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Lebesgue quasi-uniformity on textures

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Abstract

This paper considers the Lebesgue property on quasi di-uniform textures. It is well known that the quasi-uniform space with a compact topology has the Lebesgue property. This result is extended to direlational quasi-uniformities and dual dicovering quasi-uniformities. Additionally we discuss the completeness of Lebesgue di-uniformities.

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1. Introduction

A *texturing* on a set S is a point-separating, complete, completely distributive lattice $\mathcal{S}$ of subsets of S with respect to inclusion, which contains S and $\varnothing$, and for which arbitrary meet $\bigwedge$ coincides with intersection $\bigcap$ and finite joins $\bigvee$ with unions $\bigcup$. The pair $(S, \mathcal{S})$ is called a *texture*.

This definition was first introduced by L. M. Brown to represent Hutton algebras and lattices of $\mathbb{L}$ fuzzy sets in a point based setting [4]. However the development of the theory has proceeded largely independently and the work on di-uniformities has shown that it has much closer links with topological ideas than might be expected. Di-uniformity on a texture was first defined in [13] by giving descriptions in terms of direlations, dicovers and dimetrics and the concepts of completeness and total boundedness were introduced in [14]. The effect of a complementation and the relation with quasi-uniformity and uniformity were discussed in [15]. In this context the work [15] pointed out that di-uniformities provide a more unified setting for the study of quasi-uniformity and uniformity than does the classical approach.

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As is well known a quasi-uniformity is obtained by omitting the symmetry condition in the definition of a uniformity. We recall the notion of direlational uniform texture space as follows.

Definition 1.1 ([13])**.** Let $(S, \mathcal{S})$ be a texture and $\mathcal{U}$ a family of direlations on $(S, \mathcal{S})$. If $\mathcal{U}$ satisfies the conditions,

(1) $(i, I) \sqsubseteq (d, D)$ for all $(d, D) \in \mathcal{U}$. That is, $\mathcal{U} \subseteq \mathcal{RDR}$.
(2) $(d, D) \in \mathcal{U}$, $(e, E) \in \mathcal{DR}$ and $(d, D) \sqsubseteq (e, E)$ implies $(e, E) \in \mathcal{U}$.
(3) $(d, D), (e, E) \in \mathcal{U}$ implies $(d, D) \sqcap (e, E) \in \mathcal{U}$.
(4) Given $(d, D) \in \mathcal{U}$ there exists $(e, E) \in \mathcal{U}$ satisfying $(e, E) \circ (e, E) \sqsubseteq (d, D)$.
(5) Given $(d, D) \in \mathcal{U}$ there exists $(c, C) \in \mathcal{U}$ satisfying $(c, C)^{\leftarrow} \sqsubseteq (d, D)$.

then $\mathcal{U}$ is called a *direlational uniformity* on $(S, \mathcal{S})$, and $(S, \mathcal{S}, \mathcal{U})$ is known as a *direlational uniform texture*.

This definition is formally same as the usual definition of diagonal uniformity. It should be noted, that the symmetry condition (5) which guarantees a base of symmetric direlations for the direlational uniformity is quite different from the notion of symmetry for relations. In [15] an important result was obtained that a direlational uniformity on the discrete texture $(X, \mathcal{P}(X))$ corresponds not to uniformity but to quasi uniformity. When the symmetry condition (5) is removed we obtain a direlational quasi-uniform texture space $(S, \mathcal{S}, \mathcal{U}^q)$ [17].

Another representation for di-uniformities is in terms of dicovers. We recall from [2] that by a difamily we mean a set $\mathcal{C} = \{(A_j, B_j) \mid j \in J\}$ of elements of $\mathcal{S} \times \mathcal{S}$ and $\mathcal{C}$ is called a *dicover* of $(S, \mathcal{S})$ if $\bigcap_{j \in J_1} B_j \subseteq \bigvee_{j \in J_2} A_j$ for all partitions (J_1, J_2) of J. A dicover corresponds to a dual cover in the sense of [1] and this notion is related to the notion of pairs of covers with a common index used by Gantner and Steinlage [8] to characterize quasi uniformities. As in the classical case dicovers generate symmetric direlations and are not appropriate to characterize quasi di-uniformities. Hence in [17] the authors used a new notion called dual dicover to introduce dual dicovering quasi-uniformity. Below we recall these definitions.

Dual dicover([17]) A dual difamily $\mathcal{C}_d = \big\{\big((C_j^{1,1}, C_j^{1,2}),\ (C_j^{2,1}, C_j^{2,2})\big) \mid j \in J\big\}$ of elements of $(\mathcal{S} \times \mathcal{S}) \times (\mathcal{S} \times \mathcal{S})$ is called a *dual dicover of* $(S, \mathcal{S})$ if $\big\{\big(C_j^{1,1} \cap C_j^{2,1},\ C_j^{1,2} \cup C_j^{2,2}\big) \mid j \in J\big\}$ is a dicover of $(S, \mathcal{S})$.

Definition 1.2 ([17])**.** Let $(S, \mathcal{S})$ be a texture. If υ^q is a family of dual dicovers satisfying the conditions

(1) Given $\mathcal{C}_d \in \upsilon^q$ there exists an anchored dual dicover $\mathcal{D}_d \in \upsilon^q$ with $\mathcal{D}_d \prec \mathcal{C}_d$,
(2) $\mathcal{C}_d \in \upsilon^q$, $\mathcal{C}_d \prec \mathcal{D}_d$ implies $\mathcal{D}_d \in \upsilon^q$,
(3) $\mathcal{C}_d, \mathcal{D}_d \in \upsilon^q$ implies $\mathcal{C}_d \wedge \mathcal{D}_d \in \upsilon^q$,
(4) Given $\mathcal{C}_d \in \upsilon^q$ there exists $\mathcal{D}_d \in \upsilon^q$ with $\mathcal{D}_d \prec_{(\star)} \mathcal{C}_d$.

we say υ^q is a *dual dicovering quasi-uniformity* on $(S, \mathcal{S})$.

In [17] besides these definitions there is another approach by using quasi-pseudometrics. Since this work will be based on the direlational and dual dicovering representations we will omit it.

This paper is a continuation of the work [16] where Lebesgue and co-Lebesgue di-uniformities were first introduced and the relationship between Lebesgue quasi-uniformity on X and the corresponding Lebesgue di-uniformity on discrete texture $(X, \mathcal{P}(X))$ was investigated. Moreover, the notions of Lebesgue quasi di-uniformity and dual dicovering Lebesgue quasi-uniformity were introduced and discussed some of their properties. In this work our source of inspiration is [11] where the notion of pair Lebesgue quasi-uniformity was first introduced by J. Marin and S. Romaguera and we confine our attention to dual dicovering bi-Lebesgue quasi di-uniformities. The aim of this work is to continue to develop the notion of Lebesgue property on quasi di-uniform textures and investigate dicompleteness of Lebesgue di-uniform textures.

After a brief introduction, in section 2 we introduce the notion of a bi-Lebesgue quasi di-uniformity and show that on plain textures each quasi di-uniformity with a dicompact topology is a bi-Lebesgue quasi di-uniformity. We obtained the analogous result for the dual dicovering bi-Lebesgue quasi di-uniform spaces which will be defined in Definition 2.11. In section 3 we first consider the dual covering Lebesgue quasi-uniformy in the sense of Brown [1] and discuss the completeness of Lebesgue di-uniformities on the discrete texture.

General references on ditopological texture spaces include [1, 2, 3, 4, 5, 6] and constant reference will be made to [13, 14, 15, 16, 17] for definitions and results relating to di-uniformities. Our standart references for quasi uniformity are [7, 8, 9]. For the conveince of the reader we recall some more special definitions.

Let $(S, \mathcal{S})$ be a texture. For $s \in S$ the sets

$$P_s = \bigcap \{A \in \mathcal{S} \mid s \in A\} \text{ and } Q_s = \bigvee \{A \in \mathcal{S} \mid s \notin A\}$$

are called respectively, the p-sets and q-sets of $(S, \mathcal{S})$. For $A \in \mathcal{S}$ the *core* $A^\flat$ of A is given by $A^\flat = \{s \in S \mid A \not\subseteq Q_s\}$. The set $A^\flat$ does not necessarily belong to $\mathcal{S}$.

In general, a texturing of S need not be closed under set complementation, but sometimes we have a notion of complementation.

Complementation: [2] A mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$, $\forall\, A \in \mathcal{S}$ and $A \subseteq B \implies \sigma(B) \subseteq \sigma(A)$, $\forall\, A, B \in \mathcal{S}$ is called a *complementation* on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is then said to be a *complemented texture*.

Examples :

1. For any set X, $(X, \mathcal{P}(X), \pi_X)$, $\pi_X(Y) = X \setminus Y$ for $Y \subseteq X$, is the complemented *discrete texture* representing the usual set structure of X. Clearly, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$ for all $x \in X$.

2. For $\mathbb{I} = [0,1]$ define $\mathcal{J} = \{[0,t] \mid t \in [0,1]\} \cup \{[0,t) \mid t \in [0,1]\}$, $\iota([0,t]) = [0,1-t)$ and $\iota([0,t)) = [0,1-t]$, $t \in [0,1]$. Then $(\mathbb{I}, \mathcal{J}, \iota)$ is a complemented texture, which we will refer to as the *unit interval texture*. Here $P_t = [0,t]$ and $Q_t = [0,t)$ for all $t \in I$.

Ditopology: A *dichotomous topology* on $(S, \mathcal{S})$ or *ditopology* for short, is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

1. $S, \varnothing \in \tau$,
2. $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and
3. $G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies

1. $S, \varnothing \in \kappa$,
2. $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$ and
3. $K_i \in \kappa, i \in I \implies \bigcap K_i \in \kappa$.

For $A \in \mathcal{S}$ the sets $[A] = \bigcap\{K \in \kappa \mid A \subseteq K\}$ and $]A[= \bigvee\{G \in \tau \mid G \subseteq A\}$ are called the *closure* and *interior* of A.

A plain texture is one for which the texturing is closed under arbitrary unions or equivalently join coincides with union in $\mathcal{S}$. There is a considerable simplification in plain textures. We have $P_s \not\subseteq Q_s$ for each $s \in S$. Hence, for $A \in \mathcal{S}$, $s \in A$, $P_s \subseteq A$ and $A \not\subseteq Q_s$ are equivalent to each other.

One of the most useful notions in the theory of di-uniformities is that of direlation.

Direlations: [5] Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures. We use $\overline{P}_{(s,t)}$, $\overline{Q}_{(s,t)}$ to denote the p-sets and q-sets for the product texture $(S \times T, \mathcal{P}(S) \otimes \mathcal{T})$. Then:

(1) $r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *relation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies

R1 $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \implies r \not\subseteq \overline{Q}_{(s',t)}$.

R2 $r \not\subseteq \overline{Q}_{(s,t)} \implies \exists\, s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.

(2) $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *corelation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies

CR1 $\overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R$.

CR2 $\overline{P}_{(s,t)} \not\subseteq R \implies \exists\, s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.

A pair (r, R) consisting of a relation r and corelation R is called a *direlation*.

Now let (r, R) be a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. The inverses of r and R are given by

$$r^{\leftarrow} = \bigcap\{\overline{Q}_{(t,s)} \mid r \not\subseteq \overline{Q}_{(s,t)}\}, \quad R^{\leftarrow} = \bigvee\{\overline{P}_{(t,s)} \mid \overline{P}_{(s,t)} \not\subseteq R\}.$$

where $R^{\leftarrow}$ is a relation and $r^{\leftarrow}$ a corelation. The direlation $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ from $(T, \mathcal{T})$ to $(S, \mathcal{S})$ is called the *inverse* of (r, R).

For $A \subseteq S$ the *A–section of a relation* r and *A–section of a corelation* R is defined by

$$r^{\rightarrow}A = \bigcap\{Q_t \mid \forall\, s,\ r \not\subseteq \overline{Q}_{(s,t)} \implies A \subseteq Q_s\},$$
$$R^{\rightarrow}A = \bigvee\{P_t \mid \forall\, s, \overline{P}_{(s,t)} \not\subseteq R \implies P_s \subseteq A\}$$

Compositions of direlations: Let $(S, \mathcal{S})$, $(T, \mathcal{T})$, $(U, \mathcal{U})$ be textures.

(1) If p is a relation on $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and q a relation on $(T, \mathcal{T})$ to $(U, \mathcal{U})$ then their *composition* is the relation $q \circ p$ on $(S, \mathcal{S})$ to $(U, \mathcal{U})$ defined by
$$q \circ p = \bigvee \{\overline{P}_{(s,u)} \mid \exists\, t \in T \text{ with } p \not\subseteq \overline{Q}_{(s,t)} \text{ and } q \not\subseteq \overline{Q}_{(t,u)}\}.$$

(2) If P is a co-relation on $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and Q a co-relation on $(T, \mathcal{T})$ to $(U, \mathcal{U})$ then their *composition* is the co-relation $Q \circ P$ on $(S, \mathcal{S})$ to $(U, \mathcal{U})$ defined by
$$Q \circ P = \bigcap \{\overline{Q}_{(s,u)} \mid \exists\, t \in T \text{ with } \overline{P}_{(s,t)} \not\subseteq P \text{ and } \overline{P}_{(t,u)} \not\subseteq Q\}.$$

(3) With p, q; P, Q as above, the *composition* of the direlations (p, P), (q, Q) is the direlation
$$(q, Q) \circ (p, P) = (q \circ p, Q \circ P).$$

2. Lebesgue quasi di-uniform spaces

In this section we consider the Lebesgue property on quasi di-uniform textures. We introduce Lebesgue and co-Lebesgue direlational quasi-uniformities. We also define bi-Lebesgue quasi di-uniformity, dual dicovering bi-Lebesgue quasi di-uniformity and give an analog of the well known result that each quasi-uniformity compatible with a compact space is a Lebesgue quasi-uniformity.

The definition of a direlational uniformity $\mathcal{U}$ on a texture $(S, \mathcal{S})$ has been introduced in Definition 1.1. We obtain a direlational quasi-uniformity on $(S, \mathcal{S})$ by removing the symmetry condition from the definition of the direlational uniformity. Now we begin by recalling the following definition.

Definition 2.1 ([17, Definition 2.1])**.** Let $(S, \mathcal{S})$ be a texture and $\mathcal{U}^q$ a family of direlations on $(S, \mathcal{S})$. If $\mathcal{U}^q$ satisfies the conditions

(1) $(i, I) \sqsubseteq (d, D)$ for all $(d, D) \in \mathcal{U}^q$,
(2) $(d, D) \in \mathcal{U}^q$, $(e, E) \in \mathcal{DR}$ and $(d, D) \sqsubseteq (e, E)$ implies $(e, E) \in \mathcal{U}^q$,
(3) $(d, D), (e, E) \in \mathcal{U}^q$ implies $(d, D) \sqcap (e, E) \in \mathcal{U}^q$,
(4) Given $(d, D) \in \mathcal{U}^q$ there exists $(e, E) \in \mathcal{U}^q$ satisfying $(e, E) \circ (e, E) \sqsubseteq (d, D)$,

then $\mathcal{U}^q$ will be called a *direlational quasi-uniformity* on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \mathcal{U}^q)$ a *direlational quasi-uniform texture space.*

As in the classical case for the direlational quasi-uniformity $\mathcal{U}^q$ on $(S, \mathcal{S})$
$$(\mathcal{U}^q)^{\leftarrow} = \{(d, D)^{\leftarrow} : (d, D) \in \mathcal{U}^q\}$$
is also a direlational quasi-uniformity on $(S, \mathcal{S})$ and $(S, \mathcal{S}, (\mathcal{U}^q)^{\leftarrow})$ is called the *conjugate* of $(S, \mathcal{S}, \mathcal{U}^q)$ (see, [17]).

A direlational quasi-uniformity $\mathcal{U}^q$ on $(S, \mathcal{S})$ induces a *uniform ditopology* $(\tau_{\mathcal{U}^q}, \kappa_{\mathcal{U}^q})$ as follows, in exactly the same way that a direlational uniformity does [13, Lemma 4.3].

(i) $G \in \tau_{\mathcal{U}^q} \iff (G \not\subseteq Q_s \implies \exists\, (d, D) \in \mathcal{U}^q \text{ with } d[s] \subseteq G)$,

(ii) $K \in \kappa_{\mathcal{U}^q} \iff (P_s \not\subseteq K \implies \exists\,(d,D) \in \mathcal{U}^q \text{ with } K \subseteq D[s])$.

Here $d[s] = d^{\to}P_s$ and $D[s] = D^{\to}Q_s$.

When we speak of the ditopology of $(S, \mathcal{S}, \mathcal{U}^q)$ we will always mean the uniform ditopology.

In order to consider Lebesgue direlational quasi-uniformity it is necessary to recall [6] the open cover and closed cocover for the textures.

Let (τ, κ) be a ditopology on the texture $(S, \mathcal{S})$ and let $A \in \mathcal{S}$. The family $\{G_i \mid i \in I\}$ is said to be an *open cover* [6] of A if $G_i \in \tau$ for all $i \in I$ and $A \subseteq \bigvee_{i \in I} G_i$. Dually we may speak of a *closed cocover* of A, namely a family $\{F_i \mid i \in I\}$ with $F_i \in \kappa$ for all $i \in I$ satisfying $\bigcap_{i \in I} F_i \subseteq A$.

For the cocovers we need a notion of dual refinement.

Definition 2.2 ([16])**.** Let $\mathcal{K}_1, \mathcal{K}_2$ be cocovers. Then $\mathcal{K}_1$ will be called a *dual refinement* of $\mathcal{K}_2$, and write $\mathcal{K}_1 \lhd \mathcal{K}_2$ if for a given $K_2 \in \mathcal{K}_2$ there exists $K_1 \in \mathcal{K}_1$ such that $K_1 \subseteq K_2$.

Now we may give:

Definition 2.3. A direlational quasi-uniformity $\mathcal{U}^q$ on $(S, \mathcal{S})$ is called

(1) *Lebesgue direlational quasi-uniformity* provided that for each cover $\mathcal{C}$ of S which is open for the uniform ditopology there is a direlation $(r, R) \in \mathcal{U}^q$ such that $\{r[s] \mid s \in S^\flat\}$ is a refinement of $\mathcal{C}$.

(2) *Co-Lebesgue direlational quasi-uniformity* provided that for each cocover $\mathcal{K}$ of $\varnothing$ which is closed for the uniform ditopology there is a direlation $(r, R) \in \mathcal{U}^q$ such that $\mathcal{K}$ is a dual refinement of $\{R[s] \mid s \in S^\flat\}$.

In [16, Proposition 2.6] it is proved that each direlational uniformity compatible with a compact (cocompact) ditopological texture space is a Lebesgue (co-Lebesgue) direlational uniformity on $(S, \mathcal{S})$. We now have the analogous result for the direlational quasi-uniformities.

Let us recall the following definition.

Definition 2.4 ([6])**.** Let (τ, κ) be a ditopology on the texture $(S, \mathcal{S})$ and $A \in \mathcal{S}$.

(1) A is called *compact* if whenever $\{G_i \mid i \in I\}$ is an open cover of A then there is a finite subset J of I with $A \subseteq \bigcup_{j \in J} G_j$. In particular the ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called *compact* if S is compact.

(2) A is called *cocompact* if whenever $\{F_i \mid i \in I\}$ is a closed cocover of A then there is a finite subset J of I with $\bigcap_{j \in J} F_j \subseteq A$. In particular the ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called *cocompact* if $\varnothing$ is cocompact.

We recall from [6] that a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is stable (costable) if every $F \in \kappa \backslash \{S\}$ $(G \in \tau \backslash \{\varnothing\})$ is compact (cocompact). The ditopological texture space is called dicompact if it is compact, cocompact, stable and costable.

We may now give the promised result.

Proposition 2.5. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space.*

(1) *If $\mathcal{U}^q$ is a direlational quasi-uniformity compatible with a compact ditopology, then $\mathcal{U}^q$ is a Lebesgue direlational quasi-uniformity on $(S, \mathcal{S})$.*
(2) *If $\mathcal{U}^q$ is a direlational quasi-uniformity compatible with a cocompact ditopology, then $\mathcal{U}^q$ is a co-Lebesgue direlational quasi-uniformity on $(S, \mathcal{S})$.*

Proof. The proof follows from the same lines as in the proof of [16, Proposition 2.6]. □

The notion of Lebesgue quasi di-uniformity was introduced in [16] and the anchored property for the dicover was omitted in that definition. Nevertheless we now begin by introducing a new notion called bi-Lebegue quasi di-uniform space.

First let us recall the notion of an anchored dicover which plays an important role in the development of dicovering uniformities and bi-Lebesgue quasi di-uniformities.

Definition 2.6 ([13])**.** A family $\mathcal{C} \subseteq \mathcal{S} \times \mathcal{S}$ is called an *anchored dicover* if it satisfies:

(1) $\mathcal{P} \prec \mathcal{C}$, and
(2) Given $A \, \mathcal{C} \, B$ there exists $s \in S$ satisfying
   (a) $A \not\subseteq Q_u \implies \exists \, A' \, \mathcal{C} \, B'$ with $A' \not\subseteq Q_u$ and $P_s \not\subseteq B'$, and
   (b) $P_v \not\subseteq B \implies \exists \, A'' \, \mathcal{C} \, B''$ with $P_v \not\subseteq B''$ and $A'' \not\subseteq Q_s$.

A dicover $\mathcal{C} = \{(A_j, B_j) \mid j \in J\}$ is finite if the set $\{A_j \mid j \in J\}$ is finite and cofinite if the set $\{B_j \mid j \in J\}$ is finite. If $\mathcal{C}$ is defined on a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$, it is said to be *open coclosed* if $A_j \in \tau$ and $B_j \in \kappa$ and *closed co-open* if $A_j \in \kappa$ and $B_j \in \tau$ for all $j \in J$. $\mathcal{C}$ is a *refinement* of $\mathcal{D}$ if given $j \in J$ we have $L \, \mathcal{D} \, M$ so that $A_j \subseteq L$ and $M \subseteq B_j$. In this case we write $\mathcal{C} \prec \mathcal{D}$.

If (d, D) is a reflexive direlation on $(S, \mathcal{S})$ then $\gamma(d, D) = \{(d[s], D[s]) \mid s \in S^\flat\}$ is an anchored dicover of $(S, \mathcal{S})$.

Definition 2.7. Let $\mathcal{U}^q$ be a quasi di-uniformity on a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$. Then $\mathcal{U}^q$ is a *bi-Lebesque quasi di-uniformity* provided that for each open coclosed anchored dicover $\mathcal{C}$ of $(S, \mathcal{S})$ there is a direlation $(r, R) \in \mathcal{U}^q$ such that the dicover $\gamma(r, R) = \{(r[s], R[s]) \mid s \in S^\flat\}$ refines $\mathcal{C}$ and $(S, \mathcal{S}, \mathcal{U}^q)$ is called a *bi-Lebesgue quasi di-uniform texture space.*

The following important theorem, which is proved in [3] gives a characterization of dicompactness in textures.

Theorem 2.8. *The following are equivalent for $(S, \mathcal{S}, \tau, \kappa)$.*

(1) *$(S, \mathcal{S}, \tau, \kappa)$ is dicompact.*
(2) *Every closed co-open difamily with the finite exclusion property is bound.*
(3) *Every open coclosed dicover has a finite and cofinite subdicover.*

Now we may give the following theorem in the case of plain textures.

Theorem 2.9. *Let $(S, \mathcal{S}, \mathcal{U}^q)$ be a plain direlational quasi-uniform texture space such that $(\tau_{\mathcal{U}^q}, \kappa_{\mathcal{U}^q})$ is dicompact. Then $(S, \mathcal{S}, \mathcal{U}^q)$ is a bi-Lebesgue quasi di-uniform space.*

Proof. Let $\mathcal{C} = \{(A_j, B_j) \mid j \in J\}$ be an open coclosed anchored dicover of $(S, \mathcal{S})$. For each $s \in S^\flat$ and $j(s) \in J$ we have $A_{j(s)} \mathcal{C} B_{j(s)}$ with $A_{j(s)} \in \tau_{\mathcal{U}^q}$ and $B_{j(s)} \in \kappa_{\mathcal{U}^q}$. Since $(S, \mathcal{S})$ is plain and $\mathcal{C}$ is an anchored dicover, for $s' \in S^\flat$ we have $A_{j(s)} \not\subseteq Q_{s'}$ and there exists $(e_s, E_s) \in \mathcal{U}^q$ with $e_s[s'] \subseteq A_{j(s)}$. Now take $B_{j(s)} \in \kappa_{\mathcal{U}^q}$ and $P_{s'} \not\subseteq B_{j(s)}$ then we have $B_{j(s)} \subseteq E_s[s']$. Since $\mathcal{U}^q$ is a direlational quasi-uniformity there exists $(d_s, D_s) \in \mathcal{U}^q$ satisfying $(d_s, D_s) \circ (d_s, D_s) \sqsubseteq (e_s, E_s)$. Hence $\{(\,]\,d_s[s']\,[\,, [\,D_s[s']\,]\,) \mid s' \in S^\flat\}$ is an open coclosed anchored dicover of $(S, \mathcal{S})$ with $P_{s'} \subseteq \,]\,d_s[s']\,[$ and $[\,D_s[s']\,] \subseteq Q_{s'}$.

Since $(S, \mathcal{S})$ is dicompact, the open coclosed dicover $\{(\,]\,d_s[s']\,[\,, [\,D_s[s']\,]\,) \mid s' \in S^\flat\}$ has a finite cofinite subdicover $\{(\,]\,d_{s_k}[s'_k]\,[\,, [\,D_{s_k}[s'_k]\,]\,) \mid s' \in S^\flat\}$ for $k = 1, ..., n$ by Theorem 2.8.

If we set $(d, D) = \bigsqcap_{k=1}^n (d_{s_k}, D_{s_k})$, then $(d, D) \in \mathcal{U}^q$. Since $(S, \mathcal{S})$ is plain, we have $d_{s_k}[s'_k] \not\subseteq Q_{s'}$ and $P_{s'} \not\subseteq D_{s_k}[s'_k]$ for $s' \in S^\flat$ and $1 \leq k \leq n$. We shall show that $\gamma(d, D) = \{(d[s'], D[s']) \mid s' \in S^\flat\} \prec \mathcal{C}$. For the given $s' \in S^\flat$ there is $k \in \{1, 2, 3, ..., n\}$ such that $e_{s_k}[s'_k] \subseteq A_{j(s_k)}$ and $B_{j(s_k)} \subseteq E_{s_k}[s'_k]$. We need to prove $d[s'] \subseteq e_{s_k}[s'_k] \subseteq A_{j(s_k)}$ and $B_{j(s_k)} \subseteq E_{s_k}[s'_k] \subseteq D[s']$.

Now let us prove $B_{j(s_k)} \subseteq E_{s_k}[s'_k] \subseteq D[s']$. First suppose that $E_{s_k}[s'_k] \not\subseteq D[s']$. Then there exists $z \in S^\flat$ with $E_{s_k}[s'_k] \not\subseteq Q_z$ and $P_z \not\subseteq D[s']$. Because of $D = \bigsqcup_{k=1}^n D_{s_k}$ we have $D_{s_k} \subseteq D$ so $D_{s_k}[s'] \subseteq D[s']$ and we have $P_z \not\subseteq D_{s_k}[s']$. By the definition of composition of co-relations [13, Definition 1.7] we have $E_{s_k} \subseteq D_{s_k} \circ D_{s_k} \subseteq \overline{Q}_{(s'_k, z)}$.

From $P_{s'} \not\subseteq D_{s_k}[s'_k] = D_{s_k}^{\rightarrow}(Q_{s'_k})$ we obtain $\overline{P}_{(s'_k, s')} \not\subseteq D_{s_k}$ and due to $P_z \not\subseteq D_{s_k}[s']$ we have $\overline{P}_{(s', z)} \not\subseteq D_{s_k}$ by [13, Lemma 1.5].

On the other hand $E_{s_k}^{\rightarrow}(Q_{s'_k}) = E_{s_k}[s'_k] \not\subseteq Q_z$ gives $z' \in S$ with $P_{z'} \not\subseteq Q_z$ and for $t \in S^\flat$

$$\overline{P}_{(t, z')} \not\subseteq E_{s_k} \implies P_t \subseteq Q_{s'_k}. \tag{2.1}$$

by [13, Definition 1.3.(2)]. From $E_{s_k} \subseteq \overline{Q}_{(s'_k, z)}$ and $P_{z'} \not\subseteq Q_z$ we have $\overline{P}_{(s'_k, z')} \not\subseteq E_{s_k}$ and since E_{s_k} is a co-relation we have $s''_k \in S$ with $P_{s''_k} \not\subseteq Q_{s'_k}$ and $\overline{P}_{(s''_k, z')} \not\subseteq E_{s_k}$ by *CR2*. Now we may apply the implication (1) with $t = s''_k$ to give the contradiction $P_{s''_k} \subseteq Q_{s'_k}$.

(2) The proof of $d[s'] \subseteq e_{s_k}[s'_k] \subseteq A_{j(s_k)}$ is dual to the above and is omitted. □

We will use the term quasi di-uniformity [17] to refer to direlational quasi-uniformities and dual dicovering quasi-uniformities in general.

Let us recall from [17, Definition 3.8] that a dual difamily $\mathcal{C}_d$ is called an *anchored dual dicover* if (i) $\mathcal{P}_d = \{((P_s, Q_s), (P_s, Q_s))\} \mid s \in S^\flat\} \prec \mathcal{C}_d$ and (ii) $\mathcal{C}_d \prec \mathcal{C}_d^\Delta$. If (r, R) is a reflexive direlation on $(S, \mathcal{S})$ then the family $\gamma^q(r, R) = \{(\gamma(r, R), \gamma(r, R)^\leftarrow) \mid s \in S^\flat\}$ is an anchored dual dicover where $\gamma(r, R)^\leftarrow = \{(R^\leftarrow[s], r^\leftarrow[s]) \mid s \in S^\flat\}$. Moreover a dual dicover $\mathcal{C}_d$ satisfying $\left(C_j^{1,1}, C_j^{1,2}\right) \in (\tau_{\mathcal{U}^q}, \kappa_{\mathcal{U}^q})$ and $\left(C_j^{2,1}, C_j^{2,2}\right) \in (\tau_{(\mathcal{U}^q)^\leftarrow}, \kappa_{(\mathcal{U}^q)^\leftarrow})$ is called *open co-closed*.

We recall the definition of a refinement for the dual dicovers.

Definition 2.10 ([17])**.** Let $\mathcal{C}_d = \{((C_j^{1,1}, C_j^{1,2}), (C_j^{2,1}, C_j^{2,2})) \mid j \in J\}$ and $\mathcal{D}_d$ be dual dicovers. Then $\mathcal{C}_d$ is a refinement of $\mathcal{D}_d$, written $\mathcal{C}_d \prec \mathcal{D}_d$, if given $j \in J$ we have $((D^{1,1}, D^{1,2}), (D^{2,1}, D^{2,2})) \in \mathcal{D}_d$ so that

$$\begin{aligned} &(C_j^{1,1}, C_j^{1,2}) \sqsubseteq (D^{1,1}, D^{1,2}) \text{ and } (C_j^{2,1}, C_j^{2,2}) \sqsubseteq (D^{2,1}, D^{2,2}) \\ \Longleftrightarrow\ & C_j^{1,1} \subseteq D^{1,1} \ ;\ D^{1,2} \subseteq C_j^{1,2} \text{ and } C_j^{2,1} \subseteq D^{2,1} \ ;\ D^{2,2} \subseteq C_j^{2,2} \end{aligned}$$

Now let us make the following definition:

Definition 2.11. Let $(S, \mathcal{S}, \mathcal{U}^q)$ be a quasi di-uniform space. $\mathcal{U}^q$ is called a *dual dicovering bi-Lebesgue quasi di-uniformity* if for each open coclosed anchored dual dicover $\mathcal{C}_d$ of $(S, \mathcal{S}, \mathcal{U}^q)$ there is a direlation $(r, R) \in \mathcal{U}^q$ such that $\gamma^q(r, R)$ refines $\mathcal{C}_d$.

We also recall from [17, Definition 2.7] that the direlational uniformity with subbase $\mathcal{U}^q \cup (\mathcal{U}^q)^\leftarrow$ is called the *direlational uniformity associated with* $\mathcal{U}^q$ and is denoted by $\mathcal{U}^* = \mathcal{U}^q \vee (\mathcal{U}^q)^\leftarrow$.

J. Marin and S. Romaguera [11] obtained a result states that if $(X, \mathcal{U})$ is a quasi uniform space such that $(X, \tau(\mathcal{U}^*))$ is compact then $(X, \mathcal{U})$ is a pair Lebesgue quasi-uniform space.

We end this section by obtaining a similar result to the classical case.

Theorem 2.12. *Let $(S, \mathcal{S}, \mathcal{U}^q)$ be a plain quasi di-uniform texture space such that $(\tau_{\mathcal{U}^*}, \kappa_{\mathcal{U}^*})$ is dicompact. Then $(S, \mathcal{S}, \mathcal{U}^q)$ is a dual dicovering bi-Lebesque quasi di-uniform space.*

Proof. Let $\mathcal{C}_d = \left\{\left((C_j^{1,1}, C_j^{1,2}),\ (C_j^{2,1}, C_j^{2,2})\right) \mid j \in J\right\}$ be an open coclosed anchored dual dicover. For each $s \in S^\flat$ and $j(s) \in J$ we have

$$\begin{aligned} (C_{j(s)}^{1,1}, C_{j(s)}^{1,2})\, \mathcal{C}_d\, (C_{j(s)}^{2,1}, C_{j(s)}^{2,2}) \text{ with } &(C_{j(s)}^{1,1}, C_{j(s)}^{1,2}) \in (\tau_{\mathcal{U}^q} \times \kappa_{\mathcal{U}^q}) \text{ and} \\ &(C_{j(s)}^{2,1}, C_{j(s)}^{2,2}) \in (\tau_{(\mathcal{U}^q)^\leftarrow} \times \kappa_{(\mathcal{U}^q)^\leftarrow}) \end{aligned}$$

Since $(S, \mathcal{S})$ is plain and $\mathcal{C}_d$ is an anchored dual dicover we have $C_{j(s)}^{1,1} \not\subseteq Q_{s'}$ for $s' \in S^\flat$ and there exists $(d_s, D_s) \in \mathcal{U}^q$ with $d_s[s'] \subseteq C_{j(s)}^{1,1}$. In this case there exist $(r_s, R_s) \in \mathcal{U}^q$ with $(r_s, R_s)^2 \sqsubseteq (d_s, D_s)$ and ${r_s}^2[s'] \subseteq d_s[s'] \subseteq C_{j(s)}^{1,1}$ since $\mathcal{U}^q$ is a direlational quasi-uniformity.

Now take $C^{2,1}_{j(s)} \in \tau_{(\mathcal{U}^q)^{\leftarrow}}$. Since $\mathcal{C}_d$ is an anchored dual dicover and $(S, \mathcal{S})$ is plain, we have $C^{2,1}_{j(s)} \nsubseteq Q_{s'}$ for $s' \in S^\flat$. Hence, there exist $(d_s, D_s)^{\leftarrow} \in (\mathcal{U}^q)^{\leftarrow}$ and $(r_s, R_s)^{\leftarrow} \in (\mathcal{U}^q)^{\leftarrow}$ such that $(R_s{}^{\leftarrow})^2[s'] \subseteq D_s{}^{\leftarrow}[s'] \subseteq C^{2,1}_{j(s)}$.

Dually for $C^{1,2}_{j(s)} \in \kappa_{\mathcal{U}^q}$ and $C^{2,2}_{j(s)} \in \kappa_{(\mathcal{U}^q)^{\leftarrow}}$ with $P_{s'} \nsubseteq C^{1,2}_{j(s)}$ and $P_{s'} \nsubseteq C^{2,2}_{j(s)}$ we have $C^{1,2}_{j(s)} \subseteq D_s[s'] \subseteq R_s{}^2[s']$ and $C^{2,2}_{j(s)} \subseteq d_s{}^{\leftarrow}[s'] \subseteq (r_s{}^{\leftarrow})^2[s']$. Thus

$\{((r_s[s'], R_s[s']), (R_s{}^{\leftarrow}[s'], r_s{}^{\leftarrow}[s'])) \mid s' \in S^\flat\}$ is an anchored dual dicover by [17, Proposition 3.10]. Then

$\{(\,]\, r_s[s'] \cap R_s{}^{\leftarrow}[s']\, [\,), ([\, R_s[s'] \cup r_s{}^{\leftarrow}[s']\,]) \mid s' \in S^\flat\}$ is an open coclosed anchored dicover of S satisfying $P_{s'} \subseteq \,]\, r_s[s'] \cap R_s{}^{\leftarrow}[s']\, [$ and $[\, R_s[s'] \cup r_s{}^{\leftarrow}[s']\,] \subseteq Q_{s'}$

Since $(S, \mathcal{S})$ is dicompact the open coclosed dicover

$\{(\,]\, r_s[s'] \cap R_s{}^{\leftarrow}[s']\, [\,), ([\, R_s[s'] \cup r_s{}^{\leftarrow}[s']\,]) \mid s' \in S^\flat\}$ has a finite cofinite subdicover $\{(\,]\, r_{s_k}[s'_k] \cap R_{s_k}{}^{\leftarrow}[s'_k]\, [\,), ([\, R_{s_k}[s'_k] \cup r_{s_k}{}^{\leftarrow}[s'_k]\,]) \mid s' \in S^\flat\}$ for $k = 1, ..., n$ by Theorem 2.8.

Now we set $(r, R) = \prod_{k=1}^{n}(r_{s_k}, R_{s_k})$ and note that $(r, R) \in \mathcal{U}^q$. Since $(S, \mathcal{S})$ is plain we have $r_{s_k}[s'_k] \cap R^{\leftarrow}_{s_k}[s'_k] \nsubseteq Q_{s'}$ and $P_{s'} \nsubseteq R_{s_k}[s'_k] \cup r^{\leftarrow}_{s_k}[s'_k]$ for $s' \in S^\flat$ and $1 \le k \le n$.

We will complete the proof by showing $\gamma^q(r, R) \prec \mathcal{C}_d$. For the given $s' \in S^\flat$ there is $k \in \{1, 2, 3, ..., n\}$ such that $d_{s_k}[s'_k] \subseteq C^{1,1}_{j(s_k)}$, $D^{\leftarrow}_{s_k}[s'_k] \subseteq C^{2,1}_{j(s_k)}$, $C^{1,2}_{j(s_k)} \subseteq D_{s_k}[s'_k]$ and $C^{2,2}_{j(s_k)} \subseteq d^{\leftarrow}_{s_k}[s'_k]$. Now let us prove that $r[s'] \subseteq d_{s_k}[s'_k] \subseteq C^{1,1}_{j(s_k)}$. First suppose that $r[s'] \nsubseteq d_{s_k}[s'_k]$. Then there exists $z \in S^\flat$ with $r[s'] \nsubseteq Q_z$ and $P_z \nsubseteq d_{s_k}[s'_k]$. Since $r_{s_k}[s'_k] \bigcap R^{\leftarrow}_{s_k}[s'_k] \nsubseteq Q_{s'}$ we have $r_{s_k}[s'_k] \nsubseteq Q_{s'}$ and $R^{\leftarrow}_{s_k}[s'_k] \nsubseteq Q_{s'}$. Then we have $r_{s_k} \nsubseteq \overline{Q}'_{(s'_k, s)}$ and also by $r = \bigcap_{k=1}^{n} r_{s_k}$ for each $k = 1, ..., n$ we have $r \subseteq r_{s_k}$, whence $r[s'] \subseteq r_{s_k}[s']$ and we have $r_{s_k}[s'] \nsubseteq Q_z$ which gives $r_{s_k} \nsubseteq \overline{Q}_{(s', z)}$. Hence we obtain $\overline{P}_{(s'_k, z)} \subseteq r^2_{s_k} \subseteq d_{s_k}$. On the other hand $P_z \nsubseteq d_{s_k}[s'_k] = d^{\rightarrow}_{s_k} P_{s'_k}$ gives $P_z \nsubseteq Q_{z'}$ for $z' \in S^\flat$ and

$$d_{s_k} \nsubseteq \overline{Q}_{(v, z')} \implies P_{s'_k} \subseteq Q_v, \text{ for } v \in S^\flat \tag{2.2}$$

From $\overline{P}_{(s'_k, z)} \subseteq d_{s_k}$ and $P_z \nsubseteq Q_{z'}$ we have $d_{s_k} \nsubseteq \overline{Q}_{(s'_k, z')}$, and since r is a relation we have $s''_k \in S^\flat$ with $P_{s'_k} \nsubseteq Q_{s''_k}$ such that $d_{s_k} \nsubseteq \overline{Q}_{(s''_k, z')}$ by *R2*. Applying the implication (2) with $v = s''_k$ we deduce $P_{s'_k} \subseteq Q_{s''_k}$, which is a contradiction. This verifies $r[s'] \subseteq d_{s_k}[s'_k] \subseteq C^{1,1}_{j(s)}$.

Now it is easy to prove that $R^{\leftarrow}[s'] \subseteq D^{\leftarrow}_{s_k}[s'_k] \subseteq C^{2,1}_{j(s_k)}$. The other two inclusions can be shown dually and hence the proof is omitted. This completes the proof of $\gamma^q(r, R) \prec \mathcal{C}_d$, thus $(S, \mathcal{S}, \mathcal{U}^q)$ is a dual dicovering bi-Lebesque quasi di-uniform space. □

3. Completeness of Lebesgue di-uniform spaces

We conclude this paper by discussing the completeness of Lebesgue di-uniformities. The subjects of completeness and total boundedness in di-uniform spaces are discussed in [14].

In the classical theory it is known from [7, 11] that every Lebesgue uniformity is complete and every Lebesgue quasi-uniformity is convergence complete. At the beginning of this work, we expect to find an analogue result for the Lebesgue di-uniformities. However there are considerable difficulties such as the convergence of the Cauchy di-filter to obtain a similar result for the general textures. Since there is a close relationship between quasi-uniformities and di-uniformities on discrete textures so we turn our attention to the completeness of Lebesgue di-uniformities which correspond to Lebesgue quasi-uniformity on the discrete textures.

Quasi-uniform spaces can be defined in various equivalent ways; by relations that satisfy all the axioms of a uniformity except symmetry; by quasi-pseudometrics and by (pair, dual) covers. Gartner and Steinlage [8] presented a description of quasi-uniformities in terms of pairs of covers and Marin and Romaguera [11] used a similar notion called open pairs, that is $\{(G_\alpha, H_\alpha) \mid \alpha \in \mathcal{A}\}$ such that G_α is τ_Q-open and H_α is $\tau_{Q^{-1}}$-open and for each $x \in X$ there is $\alpha \in \mathcal{A}$ with $x \in G_\alpha \cap H_\alpha$ where τ_Q is the topology generated by Q and $\tau_{Q^{-1}}$ that generated by Q^{-1}.

Brown [1] independently developed a theory of quasi-uniformities by using a new concept of dual cover and showed the equivalence with the notion of open pairs mentioned in [12]. In this context the notion of Q-completeness was considered for the completeness of quasi-uniform spaces. Now we find it convenient to use the representation in terms of dual covers in this section.

Throughout this section we are interested in the concept of completeness namely Q-completeness for quasi-uniformities which is based on the use of dual covers. Since dual covers are not well known concepts, we now recall from [1, 15] some definitions and properties. However the dual covering quasi-uniformity and the equivalence with the diagonal quasi-uniformity were studied widely in [15].

Let X be a set. A family $U = \{(A_j, B_j) \mid j \in J\}$ of subsets of X is called a dual cover of X if $\bigcup\{(A_j \bigcap B_j) \mid j \in J\} = X$. If U and V are dual covers of X we say U refines V and write $U \prec V$ if whenever AUB there exists CVD satisfying $A \subseteq C$ and $B \subseteq D$. Given a binary point relation $d \in X$ we may associate with d the dual family called dual cover

$$\gamma^*(d) = \{(d[x], d^{-1}[x]) \mid x \in X\}$$

where, as usual $d[x] = \{y \in X \mid (x, y) \in d\}$ and $d^{-1}[x] = \{y \in X \mid (y, x) \in d\}$.

Now let us turn our attention to the completeness of quasi-uniformity and Lebesgue quasi-uniformity. In the literature, several authors defined various kinds of completeness on quasi-uniform spaces. We will mention particulary two of these definitions. According to [7] a quasi uniform space (X, Q) is called bicomplete if (X, Q^*) is a complete uniform space where $Q^* = Q \vee Q^{-1}$. Since Lebesgue uniformity is complete, Marin and Romaguera [11] obtained a result which states that each pair Lebesgue quasi-uniformity is bicomplete.

On the other hand a quasi-uniform space (X, Q) is convergence complete [7] provided that each Cauchy filter is τ_Q-convergent and since every Lebesgue quasi-uniformity is convergence complete, Marin and Romaguera also proved that each pair Lebesgue quasi-uniformity is convergence complete.

Starting from this point, we focus our attention on another type of completeness namely Q-completeness in the Brown's sense (see, [1]). We recall that a bifilter $\mathcal{B}$ on the set X is defined as a product of two filters $\mathcal{B}_u$ and $\mathcal{B}_v$ on X denoted by $\mathcal{B} = \mathcal{B}_u \times \mathcal{B}_v$. Any bifilter $\mathcal{B}$ is ı-regular if $F \cap G \neq \varnothing$ whenever $(F, G) \in \mathcal{B}$.

If (X, u, v) is a bitopological space and $x \in X$ then $\mathcal{B}(x) = \{(\mathcal{H}(x), \mathcal{K}(x)) \mid \mathcal{H}(x)$ is a u-nhd. and $\mathcal{K}(x)$ is a v-nhd of x$\}$ is an ı-regular bifilter which we will call the nhd. bifilter of x. The bifilter $\mathcal{B}$ converges to x if $\mathcal{B}(x) \subseteq \mathcal{B}$. If Q is a dual covering quasi-uniformity compatible with (X, u, v) then the bifilter $\mathcal{B}$ will be called Q-Cauchy if $U \cap \mathcal{B} \neq \varnothing$ for all $U \in Q$.

Definition 3.1 (see [1]). A quasi-uniform space (X, Q) is called *Q-complete* if every ı-regular Q-Cauchy bifilter is convergent in the bitopological space $(X, \tau_Q, \tau_{Q^{-1}})$.

Now we shall work with dual covers in the sense of Brown [1] instead of pair open cover and because of the equivalence of these two concepts we expect to have similar results as given in the paper of Marin and Romaguera [11]. We first give the definition of a notion *dual covering Lebesgue quasi-uniformity* which was defined by Marin and Romaguera under the name of pair Lebesgue quasi-uniformity.

Definition 3.2. Let Q be a quasi-uniformity on X. We say that Q is a *dual covering Lebesgue quasi-uniformity* if for each open dual cover $U = \{(A_j, B_j) \mid j \in J\}$ of (X, Q) there is $d \in Q$ such that the dual cover $\{(d[x], d^{-1}[x]) \mid x \in X\}$ refines $\{(A_j, B_j) \mid j \in J\}$ (i.e. for each $x \in X$ there is $j \in J$ such that $d[x] \subseteq A_j$ and $d^{-1}[x] \subseteq B_j$). Thus, we say that (X, Q) is a *dual covering Lebesgue quasi-uniform space.*

Here we recall from [16] that a quasi-uniformity Q on a set X is a Lebesgue quasi-uniformity provided that for each τ_Q-open cover $\mathcal{G}$ of X there is $d \in Q$ such that the cover $\{d[x] \mid x \in X\}$ refines $\mathcal{G}$.

Proposition 3.3. *Let (X, Q) be a dual covering Lebesgue quasi-uniform space. Then Q and Q^{-1} are Lebesgue quasi-uniformities.*

Proof. Let Q be a dual covering Lebesgue quasi-uniformity on X. We shall show that Q is a Lebesgue quasi-uniformity. Let $\{A_j : j \in J\}$ be a τ_Q-open cover of X. For each $j \in J$ let $B_j = X$. Then $\{(A_j, X) : j \in J\}$ is an open dual cover of (X, Q). So there is $d \in Q$ such that the open dual cover $\{(d[x], d^{-1}[x]) \mid x \in X\}$ refines $\{(A_j, X) : j \in J\}$ which gives the required result. Similarly we see that Q^{-1} is a Lebesgue quasi-uniformity. □

Theorem 3.4. *Every Lebesgue quasi-uniformity is Q-complete.*

Proof. Let $\mathcal{B} = \mathcal{B}_u \times \mathcal{B}_v$ be a regular Cauchy bifilter that does not converge to x. Then for each $x \in X$ we have $\mathcal{M}(x)$ a τ_Q-nhd. of x and $\mathcal{N}(x)$ a $\tau_{Q^{-1}}$-nhd. of x such that $(\mathcal{M}(x), \mathcal{N}(x)) \notin \mathcal{B}$. Also there exists $d \in Q$ such that $\{d[x] : x \in X\}$ refines $\mathcal{M}(x)$. Since $\mathcal{B} = \mathcal{B}_u \times \mathcal{B}_v$ is a Cauchy bifilter, $d[x] \in \mathcal{B}_u$ for some $x \in X$, which is a contradiction. □

The following result is clear from the above discussion so we omit the proof.

Theorem 3.5. *Let Q be a dual covering Lebesgue quasi-uniformity on X. Then both (X, Q) and (X, Q^{-1}) are Q-complete quasi-uniform spaces.*

In the remainder of this section we consider the completeness of Lebesgue di-uniformities on discrete texture $(X, \mathcal{P}(X))$. We will investigate how the relation between quasi-uniformity and di-uniformity effects the completeness of Lebesgue di-uniformities. The reader is referred to [15] for more backround material, for the benefit of the reader however we will briefly recall the necessary definitions and results.

In [15] it is shown that di-uniformities on the discrete texture correspond to quasi uniformities on X. Moreover a direlational uniformity on $(X, \mathcal{P}(X))$ corresponds to a uniformity if and only if it is complemented. Let $d \subseteq X \times X$ be a point relation then $u(d) = (d, d^{\leftarrow})$ is a direlation on $(X, \mathcal{P}(X))$ and if Q is a quasi-uniformity on X, the family $u(\mathcal{Q}) = \{(e, E) \mid \exists d \in \mathcal{Q} \text{ and } u(d) \sqsubseteq (e, E)\}$ is a direlational uniformity on the discrete texture $(X, \mathcal{P}(X))$.

Proposition 3.6 ([15]). *Let $\mathcal{Q}$ be a quasi-uniformity on X and $\mathcal{Q}^{-1}$ its conjugate. Then the direlational uniformity on $(X, \mathcal{P}(X), \pi_X)$ corresponding to $\mathcal{Q}^{-1}$ is the complement of the direlational uniformity corresponding to $\mathcal{Q}$. That is, $u(\mathcal{Q}^{-1}) = u(\mathcal{Q})'$.*

Theorem 3.7 ([15]). *Let $\mathcal{Q}$ be a quasi-uniformity on X. Then $\mathcal{Q}$ is a uniformity if and only if the corresponding di-uniformity $u(\mathcal{Q})$ on $(X, \mathcal{P}(X), \pi_X)$ is complemented.*

We can now tie the completeness of a quasi-uniformity in with the dicompleteness of a di-uniformity on $(X, \mathcal{P}(X))$.

Proposition 3.8. *The quasi-uniform space (X, Q) is Q-complete if and only if the di-uniform discrete texture space $(X, \mathcal{P}(X), u(Q))$ is dicomplete.*

Proof. It is similar to the proof of [19, Proposition 2.16]. □

Now we have the following theorems.

Theorem 3.9 ([16, Theorem 2.3]). *Let $\mathcal{Q}$ be a Lebesgue quasi-uniformity on X. Then the corresponding di-uniformity $u(\mathcal{Q})$ on $(X, \mathcal{P}(X), \pi_X)$ is a Lebesgue direlational uniformity.*

Conversely if $\mathcal{U}$ is a Lebesgue direlational uniformity on $(X, \mathcal{P}(X), \pi_X)$ then $u^{-1}(\mathcal{U})$ is a Lebesgue quasi-uniformity on X.

Theorem 3.10 ([16, Theorem 2.5])**.** *Let $\mathcal{Q}$ be Lebesgue quasi uniformity on X. Then the complement of the direlational uniformity corresponding to $\mathcal{Q}$, that is $u(\mathcal{Q})'$, is a co-Lebesgue direlational uniformity on $(X, \mathcal{P}(X), \pi_X)$.*

Conversely, if $\mathcal{U}$ is the co-Lebesgue direlational uniformity corresponding to $\mathcal{Q}^{-1}$, then $u^{-1}(\mathcal{U}')$ is a Lebesgue quasi uniformity on X.

We are now in a position to give the promised result.

Theorem 3.11. *Let Q be a Lebesgue quasi-uniformity on X. Then the corresponding Lebesgue di-uniformity $u(Q)$ on $(X, \mathcal{P}(X), \pi_X)$ is dicomplete.*

Proof. Let Q be a Lebesgue quasi-uniformity on X. We know from Theorem 3.9 that the corresponding di-uniformity $u(Q)$ on $(X, \mathcal{P}(X))$ is Lebesgue. Since every Lebesgue quasi-uniformity is Q-complete by Theorem 3.4, the corresponding Lebesgue di-uniformity $u(Q)$ is dicomplete by Proposition 3.8. □

Theorem 3.12. *Let Q be a Lebesgue quasi-uniformity on X. If Q is a uniformity, then the complement of the corresponding Lebesgue di-uniformity $u(Q)$ on $(X, \mathcal{P}(X), \pi_X)$ is dicomplete.*

Proof. If Q is a uniformity then $Q = Q^{-1}$ and $u(\mathcal{Q}^{-1}) = u(\mathcal{Q})' = u(Q)$ by Proposition 3.6 and Theorem 3.7. Then by Theorem 3.11 the co-Lebesgue di-uniformity $u(\mathcal{Q})'$ is dicomplete. □

Remark 3.13. Since every dual covering Lebesgue quasi-uniformity is a Lebesgue quasi-uniformity it is clear that the last two results hold for the dual covering Lebesgue quasi-uniformities.

4. Conclusion Remarks

In [10] Hutton gave the definition of uniformities and quasi uniformities on a Hutton algebra $\mathbb{L}^x$ using functions on $\mathbb{L}^x$. A similar representation was obtained in [18] for di-uniformities and quasi di-uniformities called difunctional uniformity and difunctional quasi-uniformity [18, Definition 2.7].

For the further studies it would be interesting to investigate the Lebesgue property on the difunctional uniformities and quasi-uniformities.

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