote by $\mathbf{BB}_d(X)$ the family of Bourbaki-bounded subsets in (X, d), then it is clear that it is a bornology in X such that $\mathbf{BB}_d(X) \subset \mathbf{B}_d(X)$. While the reverse inclusion is not true in general. For instance if we consider in $\mathbb{R}$ the $0-1$ discrete metric d, then every subset is d-bounded but only the finite ones are Bourbaki-bounded.

One of the most interesting characterization of Bourbaki-boundedness was given by Hejcman in [5]. Namely, Bourbaki-bounded subsets in the metric space (X,d) are those subsets being ρ-bounded for all the metrics ρ uniformly equivalent to d. That is,

$$\mathbf{BB}_d(X) = \bigcap \left\{ \mathbf{B}_\rho(X) : \rho \overset{u}{\simeq} d \right\}.$$

So, our second class of metrics spaces defined by Hejcman in [6] can be now introduced.

Definition 1.3 ([6])**.** A metric space (X,d) is said to be B-simple when there exists some metric ρ, uniformly equivalent to d, such that $\mathbf{BB}_d(X) = \mathbf{B}_\rho(X)$.

That is, B-simple metric spaces are those for which Bourbaki-bounded subsets can be determined ("recognized" as Hejcman says in [6]) by just only one uniformly equivalent metric. Note that, for these spaces their Bourbaki-bounded subsets are precisely the bounded subsets for a uniformly equivalent metric, in other words the Bourbaki-bounded bornology is *uniformly metrizable*. We refer to [3] where the general problem of uniformly metrizability of bornologies is studied.

Examples of B-simple metric spaces are again all normed spaces, all length spaces, and also every countable uniformly discrete spaces as $\mathbb{N}$. Nevertheless, any uncountable uniformly discrete space is not B-simple. Indeed, note that for these spaces the Bourbaki-bounded subsets are only the finite ones, and then the whole space can not be a countable union of Bourbaki-bounded subsets. Since any metric space is clearly a countable union of bounded sets, this means that Bourbaki-boundedness and boundedness are different for these spaces.

In order to give more information about B-simple spaces we present in the next result two characterizations taken from [6] and [3], respectively. Recall that for a subset A of the metric space (X,d) and $\delta > 0$, we define its δ-*enlargement* as $A^\delta_d = \bigcup_{x\in A} B_d(x,\delta)$. We say that a bornology $\mathbf{B}$ in (X,d) is *stable under uniform enlargement* if there exists some $\delta > 0$ such that $A^\delta_d \in \mathbf{B}$, for every $A \in \mathbf{B}$.

Theorem 1.4. *For a metric space (X,d) the following statements are equivalent:*

(1) *X is B-simple.*
(2) *X is a countable union of Bourbaki-bounded sets, and $\mathbf{BB}_d(X)$ is stable under uniform enlargement.*
(3) *X is a countable union of Bourbaki-bounded sets, and for some $\delta > 0$, $B^m_d(x,\delta) \in \mathbf{BB}_d(X)$, for all $x \in X$ and $m \in \mathbb{N}$.*

In relation with B-simple spaces, we will see that they can be characterized with only a countable family of uniformly equivalent metrics (Theorem 3.5). In particular, by means of these metrics we will give an useful characterization of Bourbaki-boundedness (Proposition 3.3) that will allow us to get complete answers (Theorem 3.4) to some open questions posed by Hejcman in [6]. We will finish this paper exhibiting an intermediate class of spaces that appear in the study of realcompactifications of metric spaces (see [4]). We are going to see that these spaces fit in a natural way in this context since they can be also characterized in terms of the above mentioned metrics (Theorem 4.3).

2. A common Framework

Let (X, d) be a metric space and $\varepsilon > 0$. As we have said before an ε-chain in X joining the points x and y is any collection of points $u_0, \ldots, u_m \in X$ such that, $u_0 = x$, $u_m = y$ and $d(u_i, u_{i-1}) < \varepsilon$, for $i = 1, \ldots, m$. So, we can define an equivalence relation on X as follows: $x \overset{\varepsilon}{\simeq} y$ if and only if there exists an ε-chain in X joining x and y. Every equivalence class is called an *ε-chainable component*. Note that different ε-chainable components are $\varepsilon - d$-apart, and hence they forms a uniform clopen partition of X.

Next result is the first one showing that our metric spaces have some common property. Namely, they have at most countable many ε-chainable components, for every $\varepsilon > 0$.

Proposition 2.1. *Let (X, d) a metric space. Then,*

(i) *If X is small-determined then it has finitely many ε-chainable components, for every $\varepsilon > 0$.*
(ii) *If X is B-simple then it has countably many ε-chainable components, for every $\varepsilon > 0$.*

Proof. (i) Suppose that for some $\varepsilon > 0$, X has not finitely many ε-chainable components, then we can write $X = \bigcup_{k=1}^{\infty} A_k$ as an infinite union of subsets that are $\varepsilon - d$-apart. Now, if we choose $x_k \in A_k$, and define $f(x) = k \cdot d(x_1, x_k)$ for $x \in A_k$, then f is Lipschitz in the small but not Lipschitz.

(ii) Now if X is B-simple, let ρ be a metric uniformly equivalent to d such that $\mathbf{B}_\rho(X) = \mathbf{BB}_d(X)$. Then X can be written as a countable union of Bourbaki-bounded subsets, namely

$$X = \bigcup_{k \in \mathbb{N}} B_\rho(x_0, k)$$

where $B_\rho(x_0, k)$ denotes the open ρ-ball centered in $x_0 \in X$ and radius $k \in \mathbb{N}$. Therefore, for every $\varepsilon > 0$, $B_\rho(x_0, k)$ meets at most finitely many ε-chainable components of (X, d), then it is clear that X has at most countably many of these ε-chainable components. □

Remark 2.2. Note that the condition of the metric space (X, d) to have finitely or countably many ε-chainable components, for every $\varepsilon > 0$, is equivalent to say that every uniform partition of X is finite or countable, respectively. Along the paper we will indistinctly use both equivalent conditions.

2.1. **A countable family of uniformly equivalent metrics.** Now, for a metric space (X, d) having at most countable many ε-chainable components, for every $\varepsilon > 0$, we are going to construct a countable family $\{\rho_n\}_n$ of metrics uniformly equivalent to d. Indeed, let (X, d) a metric space with this property and let $n \in \mathbb{N}$. Take $C_1, C_2, \ldots, C_i, \ldots$, the $1/n$-components in X, and choose representative points $x_1 \in C_1, x_2 \in C_2, \ldots, x_i \in C_i, \ldots$. Next, on each $1/n$-component consider the metric defined by

$$d_n(x, y) = \inf \sum_{k=1}^{m} d(u_{k-1}, u_k)$$

where the infimum is taken over all the $1/n$-chains, $x = u_0, u_1, ..., u_m = y$, joining x and y. Note that we only need to consider those chains such that, if $m \geq 2$ then $d(u_{k-1}, u_k) + d(u_k, u_{k+1}) \geq 1/n$ since otherwise, due to the triangle inequality $d(u_{k-1}, u_{k+1}) \leq d(u_{k-1}, u_k) + d(u_k, u_{k+1}) < 1/n$, the point u_k can be removed from the initial chain. We will call these chains *irreducible chains.*

It is easy to check that d_n is in fact a metric on each $1/n$-chainable component in a separately way, but we can extend it to the whole space X. For that, we define ρ_n as follows:

Definition 2.3. According to the above notation, we define for every $n \in \mathbb{N}$,

$$\rho_n(x, y) = \begin{cases} d_n(x, y) & \text{if} \quad x, y \in C_i \\ & \\ d_n(x, x_i) + d(x_i, x_1) + i + d_n(y, x_j) + d(x_j, x_1) + j & \text{if} \quad \begin{array}{l} x \in C_i, \\ y \in C_j, \\ i \neq j. \end{array} \end{cases}$$

Proposition 2.4. *Let (X, d) be a metric space for which every uniform partition is countable. Then for every $n \in \mathbb{N}$, the function $\rho_n : X \times X \to \mathbb{R}$ above defined is a metric on X.*

Proof. It is clear that only it is necessary to check the triangle inequality for ρ_n and for points x, y, z which not all of them are in the same $1/n$-chainable component. Thus we have the next three cases:

(a) If $x, y \in C_i$ and $z \in C_j$, with $i \neq j$, then

$$\rho_n(x, y) = d_n(x, y) \leq d_n(x, x_i) + d_n(x_i, y) \leq \rho_n(x, z) + \rho_n(z, y).$$

(b) If $x, z \in C_i$ and $y \in C_j$, with $i \neq j$, we have

$$\begin{aligned} \rho_n(x, y) = d_n(x, x_i) + d(x_i, x_1) + i + d_n(y, x_j) + d(x_j, x_1) + j \leq \\ \leq d_n(x, z) + d_n(z, x_i) + d(x_i, x_1) + i + d_n(y, x_j) + d(x_j, x_1) + j = \\ = \rho_n(x, z) + \rho_n(z, y). \end{aligned}$$

(c) If $x \in C_i$, $y \in C_j$ and $z \in C_k$, with $i \neq j \neq k \neq i$, then

$$\rho_n(x, y) = d_n(x, x_i) + d(x_i, x_1) + i + d_n(y, x_j) + d(x_j, x_1) + j \leq \rho_n(x, z) + \rho_n(z, y).$$

□

Note that, from the triangular inequality of the metric d, we deduce that $d(x,y) \leq \rho_n(x,y)$, for all $x,y \in X$. Thus, the identity map $id : (X,\rho_n) \to (X,d)$ is in fact Lipschitz. On the other hand, $\rho_n(x,y) = d(x,y)$ whenever $d(x,y) < 1/n$, i.e., both metrics are what is called *uniformly locally identical*, for every $n \in \mathbb{N}$ (notion defined by Janos and Williamson in [7]). Therefore the identity map $id : (X,d) \to (X,\rho_n)$ is now Lipschitz in the small. That is, d and ρ_n are not only uniformly equivalent metrics but they are Lipschitz in the small equivalent.

It must be pointed here that if we change the representative points in each $1/n$-chainable component, and we define the corresponding metric $\omega_n : X \times X \to \mathbb{R}$ similarly to ρ_n but with the new representative points, then we still have that $d(x,y) = \omega_n(x,y)$ whenever $d(x,y) < 1/n$. So that, the three metrics ρ_n, ω_n and d are uniformly locally identical. Moreover, the election of these points will be irrelevant as we can see along the paper.

In the next two results we are going to give a characterization of bounded sets for the metric ρ_n, $n \in \mathbb{N}$, as well as the relationships between all of the bornologies of ρ_n-bounded sets.

Proposition 2.5. *Let (X,d) be a metric space for which every uniform partition is countable and let $n \in \mathbb{N}$. Then $B \subset X$ is ρ_n-bounded if and only if for some $x_{i_1},\dots,x_{i_k}$ in X and $M \in \mathbb{N}$, we have*

$$B \subset \bigcup_{j=1}^{k} B_d^M(x_{i_j},1/n).$$

Proof. Suppose that $B \subset X$ is ρ_n-bounded, then for some $x_0 \in X$ and $R > 0$, we have that $B \subset B_{\rho_n}(x_0,R)$. Now, from the definition of the metric ρ_n, the set B only meets a finite number of $1/n$-chainable components, say $C_{i_1},\dots,C_{i_k}$. Let $x_{i_1},\dots,x_{i_k}$ be the corresponding representative points of these components, and take $K > \rho_n(x_0,x_{i_j})$, $j = 1,\dots,k$. Now, if $x \in B \cap C_{i_j}$ we have that

$$d_n(x,x_{i_j}) = \rho_n(x,x_{i_j}) \leq \rho_n(x,x_0) + \rho_n(x_0,x_{i_j}) < R + K.$$

Then there exists an irreducible $1/n$-chain in C_{i_j} joining x and x_{i_j}, $x = u_0,u_1,...,u_m = x_{i_j}$ such that $\sum_{l=1}^{m} d(u_{l-1},u_l) < R+K$. Since the chain is irreducible, if $m \geq 2$, every two consecutive sums satisfies $d(u_{l-1},u_l)+d(u_l,u_{l+1}) \geq 1/n$, and then $(1/n)(m-1)/2 \leq R+K$. In particular, the length of every irreducible chain must be less than M, where M is a natural number with $M > 2n(R+K)+1$. We finish, since we have that,

$$B = \bigcup_{j=1}^{k} \left(B \cap C_{i_j}\right) \subset \bigcup_{j=1}^{k} B_d^M(x_{i_j},1/n).$$

Conversely, we just need to see that every set of the form $\bigcup_{j=1}^{k} B_d^M(x_{i_j},1/n)$ is ρ_n-bounded. And for that it is enough to check that $B_d^M(x_{i_j},1/n)$ is ρ_n-bounded, for every $j = 1,\dots,k$. Indeed, let $x \in B_d^M(x_{i_j},1/n)$ then $\rho_n(x,x_{i_j}) =$

$d_n(x, x_{i_j}) \leq M/n$, that is $x \in B_{\rho_n}(x_{i_j}, M/n)$. Hence we finish since $B_d^M(x_{i_j}, 1/n) \subset B_{\rho_n}(x_{i_j}, M/n)$. □

Proposition 2.6. *For a metric space (X,d) having countable uniform partitions, we have the following chain of inclusions:*

$$\boldsymbol{B}_d(X) \supset \boldsymbol{B}_{\rho_1}(X) \supset \boldsymbol{B}_{\rho_2}(X) \supset \cdots \supset \boldsymbol{B}_{\rho_n}(X) \supset \boldsymbol{B}_{\rho_{n+1}}(X) \supset \cdots \supset \boldsymbol{BB}_d(X).$$

Proof. Note that the first inclusion is clear since $d \leq \rho_n$, for all $n \in \mathbb{N}$, as we have pointed. Regarding to the last inclusion, recall that the Bourbaki-bounded subsets are the same with uniformly equivalent metrics, and also it is true that every Bourbaki-bounded set for a given metric is a bounded set with this metric. And finally the remaining inclusions follow from the above Proposition 2.5. Indeed, for every $n \in \mathbb{N}$, if $B \in \mathbf{B}_{\rho_{n+1}}(X)$ then, for some $x_{i_1}, \dots, x_{i_k}$ in X and $M \in \mathbb{N}$, we have

$$B \subset \bigcup_{j=1}^{k} B_d^M\big(x_{i_j}, 1/(n+1)\big) \subset \bigcup_{j=1}^{k} B_d^M(x_{i_j}, 1/n).$$

Hence $B \in \mathbf{B}_{\rho_n}(X)$, as we wanted. □

Now we are ready to say which is the common framework where we are going to study our metric spaces. Namely, it will be the frame of "metric spaces for which every uniform partition is countable together with the associated family of metrics $\{\rho_n\}_n$".

3. Main Results

As we have seen, small-determined and B-simple metric spaces are into the above defined framework and we are going to see how the metrics $\{\rho_n\}$ are good to describe them.

Theorem 3.1. *A metric space (X,d) is small-determined if and only if, every uniform partition of X is finite and d is Lipschitz equivalent to ρ_n, for every $n \in \mathbb{N}$.*

Proof. Suppose X is small-determined. Then, from Proposition 2.1 and Remark 2.2, it is only necessary to check the Lipschitz equivalence between d and ρ_n, for every $n \in \mathbb{N}$. Indeed, as we have said above the identity map $id : (X, \rho_n) \to (X, d)$ is always Lipschitz, while the other identity map $id : (X, d) \to (X, \rho_n)$ is Lipschitz in the small. So, we finish taking into account that every Lipschitz in the small function defined on a small-determined space is also Lipschitz (see [2]).

Conversely, in order to see that (X,d) is small-determined, let $f \in LS_d(X)$. Then there exist $K \geq 0$ and $n_0 \in \mathbb{N}$ such that $|f(x) - f(y)| \leq K \cdot d(x,y)$ whenever $d(x,y) < 1/n_0$. We are going to see that $f : (X, \rho_{n_0}) \to \mathbb{R}$ is Lipschitz. Indeed, since X has finitely many $1/n_0$-chainable components, let

$\{x_{i_1}, ..., x_{i_k}\}$ be the finite set of representative points in the corresponding components $C_{i_1}, ..., C_{i_k}$. If $x, y \in C_{i_j}$ for $j = 1, ..., k$, then it is routine to prove (by using $1/n_0$-chains) that

$$|f(x) - f(y)| \leq K \cdot d_{n_0}(x, y) = K \cdot \rho_{n_0}(x, y).$$

On the other hand, if $x \in C_{i_j}$ and $y \in C_{i_l}$, with $j \neq l$, take $M = \max\{K, |f(x_{i_j}) - f(x_{i_l})|, j, l = 1, ..., k\}$ then

$$|f(x) - f(y)| \leq |f(x) - f(x_{i_j})| + |f(x_{i_j}) - f(x_{i_l})| + |f(x_{i_l}) - f(y)| \leq$$

$$\leq K \cdot d_{n_0}(x, x_{i_j}) + M + K \cdot d_{n_0}(x_{i_l}, y) \leq M \cdot \rho_{n_0}(x, y).$$

Finally since ρ_{n_0} and d are Lipschitz equivalent we follow that $f \in Lip_d(X)$. □

Last result can be considered as an internal characterization of the small-determined metric spaces. And we could derive, as a consequence of it, certain characterizations of them given in [2] as well as the corresponding results obtained in [8] and [9] in the context of extension of uniformly continuous functions. As an example we are going to see how obtaining easily the following result from [2]. Recall that a metric space is *uniformly connected* when it can not be the union of two sets at positive distance. For instance the space of rational numbers $\mathbb{Q}$ with the usual metric is clearly uniformly connected.

Corollary 3.2 ([2])**.** *Let (X, d) be a uniformly connected metric space. Then X is small-determined if and only if the metrics d and d_n are Lipschitz equivalent, for every $n \in \mathbb{N}$.*

Proof. The proof follows at once from Theorem 3.1. Indeed, if X is uniformly connected then, for every $n \in \mathbb{N}$, X has only one $1/n$-chainable component and therefore $\rho_n = d_n$. □

Next, before to see the relationship between B-simple metric spaces and the metrics $\{\rho_n\}_n$, we are going to give an useful characterization of Bourbaki-boundedness in this context.

Proposition 3.3. *Let (X, d) be a metric space for which every uniform partition is countable. For a subset $B \subset X$ the following are equivalent:*

1. *B is Bourbaki-bounded in X.*
2. *B is ρ_n-bounded, for every $n \in \mathbb{N}$.*

Proof. From Proposition 2.6, it is clear that (1) implies (2). Conversely, suppose B is bounded for all the metrics ρ_n. In order to see that B is Bourbaki-bounded let $\varepsilon > 0$ and take $n \in \mathbb{N}$ with $1/n < \varepsilon$. Now, from Proposition 2.5, there are $x_{i_1}, \ldots, x_{i_k}$ in X and $M \in \mathbb{N}$, such that

$$B \subset \bigcup_{j=1}^{k} B_d^M(x_{i_j}, 1/n) \subset \bigcup_{j=1}^{k} B_d^M(x_{i_j}, \varepsilon).$$

And then, B is Bourbaki-bounded, as we wanted. □

Note that above result tell us that only countably many uniformly equivalent metrics are needed to recognize Bourbaki-boundedness in metric spaces for which every uniform partition is countable. At this point we would like to recall that Hejcman in [6] wondered which spaces would have precisely this property. Moreover he also wonders if it would be reasonable to consider metrics in X not necessarily uniformly equivalent to the initial one in order to recognize Bourbaki-boundedness. Next result gives a complete answer to these open questions.

Theorem 3.4. *Let (X, d) be a metric space. Then following assertions are equivalent:*

(1) *Every uniform partition X is countable.*
(2) *For the family of metrics $\{\rho_n\}_{n\in\mathbb{N}}$, above defined, we have $\mathbf{BB}_d(X) = \bigcap_n \mathbf{B}_{\rho_n}(X)$.*
(3) *There is a family $\{\varrho_n\}_{n\in\mathbb{N}}$ of metrics uniformly equivalent to d such that $\mathbf{BB}_d(X) = \bigcap_n \mathbf{B}_{\varrho_n}(X)$.*
(4) *There is a family $\{\varrho_n\}_{n\in\mathbb{N}}$ of metrics in X such that $\mathbf{BB}_d(X) = \bigcap_n \mathbf{B}_{\varrho_n}(X)$.*

Proof. That (1) implies (2) is just Proposition 3.3. That (2) implies (3) and (3) implies (4) are obvious. Finally, suppose that (4) holds, but there is an uncountable uniform partition $\{C_i : i \in I\}$ of X. Since the partition is uniform, there exist $\varepsilon > 0$ such that C_i and C_j are $\varepsilon - d$-apart, for $i \neq j$. Now choose some point $x_i \in C_i$, for $i \in I$, and let $A = \{x_i : i \in I\}$. Note that none infinite subset of A can be Bourbaki-bounded. Now we are going to construct an infinite subset $B \subset A$ being ϱ_n-bounded, for every $n \in \mathbb{N}$, and that will contradict (4).

Indeed, let x_0 a point in X. Now, since $X = \bigcup_{k=1}^{\infty} B_{\varrho_1}(x_0, k)$, there is some open ϱ_1-ball B_1 containing an uncountable subset $A_1 \subset A$. Analogously, there is an uncountable subset $A_2 \subset A_1$ contained in some open ϱ_2-ball B_2. With an inductive process, we get a countable family of uncountable sets $A_1 \supset A_2 \supset \cdots \supset A_n \supset \ldots$, such that every $A_n \subset B_n$ for some open ϱ_n-ball B_n.

Finally, let $B = \{x_1, x_2, \ldots, x_n, \ldots\}$ be an infinite subset taking different points $x_n \in A_n$. Then B is an infinite subset of A that is ϱ_1-bounded since it is contained in B_1, B is ϱ_2-bounded since $B \setminus \{x_1\}$ is contained in A_2 and also in B_2, and so on. That is, for every $n \in \mathbb{N}$, we have that B is ϱ_n-bounded since $B \setminus \{x_1, \ldots, x_{n-1}\}$ is contained in B_n. □

Now turning to B-simple spaces we are going to characterize them by means of the metrics $\{\rho_n\}_n$.

Theorem 3.5. *A metric space (X, d) is B-simple if and only if every uniform partition is countable, and there is $n_0 \in \mathbb{N}$ such that $\mathbf{BB}_d(X) = \mathbf{B}_{\rho_n}(X)$, for $n \geq n_0$.*

Proof. One implication is clear from the definition of B-simple metric space. Conversely, if X is B-simple then every uniform partition of X is countable (Proposition 2.1 and Remark 2.2).

Now let ρ a metric uniformly equivalent to d such that $\mathbf{BB}_d(X) = \mathbf{B}_\rho(X)$. Now, from the uniform equivalence between d and ρ, there exists $n_0 \in \mathbb{N}$ such that $d(x,y) < 1/n_0$ implies $\rho(x,y) < 1$. That is, $B_d(x, 1/n) \subset B_\rho(x, 1)$, for every $x \in X$ and $n \geq n_0$. And this implies that $B_d^m(x, 1/n) \subset B_\rho(x, m)$, for every $x \in X, n \geq n_0$ and $m \in \mathbb{N}$.

We are going to see that for every $n \geq n_0$ we have that $\mathbf{BB}_d(X) = \mathbf{B}_{\rho_n}(X)$. Indeed, one inclusion is clear since $\mathbf{BB}_d(X) = \mathbf{BB}_{\rho_n}(X) \subset \mathbf{B}_{\rho_n}(X)$. On the other hand, let $B \in \mathbf{B}_{\rho_n}(X)$. Now, from Proposition 2.5, there are $x_{i_1}, \ldots, x_{i_k}$ in X and $M \in \mathbb{N}$, such that

$$B \subset \bigcup_{j=1}^{k} B_d^M(x_{i_j}, 1/n) \subset \bigcup_{j=1}^{k} B_\rho(x_{i_j}, M).$$

Then $B \in \mathbf{B}_\rho(X) = \mathbf{BB}_d(X)$ as we wanted. □

A natural question here is if it is possible to change the above condition "for every $n \geq n_0$" by the condition "for every $n \in \mathbb{N}$". Next example gives a negative answer to this question.

Example 3.6. Let $\mathbb{N}$ be the set of natural numbers, let d the $0-1$ discrete metric, and $n_0 > 1$. Consider the metric space $(\mathbb{N}, (1/n_0) \cdot d)$, then it is clear that it is a countable uniformly discrete space and therefore B-simple. On the other hand, it is easy to check (see Proposition 2.5) that for the associated metrics $\{\rho_n\}_n$ we have that

$$\mathbf{B}_{\rho_n}(\mathbb{N}) = \begin{cases} \mathcal{P}(\mathbb{N}) & \text{for } n = 1, 2, \ldots, (n_0 - 1) \\ \mathcal{P}_{finite}(\mathbb{N}) & \text{for } n \geq n_0. \end{cases}$$

And then $\mathbf{BB}_{(1/n_0)\cdot d}(\mathbb{N}) = \mathbf{B}_{\rho_n}(\mathbb{N})$ if and only if $n \geq n_0$.

We finish this section with an example, taken precisely from [6], giving a positive answer to another question posed there by Hejcman himself about the existence of non B-simple spaces with countably many uniformly equivalent metrics determining their Bourbaki-bounded subsets.

Example 3.7. Let $X = \mathbb{R}^{\mathbb{N}}$ the product metric space of countably many copies of the real line with the usual metric. We endowed X with the product metric d_∞. Since X is connected then any uniform partition has only one element, and then the Bourbaki-bounded subsets can be determined by a countable family of uniform metrics (Theorem 3.4). On the other hand, (X, d_∞) is not B-simple since it can not be a countable union of Bourbaki-bounded sets. Otherwise, suppose $X = \bigcup B_n$, where $B_n \in \mathbf{BB}_{d_\infty}(X)$. Since, for every n, the projection map $\pi_n : X \to \mathbb{R}$ is uniformly continuous, then $\pi_n(B_n)$ is a Bourbaki-bounded subset of $\mathbb{R}$, and then it must be bounded with the usual metric. Now take $x_n \in \mathbb{R} \setminus \pi_n(B_n)$ and let $x = (x_n)_n$. Then clearly $x \in X \setminus \bigcup B_n$, which is a contradiction.

4. An intermediate class of metric spaces

In this brief section we are going to present a different class of metric spaces that is also in our general framework. First of all, we are going to see the relationship between small-determined and B-simple spaces.

Proposition 4.1. *Every small-determined metric space is B-simple.*

Proof. The proof can be obtained easily using properly Theorem 3.1 and Proposition 3.3. Indeed, if (X, d) is small-determined then d and ρ_n are Lipschitz equivalent and then both metrics have the same bounded sets, for every $n \in \mathbb{N}$. Since $\mathbf{B}_d(X) = \mathbf{B}_{\rho_n}(X)$, for every n, it follows that

$$\mathbf{BB}_d(X) = \bigcap \mathbf{B}_{\rho_n}(X) = \mathbf{B}_d(X)$$

and then X is clearly B-simple. □

Note that in the above proof we have seen indirectly that if (X, d) is small-determined then it is not only B-simple but it satisfies an strong property, namely $\mathbf{BB}_d(X) = \mathbf{B}_d(X)$. In fact, for a metric space (X, d), we have the following chain of implications:

$$X\, is\ small-determined \Rightarrow \left\{ \begin{array}{c} X \text{ has the property} \\ \mathbf{BB}_d(X) = \mathbf{B}_d(X) \end{array} \right\} \Rightarrow X\, is\ B-simple.$$

To see that the above reverse implications are not true the next easy examples can be considered.

Examples 4.2. Let $\mathbb{N}$ be the set of natural numbers with the usual metric. Then it satisfies $\mathbf{BB}_d(\mathbb{N}) = \mathbf{B}_d(\mathbb{N})$ but it is not small-determined. On the other hand, the real line $\mathbb{R}$ with the metric $\hat{d} = \min\{1, d\}$ where d denotes the usual metric, is B-simple but $\mathbf{BB}_{\hat{d}}(\mathbb{R}) \neq \mathbf{B}_{\hat{d}}(\mathbb{R})$.

Last examples show that the three classes of metric spaces are different. We wonder if this intermediate property corresponds to some already known class of metric spaces. We are very interested in these spaces since, as we have seen in [4], they are precisely those spaces for which the so-called Samuel realcompactification and Lipschitz realcompactification are equivalent, and also the spaces where every real-valued uniformly continuous function is bounded on bounded sets.

We finish this paper giving a new characterization of this intermediate property by means of the metrics ρ_n. Recall that two metrics are called *boundedly equivalent* if they are the same bounded subsets.

Theorem 4.3. *For the metric space (X, d) the following assertions are equivalent.*

(1) *X has the property $\mathbf{BB}_d(X) = \mathbf{B}_d(X)$.*
(2) *Every uniform partition X is countable and d and ρ_n are boundedly equivalent, for every $n \in \mathbb{N}$.*

Proof. Obviously, condition (1) implies that X is B-simple, and therefore every uniform partition of X is countable. Moreover, from Proposition 2.6, if the first and last link of the chain coincide, then all the links are equal. That means that $\mathbf{B}_d(X) = \mathbf{B}_{\rho_n}(X)$, for every $n \in \mathbb{N}$.

That (2) implies (1) is an easy consequence of Proposition 3.3. Indeed, if d and ρ_n are boundedly equivalent, for every $n \in \mathbb{N}$, then $\mathbf{B}_d(X) = \mathbf{B}_{\rho_n}(X)$ for every $n \in \mathbb{N}$. Finally, we have $\mathbf{BB}_d(X) = \bigcap_n \mathbf{B}_{\rho_n}(X) = \mathbf{B}_d(X)$, as we wanted. □

5. Conclusions and future research

In this paper we have found a general uniform property that, in the frame of metric spaces, allows us to collect together different classes of spaces that were apparently far away. We refer to this mild property as to "having countably many elements in every uniform partition". So, spaces coming from the bornological setting, as the *B-simple* spaces, as well as from the general study of Lipschitz functions, as the *Small-determined* spaces, lie in this common context. As we have seen here, the key to obtain the main results is the fact that we can define countably many metrics uniformly equivalent to the initial one. These metrics give a way to describe bornological properties (bounded sets, Bourbaki-bounded sets) as well as some Lipschitz functions (Lipschitz in the small functions).

Moreover, along the present study a new kind of metric spaces appears in a natural way. Namely those spaces for which the Bourbaki-bounded subsets are exactly the bounded subsets. We do not know if these intermediate spaces correspond to some already studied class, and in fact we propose as future research to find more properties of them. We are very interested in these spaces since we already know that they can be characterized as those for which the Samuel realcompactification and the Lipschitz realcompactification are the same, as we have seen in our recent work [4], devoted precisely to the study of new realcompactifications for metric spaces.

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