

## Some classes of minimally almost periodic topological groups

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To the Memory of Ivan Prodanov on the Occasion of the  
30th Anniversary of his Death

### ABSTRACT

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A Hausdorff topological group  $G = (G, \mathcal{T})$  has the small subgroup generating property (briefly: has the SSGP property, or is an SSGP group) if for each neighborhood  $U$  of  $1_G$  there is a family  $\mathcal{H}$  of subgroups of  $G$  such that  $\bigcup \mathcal{H} \subseteq U$  and  $\langle \bigcup \mathcal{H} \rangle$  is dense in  $G$ . The class of SSGP groups is defined and investigated with respect to the properties usually studied by topologists (products, quotients, passage to dense subgroups, and the like), and with respect to the familiar class of minimally almost periodic groups (the m.a.p. groups). Additional classes  $\text{SSGP}(n)$  for  $n < \omega$  (with  $\text{SSGP}(1) = \text{SSGP}$ ) are defined and investigated, and the class-theoretic inclusions

$$\text{SSGP}(n) \subseteq \text{SSGP}(n+1) \subseteq \text{m.a.p.}$$

are established and shown proper.

In passing the authors also establish the presence of  $\text{SSGP}(1)$  or  $\text{SSGP}(2)$  in many of the early examples in the literature of abelian m.a.p. groups.

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1. INTRODUCTION

1.1. Conventions.

- (a) As usual, a *topological group* is a pair  $(G, \mathcal{T})$  with  $G$  a group and with  $\mathcal{T}$  a topology on  $G$  for which the maps  $G \times G \rightarrow G$  and  $G \rightarrow G$  given by  $(x, y) \rightarrow xy$  and  $x \rightarrow x^{-1}$  are continuous.
- (b) The topological spaces we hypothesize, in particular our hypothesized topological groups, are assumed to be completely regular and Hausdorff (*i.e.*, to be Tychonoff spaces). When a topology is defined or constructed on a set or a group, the Tychonoff property will be verified explicitly (if it is not obvious). In this context we recall ([24](8.4)) that in order that a topological group be a Tychonoff space, it suffices that it satisfy the Hausdorff separation property.
- (c) For  $X$  a space and  $x \in X$  we write

$$\mathcal{N}_X(x) := \{U \subseteq X : U \text{ is a neighborhood of } x\}.$$

When ambiguity is unlikely we write  $\mathcal{N}(x)$  in place of  $\mathcal{N}_X(x)$ .

- (d) The identity of a group  $G$  is denoted  $1$  or  $1_G$ ; if  $G$  is known or assumed to be abelian and additive notation is in play, the identity may be denoted  $0$  or  $0_G$ .
- (e) When  $G$  is a group and  $\kappa \geq \omega$ , we use the notations  $\bigoplus_{\kappa} G$  and  $G^{(\kappa)}$  interchangeably:

$$\bigoplus_{\kappa} G = G^{(\kappa)} := \{x \in G^{\kappa} : |\{\eta < \kappa : x_{\eta} \neq 1_G\}| < \omega\}.$$

When  $G$  is a topological group,  $\bigoplus_{\kappa} G$  has the topology inherited from  $G^{\kappa}$ .

The *minimally almost periodic* groups (briefly: the m.a.p. groups) to which our title refers are by definition those topological groups  $G$  for which every continuous homomorphism  $\phi : G \rightarrow K$  with  $K$  a compact group satisfies  $\phi[G] = \{1_K\}$ . It follows from the Gel'fand-Raïkov Theorem [17] (see [24](§22) for a detailed development and proof) that every compact group  $K$  is algebraically and topologically a subgroup of a group of the form  $\prod_{i \in I} U_i$  with each  $U_i$  a (finite-dimensional) unitary group [24](22.14). Therefore, to check that a topological group  $G$  is m.a.p. it suffices to show that each continuous homomorphism  $\phi : G \rightarrow U(n)$  with  $U(n)$  the  $n$ -dimensional unitary group satisfies  $\phi[G] = \{1_{U_n}\}$ . Similarly, since every compact abelian group  $K$  is algebraically and topologically isomorphic to a subgroup of a group of the form  $\mathbb{T}^I$  [24](22.17), to check that an abelian topological group  $G$  is m.a.p., it suffices to show that each continuous homomorphism  $\phi : G \rightarrow \mathbb{T}$  satisfies  $\phi[G] = \{1_{\mathbb{T}}\}$ .

Sometimes for convenience we denote by m.a.p. the (proper) class of m.a.p. groups, and if  $G$  is a m.a.p. group we write  $G \in \text{m.a.p.}$ . Similar conventions apply to the classes  $\text{SSGP}(n)$  ( $0 \leq n < \omega$ ) defined in Definition 3.3.

Algebraic characterizations of those abelian groups which admit an m.a.p. group topology have been achieved only recently. For a brief historical account of the literature touching this issue, see Discussion 2.1(h),(i) below.

## 2. M.A.P. GROUPS: A BRIEF HISTORICAL SURVEY

**2.1. Discussion.** With no pretense to completeness, we here discuss briefly some of the literature relating to the development of the class of m.a.p. groups.

- (a) In effect, the class m.a.p. was introduced in 1930 by von Neumann [27], who then together with Wigner [28] proved that (even in its discrete topology) the matrix group  $SL(2, \mathbb{C})$  is an m.a.p. group.
- (b) In the period 1940–1952, several workers showed that certain real topological linear spaces are m.a.p. groups; several examples, with detailed verification, are given by Hewitt and Ross [24](23.32).
- (c) In what follows we will quote at length from the 1980 paper of Prodanov [33], which showed by “elementary means” that the group  $\bigoplus_{\omega} \mathbb{Z}$  admits an m.a.p. topology.
- (d) Ajtai, Havas and Komlós [1] proved that each group  $G$  of the form  $\mathbb{Z}$ ,  $\mathbb{Z}(p^{\infty})$ , or  $\bigoplus_n \mathbb{Z}(p_n)$  (with all  $p_n \in \mathbb{P}$  either identical or distinct) admits a m.a.p. group topology.
- (e) Nienhuys [29] showed that the additive group  $\mathbb{R}$  admits an m.a.p. topology  $\mathcal{T}$  contained in its usual topology. Since  $\mathbb{Q}$  remains dense in  $(\mathbb{R}, \mathcal{T})$  and the m.a.p. property is inherited by dense subgroups, this allowed Remus [35] to infer that  $\mathbb{Q}$  admits an m.a.p. topology, hence (since the weak sum of m.a.p. groups is again a m.a.p. group) that every infinite divisible abelian group admits a m.a.p. topology. In the same paper [35], Remus showed that every free abelian group admits an m.a.p. topology.
- (f) Protasov [34] and Remus [35] asked whether every infinite abelian group admits an m.a.p. group topology; the question was deftly settled in the negative by Remus [36] with the straightforward observation that for distinct  $p, q \in \mathbb{P}$ , every group topology on the infinite group  $G := \mathbb{Z}(p) \times (\mathbb{Z}(q))^{\kappa}$  (with  $\kappa \geq \omega$ ) has the property that the homomorphism  $x \rightarrow qx$  maps  $G$  continuously onto the compact group  $\mathbb{Z}(p)$ . (See [3](3.J), [5](4.6) for additional discussion.)
- (g) In view of the cited examples  $\mathbb{Z}(p) \times (\mathbb{Z}(q))^{\kappa}$  of Remus [36], it was natural for Comfort [3](3.J.1) to ask: (1) Does every abelian group which is not torsion of bounded order admit an m.a.p. topology? (2) What about the countable case?
- (h) Gabrielyan [14], [16] answered Question (g)(2) affirmatively, showing that indeed the witnessing topology may be chosen complete in the sense that every Cauchy net converges. Gabrielyan [15] showed further that an abelian torsion group of bounded order admits an m.a.p. topology if and only if each of its leading Ulm-Kaplansky invariants is infinite. (The reader unfamiliar with the Ulm-Kaplansky invariants might consult [13](§77); those cardinals also play a significant role in [4] in a setting closely related to the present paper.)

(i) Dikranjan and Shakhmatov [8] gave a definitive positive answer to Question (g)(1) in its fullest generality, indeed they gave several equivalent conditions on a (not necessarily abelian) group  $G$  which are necessary and sufficient that  $G$  admit an m.a.p. topology. Among those conditions, evidently satisfied by each abelian group not torsion of bounded order, are these:

- (1)  $G$  is connected in its Zariski topology;
- (2)  $m \in \mathbb{Z} \Rightarrow mG = \{0\}$  or  $|mG| \geq \omega$ ;
- (3) the group  $\text{fin}(G)$  is trivial, i.e.,  $\text{fin}(G) = \{0\}$ . (The group  $\text{fin}(G)$ , whose study was initiated in [7](4.4) and continued in [4](§2), may be defined by the relation

$$\text{fin}(G) = \langle \bigcup \{mG : m \in \mathbb{Z}, |mG| < \omega\} \rangle.$$

Detailed subsequent analysis of the theorems and techniques of [8] have allowed those authors to answer the following two questions in the negative; these questions were posed in [19] and in a privately circulated pre-publication copy of the present manuscript.

- (1) Let  $G$  be a group with a normal subgroup  $K$  for which  $K$  and  $G/K$  admit topologies  $\mathcal{U}$  and  $\mathcal{V}$  respectively such that  $(K, \mathcal{U}) \in \text{m.a.p}$  and  $(G/K, \mathcal{W}) \in \text{m.a.p}$ . Is there then necessarily a group topology  $\mathcal{T}$  on  $G$  such that ( $K$  is closed in  $(G, \mathcal{T})$  and)  $(K, \mathcal{U}) = (K, \mathcal{T}|_K)$  and  $(G/K, \mathcal{U}) = (G/K, \mathcal{T}_q)$  with  $\mathcal{T}_q$  the quotient topology?
- (2) Let  $G$  be a group with a normal subgroup  $K$  such that both  $K$  and  $G/K$  admit m.a.p. topologies. Must  $G$  admit a m.a.p. topology?

### 3. SSGP GROUPS: SOME GENERALITIES

**Definition 3.1.** Let  $G = (G, \mathcal{T})$  be a topological group and let  $A \subseteq G$ . Then  $A$  topologically generates  $G$  if  $\langle A \rangle$  is dense in  $G$ .

**Definition 3.2.** A Hausdorff topological group  $G = (G, \mathcal{T})$  has the *small subgroup generating property* if for every  $U \in \mathcal{N}(1_G)$  there is a family  $\mathcal{H}$  of subgroups of  $G$  such that  $\bigcup \mathcal{H} \subseteq U$  and  $\bigcup \mathcal{H}$  topologically generates  $G$ .

A (Hausdorff) topological group with the small subgroup generating property is said to *have the SSGP property*, or to *be an SSGP group*, or simply to *be SSGP*.

Now for  $0 \leq n < \omega$  the classes  $\text{SSGP}(n)$  are defined as follows.

**Definition 3.3.** Let  $G = (G, \mathcal{T})$  be a Hausdorff topological group. Then

- (a)  $G \in \text{SSGP}(0)$  if  $G$  is the trivial group.
- (b)  $G \in \text{SSGP}(n+1)$  for  $n \geq 0$  if for every  $U \in \mathcal{N}(1_G)$  there is a family  $\mathcal{H}$  of subgroups of  $G$  such that
  - (1)  $\bigcup \mathcal{H} \subseteq U$ ,
  - (2)  $H := \overline{\langle \bigcup \mathcal{H} \rangle}$  is normal in  $G$ , and
  - (3)  $G/H \in \text{SSGP}(n)$ .

*Remarks 3.4.*

- (a) For  $0 \leq n < \omega$  the class-theoretic inclusion  $\text{SSGP}(n) \subseteq \text{SSGP}(n+1)$  holds, hence  $\text{SSGP}(n) \subseteq \text{SSGP}(m)$  when  $n < m < \omega$ . To see this, note that when  $G \in \text{SSGP}(n)$  and  $U \in \mathcal{N}(1_G)$  then we have, taking  $\mathcal{H} := \{\{1_G\}\}$ , that  $H := \overline{\langle \bigcup \mathcal{H} \rangle} = \{1_G\}$  and  $G/H \simeq G \in \text{SSGP}(n)$ , so indeed  $G \in \text{SSGP}(n+1)$ .
- (b) Clearly the class  $\text{SSGP}$  of Definition 3.2 coincides with the class  $\text{SSGP}(1)$  of Definition 3.3.

A topological group  $G$  is said to be *precompact* if  $G$  is a (dense) topological subgroup of a compact group. It is a theorem of Weil [40] that a topological group  $G$  is precompact if and only if  $G$  is *totally bounded* in the sense that for each  $U \in \mathcal{N}(1_G)$  there is finite  $F \subseteq G$  such that  $G = FU$ .

It is obvious that a precompact group  $G$  with  $|G| > 1$  is not m.a.p. Indeed if  $G$  is dense in the compact group  $\overline{G}$  then the continuous function  $\text{id} : G \hookrightarrow \overline{G}$  does not satisfy  $\text{id}[G] = \{1_{\overline{G}}\}$ .

**Theorem 3.5.** *The class-theoretic inclusion  $\text{SSGP}(n) \subseteq \text{m.a.p.}$  holds for each  $n < \omega$ .*

*Proof.* The proof is by induction on  $n$ . Clearly if  $G \in \text{SSGP}(0)$  and  $\phi \in \text{Hom}(G, U(m))$  then  $\phi[G] = \{1_{U(m)}\}$ , so  $G \in \text{m.a.p.}$  Suppose now that  $\text{SSGP}(n) \subseteq \text{m.a.p.}$ , let  $G$  be a topological group such that  $G \in \text{SSGP}(n+1)$ , and let  $\phi : G \rightarrow U(m)$  be a continuous homomorphism. Choose  $V \in \mathcal{N}(1_{U(m)})$  so that  $V$  contains no subgroups of  $U(m)$  other than  $\{1_{U(m)}\}$ . Then  $U := \phi^{-1}[V] \in \mathcal{N}(1_G)$ , and  $\phi$  maps every subgroup of  $U$  to  $1_{U(m)}$ . Let  $\mathcal{H}$  be a family of subgroups of  $G$  such that  $\bigcup \mathcal{H} \subseteq U$  and  $H := \overline{\langle \mathcal{H} \rangle}$  is normal in  $G$ , with  $G/H \in \text{SSGP}(n)$ . Since a homomorphism maps subgroups to subgroups we have  $\phi[H] = \{1_{U(m)}\}$ . It follows that  $\phi$  defines a continuous homomorphism  $\tilde{\phi} : G/H \rightarrow U(m)$  (given by  $\tilde{\phi}(xH) := \phi(x)$ ). By the induction hypothesis,  $\tilde{\phi}$  is the trivial homomorphism, so  $\phi$  is trivial as well; the relation  $G \in \text{m.a.p.}$  follows.  $\square$

Now in 3.6–3.18 we clarify what we do and do not know about the classes of groups mentioned in Theorem 3.5. (The reader familiar with the literature may recall that a group as hypothesized in Lemma 3.6 is referred to frequently as a *group with no small subgroups*.)

**Lemma 3.6.** *Let  $G$  be a nontrivial (Hausdorff) topological group for which some  $U \in \mathcal{N}(1_G)$  contains no subgroup other than  $\{1_G\}$ . Then there is no  $n < \omega$  such that  $G \in \text{SSGP}(n)$ .*

*Proof.* Clearly  $G \notin \text{SSGP}(0)$ . Suppose there is a minimal  $n > 0$  such that  $G \in \text{SSGP}(n)$ , and let  $U \in \mathcal{N}(1_G)$  be as hypothesized. Then the only choice for  $\mathcal{H}$  (as required in Definition 3.3) is  $\mathcal{H} := \{\{1_G\}\}$ , yielding  $H = \overline{\langle \bigcup \mathcal{H} \rangle} = \{1_G\}$ . Thus,  $G/H = G \in \text{SSGP}(n-1)$ , which contradicts the assumption that  $n$  is minimal.  $\square$

Evidently Lemma 3.6 furnishes a plethora of topological groups which belong to none of the classes  $\text{SSGP}(n)$ . The interested reader will readily augment the following incomplete list.

**Corollary 3.7.** *Let  $G$  be a nontrivial topological group which is either discrete or a Lie group. Then there is no  $n < \omega$  such that  $G \in \text{SSGP}(n)$ .*

For another statement in parallel with Corollary 3.7, see Theorem 3.16 below.

**Definition 3.8.** Let  $G$  be a group and let  $1_G \notin C \subseteq G$ . Then  $C$  *cogenerates*  $G$  if every subgroup  $H$  of  $G$  such that  $|H| > 1$  satisfies  $H \cap C \neq \emptyset$ .

**Theorem 3.9.** *Let  $G$  be a nontrivial finitely cogenerated topological group. Then there is no  $n < \omega$  such that  $G \in \text{SSGP}(n)$ .*

*Proof.* Let  $C$  be a finite set of cogenerators for  $G$ , and choose  $U \in \mathcal{N}(1_G)$  such that  $U \cap C = \emptyset$ . Then  $U$  contains no subgroup other than  $\{1_G\}$ , and the statement follows from Lemma 3.6.  $\square$

We have noted for every  $n < \omega$  the class-theoretic inclusion  $\text{SSGP}(n) \subseteq \text{m.a.p.}$  On the other hand, there are many examples of  $G \in \text{m.a.p.}$  such that  $G \in \text{SSGP}(n)$  for no  $n < \omega$ . But more is true: There are groups which admit an m.a.p. topology which do not for any  $n < \omega$  admit an  $\text{SSGP}(n)$  topology. Indeed from Corollary 3.14 and Theorem 3.9 respectively we see that the groups  $G = \mathbb{Z}$  and  $G = \mathbb{Z}(p^\infty)$  admit no  $\text{SSGP}(n)$  topology; while Ajtai, Havas, and Komlós [1], and later Zelenyuk and Protasov [41], have shown the existence of m.a.p. topologies for  $\mathbb{Z}$  and for  $\mathbb{Z}(p^\infty)$ .

In Theorem 3.13 we show that in the context of abelian groups, Theorem 3.9 can be strengthened. We use the following basic facts from the theory of abelian groups.

**Lemma 3.10.** *A finitely cogenerated group is the direct sum of finite cyclic  $p$ -groups and groups of the form  $\mathbb{Z}(p^\infty)$ , hence is torsion ([12](3.1 and 25.1)).*

**Lemma 3.11.** *A finitely generated abelian group is the direct sum of cyclic groups and cyclic torsion groups ([12](15.5)).*

**Lemma 3.12.** *If  $G$  is a finitely generated abelian group and  $H$  is a torsionfree subgroup then there is a decomposition  $G = K \oplus T$  where  $T$  is the torsion subgroup of  $G$ ,  $K$  is torsionfree, and  $H \subseteq K$  ([12](Chapter III)).*

**Theorem 3.13.** *A nontrivial abelian group which is the direct sum of a finitely generated group and a finitely cogenerated group does not admit an  $\text{SSGP}(n)$  topology for any  $n < \omega$ .*

*Proof.* We proceed by induction on the torsionfree rank,  $r_0(G)$ . Suppose first that  $r_0(G) = 0$ . Then  $G$  is finitely co-generated and does not admit an  $\text{SSGP}(n)$  topology by Theorem 3.9. Now suppose that the theorem has been proved up to rank  $r - 1$  and we have  $r_0(G) = r \geq 1$  and  $G = F \oplus T$ , with  $F$  finitely generated and  $T$  finitely co-generated. Using Lemmas 3.10 and 3.11, we rewrite  $G$  in the

form  $G = F' \oplus T'$  where  $T'$  is the (finitely cogenerated) torsion subgroup and  $F'$  is free. Then  $r_0(F') = r_0(G) = r$ . Let  $a \in F'$  be an element of infinite order and choose  $U \in \mathcal{N}(0)$  so that  $a \notin \overline{U}$  and so that  $\overline{U} \cap C = \emptyset$  where  $C$  is a finite set of cogenerators of  $T'$  (with  $0 \notin C$ ). If all subgroups contained in  $U$  are torsion, then each such subgroup is a subgroup of  $T'$  and is therefore the zero subgroup, since it misses  $C$ . In that case, by Lemma 3.6 there is no  $n < \omega$  such that  $G \in \text{SSGP}(n)$ . Alternatively, if  $U$  contains a cyclic subgroup  $H$  of infinite order, we have  $r_0(\overline{H}) > 0$ . Furthermore, since  $\overline{H} \subseteq \overline{U}$ , we have  $\overline{H} \cap T' = \{0\}$ . It follows from Lemma 3.12 that there is a decomposition  $G = F'' \oplus T'$  which is isomorphic to the original decomposition and is such that  $\overline{H} \subseteq F''$ . Since a quotient of a finitely generated group is also finitely generated, it follows that  $F''/\overline{H}$  is finitely generated. Then we have  $G/\overline{H} = (F''/\overline{H}) \oplus T'$ . Since  $r_0(G) = r_0(\overline{H}) + r_0(G/\overline{H})$  (see for example [12](§16, Ex. 3(d))), we also have  $r_0(G/\overline{H}) < r$ . Further, the group  $G/\overline{H}$  is nontrivial since  $\overline{H} \subseteq \overline{U}$  and  $a \notin \overline{U}$ . It follows from the induction assumption that  $G/\overline{H}$  does not admit an  $\text{SSGP}(n)$  topology, and so by Theorem 3.15(b) (below), neither does  $G$ .  $\square$

**Corollary 3.14.** *The group  $\mathbb{Z}$  does not admit an  $\text{SSGP}(n)$  topology for any  $n < \omega$ .*

The following theorem lists several inheritance properties for groups in the classes  $\text{SSGP}(n)$ .

**Theorem 3.15.**

- (a) *If  $K$  is a closed normal subgroup of  $G$ , with  $K \in \text{SSGP}(n)$  and  $G/K \in \text{SSGP}(m)$  then  $G \in \text{SSGP}(m+n)$ .*
- (b) *If  $G \in \text{SSGP}(n)$  and  $\pi : G \rightarrow B$  is a continuous homomorphism from  $G$  onto  $B$ , then  $B \in \text{SSGP}(n)$ . In particular, if  $K$  is a closed normal subgroup of  $G \in \text{SSGP}(n)$  then  $G/K \in \text{SSGP}(n)$ .*
- (c) *If  $K$  is a dense subgroup of  $G$  and  $K \in \text{SSGP}(n)$  then  $G \in \text{SSGP}(n)$ .*
- (d) *If  $G_i \in \text{SSGP}(n)$  for each  $i \in I$  then  $\bigoplus_{i \in I} G_i \in \text{SSGP}(n)$  and  $\prod_{i \in I} G_i \in \text{SSGP}(n)$ .*

*Proof.* We proceed in each case by induction on  $n$ . Each statement is trivial when  $n = 0$ . We address (a), (b), (c) and (d) in order, assuming in each case for  $1 \leq n < \omega$  that the statement holds for  $n - 1$ .

(a) Let  $U \in \mathcal{N}(1_G)$ , so that  $U \cap K \in \mathcal{N}(1_K)$ . Then there is a family  $\mathcal{H}$  of subgroups of  $K$  such that  $\bigcup \mathcal{H} \subseteq U \cap K$  and  $K/H \in \text{SSGP}(n-1)$  where  $H := \overline{\langle \mathcal{H} \rangle}^K = \overline{\langle \mathcal{H} \rangle}^G$ . Since  $G/K$  is topologically isomorphic with  $(G/H)/(K/H)$ , we have  $(G/H)/(K/H) \in \text{SSGP}(m)$  along with  $K/H \in \text{SSGP}(n-1)$ . Then by the induction hypothesis,  $G/H \in \text{SSGP}(m+n-1)$ . Since  $\bigcup \mathcal{H} \subseteq (U)$  with  $U$  arbitrary, we have  $G \in \text{SSGP}(m+n)$ , as required.

(b) Given  $G \in \text{SSGP}(n)$  and continuous  $\pi : G \rightarrow B$ , let  $U \in \mathcal{N}(1_B)$ . Then  $\pi^{-1}[U] \in \mathcal{N}(1_G)$  and there is a family  $\mathcal{H}$  of subgroups of  $G$  such that  $\bigcup \mathcal{H} \subseteq (\pi^{-1}[U])$  and  $G/\overline{\langle \mathcal{H} \rangle} \in \text{SSGP}(n-1)$ . Let  $\tilde{\mathcal{H}}$  be the family of subgroups of  $B$  given by  $\tilde{\mathcal{H}} := \{\pi[L] : L \in \mathcal{H}\}$ . Then  $\bigcup \tilde{\mathcal{H}} \subseteq (U)$ . Set  $H := \overline{\langle \mathcal{H} \rangle}$  and set



$\tilde{H} := \overline{\langle \cup \tilde{\mathcal{H}} \rangle}$ . Then  $\tilde{H}$  is normal in  $B$  since by assumption  $H$  is normal in  $G$ . By invoking the induction hypothesis we will show that  $B/\tilde{H} \in \text{SSGP}(n-1)$  and thus that  $B \in \text{SSGP}(n)$ . Note that  $H \subseteq \pi^{-1}[\tilde{H}]$  since  $\langle \cup \mathcal{H} \rangle \subseteq \pi^{-1}[\langle \cup \tilde{\mathcal{H}} \rangle]$ ; further,  $\pi^{-1}[\tilde{H}]$  is closed. We have then that  $G/H \in \text{SSGP}(n-1)$  so by induction,  $(G/H)/(\pi^{-1}[\tilde{H}]/H) \in \text{SSGP}(n-1)$  and this is topologically isomorphic with  $G/\pi^{-1}[\tilde{H}]$  by the second topological isomorphism theorem. Now, we claim that the algebraic isomorphism  $\tilde{\pi} : G/\pi^{-1}[\tilde{H}] \rightarrow B/\tilde{H}$  induced by  $\pi$  is continuous (though it may not be open). Clearly,  $\pi$  maps cosets of  $\pi^{-1}[\tilde{H}]$  to cosets of  $\tilde{H}$ . If  $\tilde{V}$  is an open union of cosets of  $\tilde{H}$ , then  $\pi^{-1}[\tilde{V}]$  is an open union of cosets of  $\pi^{-1}[\tilde{H}]$  and the claim follows. Again, by the induction hypothesis, since  $\tilde{\pi}$  is continuous and surjective, we now conclude that  $B/\tilde{H} \in \text{SSGP}(n-1)$  and thus  $B \in \text{SSGP}(n)$ , as required.

(c) Given  $G$  and  $K$  as hypothesized, let  $U \in \mathcal{N}(1_G)$ . Since  $U \cap K \in \mathcal{N}(1_K)$ , there is a family  $\mathcal{H}$  of subgroups of  $K$  such that  $\cup \mathcal{H} \subseteq U \cap K$  and  $K/H \in \text{SSGP}(n-1)$ , where  $H := \overline{\langle \cup \mathcal{H} \rangle}^K$ . Note that  $H = \overline{H}^G \cap K$ . Let  $\phi : K\overline{H}/\overline{H} \rightarrow K/(K \cap \overline{H})$  be the natural isomorphism from the first (algebraic) isomorphism theorem for groups. The corresponding theorem for topological groups says that  $\phi$  is an open map, i.e.,  $\phi^{-1}$  is a continuous map. Then from part (a) of this theorem,  $K\overline{H}/\overline{H} \in \text{SSGP}(n-1)$ . Now  $K\overline{H}/\overline{H}$  is dense in  $G/\overline{H}$ , because the subset of  $G$  that projects onto the closure of  $K\overline{H}/\overline{H}$  must be closed and must contain  $K\overline{H}$ . Then  $G/\overline{H} \in \text{SSGP}(n-1)$  by the induction hypothesis. Since  $\overline{H}^G = \overline{\langle \cup \mathcal{H} \rangle}^G$  we have  $G \in \text{SSGP}(n)$ , as required.

(d) Since  $\bigoplus_{i \in I} G_i$  is dense in  $\prod_{i \in I} G_i$ , it suffices by part (c) to treat the case  $G := \bigoplus_{i \in I} G_i$ . Let  $U \in \mathcal{N}_G(1_G)$ , say  $U = \bigoplus_{i \in I} U_i$  where  $U_i \in \mathcal{N}(1_{G_i})$  and  $U_i = G_i$  for  $i > N_U$ . Since each  $G_i \in \text{SSGP}(n)$ , we have, for each  $i \in I$ , a family  $\mathcal{H}_i$  of subgroups of  $G_i$  such that  $\cup \mathcal{H}_i \subseteq U_i$  and  $G_i/H_i \in \text{SSGP}(n-1)$  where  $H_i := \overline{\langle \cup \mathcal{H}_i \rangle}$ . Now consider the family of subgroups of  $G$  given by  $\mathcal{H} := \{L \subseteq G : L = \bigoplus_{i \in I} L_i \text{ with } L_i \in \mathcal{H}_i\}$ . Then  $\cup \mathcal{H} \subseteq U$ ,  $\langle \cup \mathcal{H} \rangle$  is identical to  $\bigoplus_{i \in I} \langle \cup \mathcal{H}_i \rangle$ , and  $H := \overline{\langle \cup \mathcal{H} \rangle}$  is identical to  $\bigoplus_{i \in I} H_i$ . We also have that  $G/H$  is topologically isomorphic with  $\bigoplus_{i \in I} G_i/H_i$  (cf. [24](6.9)). From the induction hypothesis we have  $G/H \in \text{SSGP}(n-1)$ , so  $G \in \text{SSGP}(n)$ , as required.  $\square$

We give a noteworthy consequence of Theorem 3.15(b).

**Theorem 3.16.** *Let  $G$  be a topological group which contains a proper open normal subgroup. Then there is no  $n < \omega$  such that  $G \in \text{SSGP}(n)$ .*

*Proof.* If  $G$  is a counterexample with proper open normal subgroup  $U$ , then by Theorem 3.15(b) we have  $G/U \in \text{SSGP}(n)$  with  $G/U$  discrete, contrary to Corollary 3.7.  $\square$

*Remark 3.17.* Certain other tempting statements of inheritance or permanence type, parallel in spirit to those considered in Theorem 3.15, do not hold in general. We give some examples.



- (a) We show below, using a construction of Hartman and Mycielski [22] and of Dierolf and Warken [6], that a closed subgroup of an SSGP group may lack the SSGP( $n$ ) property for every  $n < \omega$ . Indeed, every topological group can be embedded as a closed subgroup of an SSGP group (Theorem 3.20).
- (b) The conclusion of part (a) of Theorem 3.15 can fail when  $s < m + n$  replaces  $m + n$  in its statement. For example, the construction used in Lemma 4.5 shows that a topological group  $G \notin \text{SSGP}(n)$  may have a closed normal subgroup  $K \in \text{SSGP}(1)$  with also  $G/K \in \text{SSGP}(n)$ , so  $s = m + n$  is minimal when  $m = 1$ . We did not pursue the issue of minimality of  $m + n$  in Theorem 3.15(a) for arbitrary  $m, n > 1$ .
- (c) The converse to Theorem 3.15(c) can fail. In [19] a certain monothetic m.a.p. group constructed by Glasner [18] is shown to have SSGP, but we noted above in Corollary 3.14 that  $\mathbb{Z}$  admits an SSGP( $n$ ) topology for no  $n < \omega$ .

In contrast to that phenomenon, it should be mentioned that (as has been noted by many authors) in the context of m.a.p. groups, a dense subgroup  $H$  of a topological group  $G$  satisfies  $H \in \text{m.a.p.}$  if and only if  $G \in \text{m.a.p.}$  Thus in particular in the case of Glasner’s monothetic group, necessarily the dense subgroup  $\mathbb{Z}$  inherits an m.a.p. topology.

We now restrict our discussion to abelian groups and to the class  $\text{SSGP} = \text{SSGP}(1)$ , and examine which specific abelian groups do and do not admit an SSGP topology. We have already noted (Theorem 3.13) that the product of a finitely cogenerated abelian group with a finitely generated abelian group does not admit an SSGP topology even though it may admit an m.a.p. topology. We now give additional examples of abelian groups which admit not only an m.a.p. topology but also an SSGP topology.

**Theorem 3.18.** *The following abelian groups admit an SSGP topology.*

- (a)  $\mathbb{Q}$ , and those subgroups of  $\mathbb{Q}$  in which some primes are excluded from denominators, as long as an infinite number of primes and their powers are allowed;
- (b)  $\mathbb{Q}/\mathbb{Z}$  and  $\mathbb{Q}'/\mathbb{Z}$  where  $\mathbb{Q}'$  is a subgroup of  $\mathbb{Q}$  as in described in (a);
- (c) direct sums of the form  $\bigoplus_{i < \omega} \mathbb{Z}_{p_i}$  where the primes  $p_i$  all coincide or all differ;
- (d)  $\mathbb{Z}^{(\omega)}$  (the direct sum);
- (e)  $\mathbb{Z}^\omega$  (the full product);
- (f)  $G^{(\alpha)}$  for  $|G| > 1$  and  $\alpha \geq \omega$ ;
- (g)  $F^\alpha$  for  $1 < |F| < \omega$  and  $\alpha \geq \omega$ ;
- (h) arbitrary sums and products of groups which admit an SSGP topology.

Item (h) is a special case of Theorem 3.15(d), and item (e) is demonstrated in the second author’s paper [20]. The “coincide” case of item (c) follows from item (f), the “differ” case is established below in Theorem 4.9. Theorem 3.22((c) and (d)) below demonstrates the validity of item (f) for  $\omega \leq \alpha \leq \mathfrak{c}$ . This together

with (h) gives (d) and (f) in full generality. Item (g) then follows from the relation  $F^\alpha \simeq \bigoplus_{2^\alpha} F$  ([12](8.4, 8.5)). The remaining items are demonstrated in [19].

We note that from items (a) and (h) of Theorem 3.18 and the familiar algebraic structure theorem  $\mathbb{R} = \bigoplus_{\mathfrak{c}} \mathbb{Q}$  ([12](p.105), [24](A.14)) it follows that  $\mathbb{R}$  admits an SSGP topology.

There are many examples of nontrivial SSGP(1) groups (that is, of SSGP groups). It has been shown by Hartman and Mycielski [22] that every topological group  $G$  embeds as a closed subgroup into a connected, arcwise connected group  $G^*$ ; two decades later Dierolf and Warken [6], working independently and without reference to [22], found essentially the same embedding  $G \subseteq G^*$  and showed that  $G^* \in \text{m.a.p.}$ . Indeed the arguments of [6] show with minimal additional effort that  $G^* \in \text{SSGP}$  (of course with property SSGP not yet having been defined or named). We now describe the construction and we supply briefly the necessary details.

**Definition 3.19.** Let  $G$  be a Hausdorff topological group. Then algebraically  $G^*$  is the group of step functions  $f : [0, 1) \rightarrow G$  with finitely many steps, each of the form  $[a, b)$  with  $0 \leq a < b \leq 1$ . The group operation is pointwise multiplication in  $G$ . The topology  $\mathcal{T}$  on  $G^*$  is the topology generated by (basic) neighborhoods of the identity function  $1_{G^*} \in G^*$  of the form

$$N(U, \epsilon) := \{f \in G^* : \lambda(\{x \in [0, 1) : f(x) \notin U\}) < \epsilon\},$$

where  $\epsilon > 0$ ,  $U \in \mathcal{N}_G(1_G)$ , and  $\lambda$  denotes the usual Lebesgue measure on  $[0, 1)$ .

**Theorem 3.20.** *Let  $G$  be a topological group. Then*

- (a)  $G$  is closed in  $G^* = (G^*, \mathcal{T})$ ;
- (b)  $G^*$  is arcwise connected; and
- (c)  $G^* \in \text{SSGP}$ .

*Proof.* Note first that the association of each  $x \in G$  with the function  $x^* \in G^*$  (the function given by  $x^*(r) := x$  for all  $r \in [0, 1)$ ) realizes  $G$  algebraically as a subgroup of  $G^*$ . Furthermore the map  $x \rightarrow x^*$  is a homeomorphism onto its range, since for  $\epsilon < 1$ ,  $U \in \mathcal{N}_G(1_G)$  and  $x \in G$  one has

$$x \in U \Leftrightarrow x^* \in N(U, \epsilon).$$

- (a) Let  $f_0 \in G^*$  and  $f_0 \notin G$ . There are distinct (disjoint) subintervals of  $[0, 1)$  on which  $f_0$  assumes distinct values  $g_0, g_1 \in G$  respectively. By the Hausdorff property there is  $U \in \mathcal{N}_G(1_G)$  such that  $g_1U \cap g_2U = \emptyset$ . Choose  $\epsilon$  smaller than the measure of either of the two indicated intervals. Then  $f_0N(U, \epsilon)$  is a neighborhood of  $f_0$  such that  $f_0N(U, \epsilon) \cap G = \emptyset$ . Therefore,  $G$  is closed in  $G^*$ .
- (b) Let  $f \in G^*$  and for each  $t \in [0, 1)$  define  $f_t : [0, 1) \rightarrow G$  by  $f_t(x) = f(x)$  for  $0 \leq x < t$  and  $f_t(x) = 1_G$  for  $t \leq x < 1$ ; and define  $f_1 := f$ . Then  $t \mapsto f_t$  is a continuous map from  $[0, 1]$  to  $G^*$  such that  $f_0 = 1_{G^*}$  and  $f_1 = f$ . To show that the map is continuous, let  $f_tN(U, \epsilon)$  be a

basic neighborhood of  $f_t$  and let  $s \in (t - \epsilon/4, t + \epsilon/4) \cap [0, 1]$ . Then  $f_s \in f_t N(U, \epsilon)$ , since

$$\lambda(\{x \in [0, 1] : f_s(x) - f_t(x) \in U\}) < \epsilon.$$

We conclude that  $G^*$  is arcwise connected.

- (c) Let  $N(U, \epsilon) \in \mathcal{N}(1_{G^*})$ , and for each interval  $I = [t_0, t_1] \subseteq [0, 1]$  with  $t_1 - t_0 < \epsilon$  let

$$F(I) := \{f \in G^* : f \text{ is constant on } I, f \equiv 1_G \text{ on } [0, 1] \setminus I\}.$$

Then  $F(I)$  is a subgroup of  $G^*$  and  $F(I) \subseteq N(U, \epsilon)$ , and with  $\mathcal{H}_\epsilon := \{F(I)\}$  we have that each  $f \in G^*$  is the product of finitely many elements from  $\bigcup \mathcal{H}_\epsilon$ —i.e.,  $f \in \langle \bigcup \mathcal{H}_\epsilon \rangle \subseteq \overline{\langle \bigcup \mathcal{H}_\epsilon \rangle}$ . It follows that  $G^* \in \text{SSGP}$ , as asserted. □

For later use we identify certain subgroups  $G_A^*$  of  $G^*$  that retain properties (a) and (c) (but not (b)) of Theorem 3.20. When  $G$  and  $A$  are countable the group  $G_A^*$  also is countable. An equivalent definition and some related consequences can also be found in [7] and in [19] (Definition 2.3.1). Here is the relevant definition.

**Definition 3.21.** Let  $G$  be a topological group and let  $A \subseteq [0, 1]$  where  $A$  is dense in  $[0, 1]$  and  $0 \in A$ . Then  $G_A^* = (G_A^*, \mathcal{T})$  is the subgroup of  $(G^*, \mathcal{T})$  obtained by restriction of step functions on  $[0, 1]$  to those steps  $[a, b)$  such that  $a, b \in A \cup \{1\}$ ,  $a < b$ .

**Theorem 3.22.** Let  $G$  be a topological group. Then

- (a)  $G$  is closed in  $G_A^* = (G_A^*, \mathcal{T})$ ;
- (b)  $G_A^*$  is dense in  $G^*$ ;
- (c)  $G_A^* \in \text{SSGP}$ ; and
- (d) if  $G$  is abelian, then the groups  $G_A^*, G^{(\alpha)}$  (with  $\alpha = |A|$ ) are isomorphic as groups.

*Proof.* With the obvious required change, the proofs of (a) and (c) coincide with the corresponding proofs in Theorem 3.20.

- (b) Let  $f \in G^*$  have  $n$  steps ( $n < \omega$ ) and let  $f \cdot N(U, \epsilon) \in \mathcal{N}_{G^*}(f)$ . Then there is  $\tilde{f} \in f \cdot N(U, \epsilon) \cap G_A^*$  such that  $\tilde{f}$  has step end-points in  $A \cup \{1\}$ , each within  $\epsilon/n$  of the corresponding end-point for  $f$ .
- (d) We give an explicit isomorphism.  $G^{(\alpha)}$  can be expressed as the set of functions  $\phi : A \rightarrow G$  with finite support and pointwise addition. Each such function is the sum of finitely many elements of the form  $\phi_{a,g}$  with  $a \in A$ ,  $g \in G$ ,  $\phi_{a,g}(a) = g$  and  $\phi_{a,g}(x) = 0$  for  $x \neq a$ . Now we define corresponding functions  $f_{a,g} \in G_A^*$ . Let  $f_{0,g}(x) = g$  for all  $x \in [0, 1]$ , and for  $a > 0$  let  $f_{a,g}$  be the two-step function defined by  $f_{a,g}(x) = g$  for  $0 \leq x < a$  and  $f_{a,g}(x) = 0$  for  $a \leq x < 1$ . Then the map  $\phi_{a,g} \mapsto f_{a,g}$  extends linearly to an isomorphism from  $G^{(\alpha)}$  onto  $G_A^*$ .

□

*Remark 3.23.* Note that Theorem 3.15(c) cannot be used to prove (c) from (b) in Theorem 3.22. Note also that the isomorphism given in the proof of (d) provides a way of imposing an SSGP topology on  $G^{(\alpha)}$  for  $\omega \leq \alpha \leq c$  and  $G$  an abelian group. When  $G$  is nonabelian the corresponding mapping still exists but it need not be an isomorphism. In that case, each element of  $G_A^*$  is a product of two-step functions, but the order in which they are multiplied can affect the product.

Some other SSGP groups arise as a consequence of the following fact.

**Theorem 3.24.** *Let  $G = (G, \mathcal{T})$  be a (possibly nonabelian) torsion group of bounded order such that  $(G, \mathcal{T})$  has no proper open subgroup. Then  $G \in \text{SSGP}$ .*

*Proof.* There is an integer  $M$  which bounds the order of each  $x \in G$ , and then  $N := M!$  satisfies  $x^N = 1_G$  for each  $x \in G$ .

We must show: Each  $U \in \mathcal{N}(1_G)$  contains a family  $\mathcal{H}$  of subgroups such that  $\langle \bigcup \mathcal{H} \rangle$  is dense in  $G$ . Given such  $U$ , let  $V \in \mathcal{N}(1_G)$  satisfy  $V^N \subseteq U$ . For each  $x \in V$  we have  $x^k \in U$  for  $0 \leq k \leq N$ , hence  $x \in V \Rightarrow \langle x \rangle \subseteq U$ . Thus with  $\mathcal{H} := \{ \langle x \rangle : x \in V \}$  we have:  $\mathcal{H}$  is a family of subgroups of  $U$  (that is, of subsets of  $U$  which are subgroups of  $G$ ). Then  $V \subseteq \bigcup \mathcal{H}$ , so  $G = \langle V \rangle \subseteq \langle \bigcup \mathcal{H} \rangle$ —the first equality because  $\langle V \rangle$  is an open subgroup of  $G$ . □

In Corollaries 3.25 and 3.28 we record two consequences of Theorem 3.24.

**Corollary 3.25.** *If  $(G, \mathcal{T})$  is a (possibly nonabelian) connected torsion group of bounded order, then  $(G, \mathcal{T}) \in \text{SSGP}$ .*

*Proof.* A connected group has no proper open subgroup, so Theorem 3.24 applies. □

**Lemma 3.26.** *Let  $G \in \text{m.a.p.}$  and  $G$  abelian. Then  $G$  does not contain a proper open subgroup.*

*Proof.* Suppose that  $H$  is a proper open subgroup of  $G$ . Since  $G/H$  is a nontrivial abelian discrete (and therefore locally compact) group, there is a nontrivial (continuous) homomorphism  $\phi : G/H \rightarrow \mathbb{T}$ . Then the composition of  $\phi$  with the projection map from  $G$  to  $G/H$  is a nontrivial continuous homomorphism from  $G$  to a compact group, contradicting the m.a.p. property of  $G$ . □

*Remark 3.27.* We are grateful to Dikran Dikranjan for the helpful reminder that Lemma 3.26 fails when the “abelian” hypothesis is omitted. Examples to this effect abound, samples including: (a) the infinite algebraically simple groups whose only group topology is the discrete topology, as concocted by Shelah [38] under [CH], and by Hesse [23] and Ol’shanskii [30] (and later by several others) in [ZFC]; and (b) such matrix groups as  $SL(2, \mathbb{C})$ , shown by von Neumann [27] to be m.a.p. even in the discrete topology (the later treatments [28], [24](22.22(h)) and [2](9.11) of this specific group follow closely those of [27]). In connection with this comment, note however that Theorem 3.16 above is an appropriate nonabelian analogue of Lemma 3.26 for SSGP( $n$ ) groups.

**Corollary 3.28.** *For an abelian torsion group  $G$  of bounded order, these conditions are equivalent for each group topology  $\mathcal{T}$  on  $G$ .*

- (a)  $(G, \mathcal{T}) \in \text{SSGP}$ ;
- (b)  $(G, \mathcal{T}) \in \text{m.a.p.}$ ; and
- (c)  $(G, \mathcal{T})$  has no proper open subgroup.

*Proof.* The implications (a)  $\Rightarrow$  (b), (b)  $\Rightarrow$  (c), and (c)  $\Rightarrow$  (a) are given respectively by Theorem 3.5, Lemma 3.26, and Theorem 3.24.  $\square$

*Remark 3.29.* It is worthwhile to note that connected torsion groups of bounded order, as hypothesized in Theorem 3.25, do exist. Here we give two quite different proofs that for  $0 < n < \omega$  there is a nontrivial connected torsion group of exponent  $n$ . Of these (A), as remarked by the referee, uses the construction given in Theorem 3.20; while (B), drawing freely on the expositions [37] and [2](2.3–2.4), derives from the “free topological group” constructions first given by Markov [25], [26] and Graev [21].

- (A) Let  $G$  be a group of exponent  $n$  (for example,  $G = \mathbb{Z}(n)$ ) and define  $G^*$  as in Definition 3.19. Since algebraically  $G^* \subseteq G^{(0,1)}$ , also  $G^*$  has exponent  $n$ ; and  $G^*$  is connected by Theorem 3.20(b).
- (B) Let  $X$  be a Tychonoff space and let

$$G := \{\sum_{i=1}^N k_i x_i : k_i \in \mathbb{Z}, N < \omega, x_i \in X\}$$

be the free abelian topological group on the alphabet  $X$  with  $0_G = 0$ , and for continuous  $f : X \rightarrow H$  with  $H$  a topological abelian group define  $\bar{f} : G \rightarrow H$  by  $\bar{f}(\sum_{i=1}^N k_i x_i) = \sum_{i=1}^N k_i f(x_i) \in H$ . It is easily checked, as in the sources cited, that (a) in the (smallest) topology  $\mathcal{T}$  making each such  $\bar{f}$  continuous,  $(G, \mathcal{T})$  is a (Hausdorff) topological group; (b) the map  $x \rightarrow 1 \cdot x$  from  $X$  to  $G$  maps  $X$  homeomorphically onto a closed topological subgroup of  $G$ ; and (c)  $G$  is connected if (and only if)  $X$  is connected.

Now take  $X$  compact connected and fix  $n$  such that  $0 < n < \omega$ . It suffices to show that (1)  $nX$  is a proper closed subset of  $G$ , and (2) every proper closed subset  $F \subseteq X$  generates a proper closed subgroup  $\langle F \rangle$  of  $(G, \mathcal{T})$ ; for then the group  $G/\langle nX \rangle$  will be as desired, since  $a \in G \Rightarrow na \in \langle nX \rangle$ .

- (1)  $nX$  is compact in  $G$ , hence closed. Define  $f_0 : X \rightarrow \mathbb{R}$  by  $f_0 \equiv 1$ ; then  $\bar{f}_0 \equiv n$  on  $nX$ , while for  $x \in X$  we have  $\bar{f}_0((n+1)x) = n+1$ , so  $(n+1)x \notin nX$ .
- (2) Given  $x \in X \setminus F$  choose continuous  $f_1 : X \rightarrow \mathbb{R}$  such that  $f_1(x) = 1$ ,  $f_1 \equiv 0$  on  $F$ . Then  $\bar{f}_1 \equiv 0$  on  $\langle F \rangle$  and  $\bar{f}_1(x) = 1$ , so  $x = 1 \cdot x \notin \langle F \rangle$ ; so  $\langle F \rangle$  is proper in  $G$ . If  $a = \sum_{i=1}^N k_i x_i \in G \setminus \langle F \rangle$  there is  $i_0$  such that  $x_{i_0} \notin F$ , and with continuous  $f_2 : X \rightarrow \mathbb{R}$  such that  $f_2(x_{i_0}) = 1$ ,  $f_2(x_i) = 0$  for  $i \neq i_0$  and  $f_2 \equiv 0$  on  $F$  we have  $\bar{f}_2(a) = k_{i_0}$  and  $\bar{f}_2 \equiv 0$  on  $\langle F \rangle$ . Then  $U := \bar{f}_2^{-1}(k_{i_0} - 1/3, k_{i_0} + 1/3) \in \mathcal{N}_G(a)$  and  $U \cap \langle F \rangle = \emptyset$ ; so  $\langle F \rangle$  is closed in  $G$ .

#### 4. SSGP(N) GROUPS: SOME SPECIFICS

The question naturally arises whether for  $n < \omega$  the class-theoretic inclusion  $\text{SSGP}(n) \subseteq \text{SSGP}(n + 1)$  is proper. That issue is addressed below in Theorem 4.6. *En route* to that we describe one of the earliest examples of an abelian m.a.p. group, constructed by Prodanov [33]; his proof illustrates use of the  $\text{SSGP}(2)$  property to prove m.a.p. (Of course, the classes  $\text{SSGP}(n)$  had not been formally defined in 1980.) We credit the referee with posing a useful question which allowed us eventually to prove that Prodanov’s group  $(G, \mathcal{T})$  satisfies not only  $(G, \mathcal{T}) \in \text{SSGP}(2)$  (Theorem 4.2) but even  $(G, \mathcal{T}) \in \text{SSGP}(1)$  (Theorem 4.3); this corrects a misstatement given in [19] and in an early version of this paper.

Algebraically, Prodanov’s group  $G$  is the group  $G := \mathbb{Z}^{(\omega)} = \bigoplus_{\omega} \mathbb{Z}$ . We begin our verification that  $(G, \mathcal{T})$  is as desired by quoting directly from Prodanov [33]. For this, denote by  $\{e_m : 1 \leq m < \omega\}$  the canonical basis for  $G$  and use induction to define a sequence of finite subsets of  $\mathbb{Z}^{(\omega)}$ :

“Let  $A_1 = \{e_1 - e_2, e_2\}$ , and suppose that the sets  $A_1, A_2, \dots, A_{m-1}$  ( $m = 2, 3, \dots$ ) are already defined. By  $\alpha_m$  we denote an integer so large that the  $s$ -th co-ordinates of all elements of  $A_1 \cup A_2 \cup \dots \cup A_{m-1}$  are zero for  $s \geq \alpha_m$ . Now we define  $A_m$  to consist of all differences

$$(1) e_{i+k\alpha_m} - e_{i+(k+1)\alpha_m} \quad (1 \leq i \leq m, 0 \leq k \leq 2^{m-1} - 1)$$

and of the elements

$$(2) e_{i+2^{m-1}\alpha_m} \quad (1 \leq i \leq m).$$

Thus the sequence  $\{A_m\}_{m=1}^{\infty}$  is defined.

Now for arbitrary  $n \geq 1$  we define

$$(3) U_n := (n + 1)! \mathbb{Z}^{(\omega)} \pm A_n \pm 2A_{n+1} \pm \dots \pm 2^l A_{n+l} \pm \dots$$

(By the notation of (3) Prodanov means that  $U_n$  consists of those elements of  $\mathbb{Z}^{(\omega)}$  which can be represented as a finite sum consisting of an element divisible by  $(n + 1)!$  plus at most one element of  $A_n$  with arbitrary sign, plus at most two elements of  $A_{n+1}$  with arbitrary signs, plus at most four elements of  $A_{n+2}$  with arbitrary signs, and so on.)

“It follows directly from that definition that the sets  $U_n$  are symmetric with respect to 0, and that  $U_{n+1} + U_{n+1} \subset U_n$  ( $n = 1, 2, \dots$ ). Therefore they form a fundamental system of neighborhoods of 0 for a group topology  $\mathcal{T}$  on  $\mathbb{Z}^{(\omega)}$ .”

Since we need it later, we give a careful proof of an additional fact outlined only briefly by Prodanov [33].

**Theorem 4.1.** *The group  $\mathbb{Z}^{(\omega)}$  with the group topology  $\mathcal{T}$  defined above is Hausdorff.*

*Proof.* It suffices to show  $\bigcap_{n < \omega} U_n = \{0\}$ .

Let  $0 \neq g = \sum_i a_i e_i$  where the  $e_i$  form the canonical basis. Then there is a least integer  $r$  such that  $a_i = 0$  for  $i \geq r + 1$ , and there is a least integer  $s$  such that  $(s + 1)!$  does not divide  $g$ . Let  $n := \max(r, s)$ . We claim that if  $n > 0$  then  $g \notin U_n$ . Suppose otherwise. Since  $(n + 1)!$  does not divide  $g$ , there is some  $p \leq n$  such that  $(n + 1)!$  does not divide  $a_p$ . Thus any representation of  $g$  in the form (3) must include, for at least one  $m > n$ , one or more terms of the form  $\pm(e_p - e_{p+\alpha_m})$ , all with the same sign. This means that the components  $\pm e_{p+\alpha_m}$  must be cancelled by the same components from additional terms of the form  $\pm(e_{p+\alpha_m} - e_{p+2\alpha_m})$ . This chain of implications continues until  $k$  reaches its maximum value,  $2^{m-1} - 1$ , with the inclusion of terms  $\pm(e_{p+(2^{m-1}-1)\alpha_m} - e_{p+2^{m-1}\alpha_m})$ . Finally, the components  $\pm e_{p+2^{m-1}\alpha_m}$  must be cancelled by terms of type (2) with  $i = p$  and the same value of  $m$ . This means that we have necessarily included at least  $2^{m-1} + 1$  elements from  $A_m$  in our expansion of  $g$ , contradicting the requirement that no more than  $2^{m-n}$  elements of  $A_m$  be included as summands for such representations of  $g \in U_n$ .  $\square$

**Theorem 4.2.** *Prodanov's group  $(G, \mathcal{T})$  satisfies  $(G, \mathcal{T}) \in \text{SSGP}(2)$ .*

*Proof.* To see that  $(\mathbb{Z}^{(\omega)}, \mathcal{T}) \in \text{SSGP}(2)$ , we show first that each  $U_n$  generates  $\mathbb{Z}^{(\omega)}$ . It suffices to show that each  $e_i \in \langle U_n \rangle$  ( $1 \leq i < \omega$ ). Choosing  $m$  such that  $m \geq i$  and  $m \geq n$ , we have from (1) and (2) above that

$$e_i = [\sum_{k=0}^{2^{m-1}-1} (e_{i+k\alpha_m} - e_{i+(k+1)\alpha_m})] + e_{i+2^{m-1}\alpha_m} \in \langle A_m \rangle \subseteq \langle U_m \rangle \subseteq \langle U_n \rangle.$$

Thus  $(\mathbb{Z}^{(\omega)}, \mathcal{T})$  has no proper open subgroups, so with  $H_n := (n + 1)!\mathbb{Z}^{(\omega)}$  and  $G_n := \mathbb{Z}^{(\omega)} / \overline{H_n}$  we have that  $H_n \subseteq U_n$ ,  $G_n$  is of bounded order, and also  $G_n$  has no proper open subgroups. Then  $G_n \in \text{SSGP}$  by Theorem 3.24, so  $(\mathbb{Z}^{(\omega)}, \mathcal{T}) \in \text{SSGP}(2)$ .  $\square$

Prodanov proves explicitly that  $(G, \mathcal{T}) \in \text{m.a.p.}$ . Theorem 4.2 exploits his argument insofar as it applies to our more demanding context. Then of course, the following Theorem 4.3 strengthens Theorem 4.2.

**Theorem 4.3.** *Prodanov's group  $(G, \mathcal{T})$  satisfies  $(G, \mathcal{T}) \in \text{SSGP}(1)$ .*

*Proof.* Since we have already shown that  $(G, \mathcal{T})$  is Hausdorff, it remains only to show that every  $U_n \in \mathcal{N}_{0_G}$  contains a family  $\mathcal{H}$  of subgroups such that  $G = \overline{\langle \mathcal{H} \rangle}$ . In fact, we show that  $\mathcal{H}$  may be chosen so that  $\langle \bigcup \mathcal{H} \rangle = G$ . With  $n$  given, choose  $k$  such that  $2^k \geq (n + 1)! - 1$  and let  $x \in A_m$  with  $m := n + k$ . We claim that  $\langle x \rangle \subseteq U_n$ . Since  $U_n$  contains elements formed from sums which include up to  $2^k$  elements from  $A_{n+k}$ , each  $y \in \langle x \rangle$  can be written in the form  $y = rx + sx$  with  $r, s \in \mathbb{Z}$ , where  $rx \in (n + 1)!\mathbb{Z}^{(\omega)}$  and  $sx \in 2^{m-n}A_m$ . Since  $e_i \in \langle A_m \rangle$  for all  $i$  and  $m$  (by the proof of Theorem 4.2), we have that  $e_i \in \langle \bigcup \mathcal{H} \rangle$  for every  $i < \omega$ , where  $\mathcal{H}$  includes all subgroups of the form  $\langle x \rangle$  with  $x \in A_m$  and  $2^{m-n} \geq (n + 1)! - 1$ .  $\square$



Now as promised we construct a family of topological groups demonstrating that the class-theoretic inclusion  $SSGP(n) \subseteq SSGP(n + 1)$  is proper for each  $n$ . For  $n = 0$  this is clear since we already have many examples of nontrivial groups in the class  $SSGP(1)$ . The strategy is to find particular groups  $G_n \in SSGP(n)$  and  $H \in SSGP(1)$  and then construct a topology for  $H \oplus G_n$  in such a way that (a)  $H$  with its original topology is a closed normal subgroup, (b)  $(H \oplus G_n)/H$  is topologically isomorphic with  $G_n$ , (c)  $H \oplus G_n \notin SSGP(n)$ , and (d)  $H \oplus G_n \in SSGP(n + 1)$ . Such a construction in the case  $n = 1$  was given in the second author’s dissertation [19]; our task here is to generalize that construction to arbitrary  $n$ .

The properties of  $G_n$  that will be required in the induction step are that  $G_n = (G_n, \mathcal{T}_n)$  is abelian, countable and torsionfree, with the group topology  $\mathcal{T}_n$  defined by a metric. For convenience, we assume that the maximum distance is 1. These properties are satisfied in the case  $n = 0$  by the trivial group, but it is more illuminating to begin the induction with  $G_1$  rather than  $G_0$ .

Let  $H$  be the topological group  $\mathbb{Z}_A^*$  as in Definition 3.21, where  $\mathbb{Z}$  has the discrete topology and

$$A := \{x \in [0, 1) : x = \frac{t}{2^m} \text{ for } m, t < \omega \text{ and } 0 \leq t < 2^m\}.$$

Set  $G_1 = H$ . It is clear that  $G_1$  is abelian, countable and torsionfree, and is not the trivial group.  $G_1$  also has  $SSGP(1)$  (Theorem 3.22). The topology on  $H = G_1 = \mathbb{Z}_A^*$ , defined as in Definitions 3.19 and 3.21 can be seen as a metric topology given by the norm  $\|h\| := \lambda(Supp(h))$  for  $h \in H$ , where  $Supp(h)$  is the support of  $h$  as a function on  $[0, 1)$ . (Here the “norm” designation follows historical precedent; we use it both out of respect and for convenience, but we do not require that  $\|Ng\| = |N| \cdot \|g\|$ .)

Fix  $n > 1$  and suppose there is a countable, torsionfree abelian group  $G_{n-1}$  with a metric  $\rho$  that defines a group topology on  $G_{n-1}$  such that  $G_{n-1} \in SSGP(n - 1)$  and  $G_{n-1} \notin SSGP(n - 2)$ . Now, define (algebraically)  $G_n := H \oplus G_{n-1}$ ; we give  $G_n$  a metric topology  $\mathcal{T}_n$  which is different from the product topology, using a technique taken from M. Ajtai, I. Havas, and J. Komlós [1]. We create a metric group topology on  $G_n$  starting with a function  $\nu : S \rightarrow \mathbb{R}^+$ , where  $S$  is a specified generating set for  $G_n$  with  $0 \notin S$ .  $S$  will typically be highly redundant as a generating set. We refer to  $\nu$  together with the generating set  $S$  as a “provisional norm” (in terms of which a norm on  $G_n$  will be defined). For  $x \in G_n$  we write  $x = (h, g)$  with  $h \in H$  and  $g \in G_{n-1}$ .

We designate a double sequence of generating functions  $e_{m,t} \in H$  for  $m, t < \omega$  and  $t < 2^m$ :

$$e_{m,t}(x) = 1 \text{ for } x \in \left[\frac{t}{2^m}, \frac{t+1}{2^m}\right) \quad \text{and} \quad e_{m,t}(x) = 0 \text{ otherwise.}$$

We note that  $\|p \cdot e_{m,t}\| = \frac{1}{2^m}$  for all  $p \in \mathbb{Z}$ ,  $p \neq 0$ . We also name a basic set of neighborhoods of  $0 \in G_{n-1}$  and we label all the non-zero elements in each neighborhood:

$$U_m := \{g \in G_{n-1} : \|g\| \leq \frac{1}{2^m}\} \text{ for } m < \omega$$

$$U_m \setminus \{0\} = \{g_{m,t} : t < \omega\}.$$

In addition, let

$$r(m, t) : \omega \times \omega \rightarrow \omega$$

be an arbitrary bijection. We set  $S$  to be the collection of elements of the following two types:

- (1)  $(p \cdot e_{m,t}, 0)$  for  $p \in \mathbb{Z} \setminus \{0\}$  and  $m, t < \omega$  with  $t < 2^m$
- (2)  $(f_r, g_{m,t})$  for  $m < \omega, 0 \leq t < \omega$  and  $r = r(m, t)$ ,

$$\text{where } f_r = \sum_{i=0}^{2^{r-1}-1} e_{r,2^i}.$$

The set of functions  $f_r$  are linearly independent, so the set of elements of type (2) is also linearly independent. We use that fact along with the fact that each  $f_r$  has support of measure  $\frac{1}{2}$ . Now, we make the provisional norm assignments,

- (1)  $\nu((p \cdot e_{m,t}, 0)) = \|p \cdot e_{m,t}\|_H = \frac{1}{2^m}$
- (2)  $\nu((f_r, g_{m,t})) = \|g_{m,t}\|_{G_{n-1}} \leq \frac{1}{2^m}$  with  $r = r(m, t)$ .

Notice that (1) gives the same provisional norm to every nonzero element in a subgroup of  $G_n$ , whereas the assignments given by (2) are for a linearly independent set of elements of  $G_n$ .

Now we define a seminorm  $\|\cdot\|$  on  $G_n$  in terms of the provisional norm  $\nu$ .

**Definition 4.4.** For  $g \in G_n$ ,

$$\|g\| := \inf \left\{ \sum_{i=1}^N |\alpha_i| \nu(s_i) : g = \sum_{i=1}^N \alpha_i s_i, s_i \in S, \alpha_i \in \mathbb{Z}, N < \omega \right\} \cup \{1\}.$$

This defines a seminorm because  $S$  generates  $G_n$  and because the use of the infimum in the definition guarantees that the triangle inequality will be satisfied. Therefore, the neighborhoods of 0 defined by this seminorm will generate a (possibly non-Hausdorff) group topology on  $G_n$ .

Now in Lemma 4.5 we use the notation and definition just introduced.

- Lemma 4.5.** (a)  $G_n$  is a torsionfree, countable abelian group;  
 (b) the seminorm on  $G_n$  is a norm (resulting in a Hausdorff metric);  
 (c)  $G_n \in \text{SSGP}(n)$ ; and  
 (d)  $G_n \notin \text{SSGP}(n - 1)$ .

*Proof.* (a) is clear.

(b) To show that  $\|\cdot\|$  is a norm on  $G_n$ , we need to show that for  $0 \neq x \in G_n$  we have  $\|x\| > 0$ . Let  $x = (h, g)$ . If  $g \neq 0$  then an expansion of  $(h, g)$  by elements of  $S$  must include at least one element of type (2). For those elements, we have  $\nu((f_r, g)) = \|g\|$ . Because the metric on  $G_{n-1}$  satisfies the triangle inequality, any expansion of  $(h, g)$  by elements of  $S$  must yield a value of  $\|g\|$  or greater for the expression within the curly brackets in Definition 4.4. We conclude that  $\|(h, g)\|_{G_n} \geq \|g\|_{G_{n-1}}$ . On the other hand, if  $g = 0$  then there is an expression for  $(h, 0)$  in terms of elements of  $S$  of type  $(p \cdot e_{m,t}, 0)$  such

that  $\sum_{i=1}^N |\alpha_i| \nu(s_i) = \|h\| = \lambda(\text{supp}(h))$  and this value is minimal. If, instead, the expansion includes elements of type  $s = (f_r, g_{m,t})$  then there is a minimal  $\nu$ -value such a term can have. This is because there is a minimal size,  $\frac{1}{2^M}$ , for an interval on which  $h$  is constant. An expansion of  $(h, 0)$  by elements of  $S$  that includes an element  $s = (f_r, g_{m,t})$  such that  $r(m,t) > M$  would also have to include  $2^{r-1}$  elements of  $S$  of the form  $(e_{r,i}, 0)$ , each with coefficient  $-1$ . The contribution of these terms to the sum  $\sum_{i=1}^N |\alpha_i| \nu(s_i)$  is greater than or equal to  $\frac{1}{2}$ . Such an expansion cannot affect the infimum. So if  $\|(h, 0)\| \neq \|h\|_H$  then  $\|(h, 0)\| \geq \min(\{\|g_{m,t}\|_{G_n} : r(m,t) < M\})$ . We conclude that  $\|(h, g)\|$  is bounded away from 0 except when  $(h, g) = (0, 0)$ .

(c) We show next that  $G_n \in \text{SSGP}(n)$ . We claim first that the subgroup  $H \times \{0\}$  of  $G_n$  is an SSGP group in the topology inherited from  $G_n$ . This is clear because from the provisional norm assignment,  $\nu((p \cdot e_{m,t}, 0)) = \|p \cdot e_{m,t}\|_H$ , it follows that  $\|(h, 0)\| \leq \|h\|_H$  for each  $h \in H$ , so any  $\epsilon$ -neighborhood of  $(0, 0)$  contains a family  $\mathcal{H}$  of subgroups such that  $(\bigcup \mathcal{H}) = H \times \{0\} = \{(h, 0) : h \in H\}$ . We will show that the quotient topology for  $G_n/(H \times \{0\})$  coincides with the original topology for  $G_{n-1}$  (which also implies that  $H \times \{0\}$  is closed in  $G_n$ ). For each  $g \in G_{n-1}$  there is an  $h \in H$  such that  $\|(h, g)\| = \|g\|_{G_{n-1}}$ , namely  $h = f_r$  where  $g = g_{m,t}$  and  $r = r(m,t)$ . (There are, in general, many such pairs  $m, t$  and corresponding  $f_r$ .) On the other hand, as we showed above,  $\|(h, g)\| \geq \|g\|_{G_{n-1}}$  for each  $h \in H$ . We conclude that  $g$  is in the  $\epsilon$ -neighborhood of  $0 \in G_{n-1}$  if and only if there is  $h \in H$  such that  $(h, g)$  is in the  $\epsilon$ -neighborhood of  $(0, 0) \in G_n$ . In other words, the neighborhoods of 0 in  $G_{n-1}$  coincide with the projections onto  $G_n/(H \times \{0\})$  of the neighborhoods of  $(0, 0)$  in  $G_n$ . Thus the topologies of  $G_n/(H \times \{0\})$  and  $G_{n-1}$  coincide. (Note, however, that the *subgroup* topology on  $G_{n-1}$  does *not* coincide with its original topology.) Since by assumption  $G_{n-1} \in \text{SSGP}(n-1)$ , we have indeed that  $G_n \in \text{SSGP}(n)$ .

(d) It remains to show  $G_n \notin \text{SSGP}(n-1)$ . Suppose the contrary. Then every  $\epsilon$ -neighborhood  $U_\epsilon$  of  $(0, 0) \in G_n$  (say with  $\epsilon < \frac{1}{4}$ ) contains a family  $\mathcal{K}_\epsilon$  of subgroups such that  $G_n/\overline{(\bigcup \mathcal{K}_\epsilon)} \in \text{SSGP}(n-2)$ . Let  $G \in \mathcal{K}_\epsilon$  and  $(h, g) \in G$  with  $g \neq 0$ . For  $|N| < \omega$  we must have  $\|(Nh, Ng)\| < \epsilon$ , so each  $(Nh, Ng)$  has an expansion

$$\begin{aligned} (Nh, Ng) &= \sum_{i=1}^{M_N} \alpha_{N,i} (h_i, 0) + \sum_{j=1}^{L_N} \beta_{N,i} (h'_i, g_i) \quad \text{such that} \\ \sum_{i=1}^{M_N} \eta(\alpha_{N,i}) \nu((h_i, 0)) + \sum_{j=1}^{L_N} |\beta_{N,i}| \nu((h'_i, g_i)) &< \epsilon \end{aligned}$$

where each  $(h_i, 0), (h'_i, g_i) \in S$  and where  $\eta : \mathbb{Z} \rightarrow \{0, 1\}$  is defined by  $\eta(k) := 1$  when  $k \neq 0$  and  $\eta(k) := 0$  when  $k = 0$ .

We consider two cases.

Case 1. In the expansion above, each coefficient  $\beta_{N,i}$  has the form  $\beta_{N,i} = N\beta_{1,i}$ . Then clearly for sufficiently large  $|N|$  we have  $\|(Nh, Ng)\| > \frac{1}{4} > \epsilon$ .

Case 2. There is  $N < \omega$  for which, for some  $k$ , the expansion for  $(Nh, Ng)$  is such that  $\beta_{N,k} \neq N\beta_{1,k}$ . For the  $H$ -component of the given expansion of  $(Nh, Ng)$ , we can write

$$Nh = \sum_{i=1}^M \alpha_{N,i} h_i + \sum_{i=1}^L \beta_{N,i} h'_i$$

where  $M = \max(M_1, M_N)$  and  $L = \max(L_1, L_N)$ . Multiplying the specified expansion of  $(h, g)$  by the number  $N$ , we also have

$$Nh = \sum_{i=1}^M N\alpha_{1,i} h_i + \sum_{i=1}^L N\beta_{1,i} h'_i.$$

Equating the two expansions and rearranging, we can write

$$\sum_{j=1}^L (\beta_{N,j} - N\beta_{1,j}) h'_j = \sum_{i=1}^M (N\alpha_{1,i} - \alpha_{N,i}) h_i$$

where, for the specified index  $k$ , we have  $(\beta_{N,k} - N\beta_{1,k}) h'_j \neq 0$ . Since the  $h'_j$  are linearly independent, each  $h'_j$  that has a nonzero coefficient in the expression above must be balanced by terms on the right. This implies that  $\sum_{i=1}^M \eta(N\alpha_{1,i} - \alpha_{N,i}) \geq \frac{1}{2}$ , which in turn means that either  $\sum_{i=1}^M \eta(\alpha_{1,i}) \geq \frac{1}{4}$  or  $\sum_{i=1}^M \eta(\alpha_{N,i}) \geq \frac{1}{4}$ . We conclude that  $\|(h, g)\| \geq \frac{1}{4} > \epsilon$  or  $\|(Nh, Ng)\| \geq \frac{1}{4} > \epsilon$ , contradicting  $G \in \mathcal{K}_\epsilon$ .

We conclude that for  $\epsilon < \frac{1}{4}$  we must have  $\bigcup \mathcal{K}_\epsilon \subseteq H$ , so that  $\overline{\langle \bigcup \mathcal{K}_\epsilon \rangle}$  is a closed subgroup of  $H$ . In such cases we have  $G_n / \overline{\langle \bigcup \mathcal{K}_\epsilon \rangle} \notin \text{SSGP}(n-2)$  because, by Theorem 3.15(b),  $G_n / \overline{\langle \bigcup \mathcal{K}_\epsilon \rangle} \in \text{SSGP}(n-2)$  would imply that  $(G_n / \overline{\langle \bigcup \mathcal{K}_\epsilon \rangle}) / (H / \overline{\langle \bigcup \mathcal{K}_\epsilon \rangle}) \in \text{SSGP}(n-2)$  or, equivalently, that  $G_n / H \simeq G_{n-1} \in \text{SSGP}(n-2)$ , contrary to the induction assumption for  $G_{n-1}$ .  $\square$

We emphasize the essential content of Lemma 4.5.

**Theorem 4.6.** *For  $1 \leq n < \omega$  there is an abelian topological group  $G$  such that  $G \in \text{SSGP}(n)$  and  $G \notin \text{SSGP}(n-1)$ .*

In the paragraph following Theorem 3.9 we noted the existence of abelian topological groups  $G \in \text{m.a.p.}$  such that  $G \in \text{SSGP}(n)$  for no  $n < \omega$ . ( $\mathbb{Z}$  and  $\mathbb{Z}(p^\infty)$ , appropriately topologized, are examples.) In the following corollary we note the availability of other examples to the same effect.

**Corollary 4.7.** *There is a group  $G$  of the form  $G = \prod_{k < \omega} G_k$  such that*

- (a) *for each  $k < \omega$  there is  $n_k < \omega$  such that  $G_k \in \text{SSGP}(n_k)$ ;*
- (b) *there is no  $n$  such that  $G \in \text{SSGP}(n)$ .*

*Proof.* Using Theorem 4.6, for  $k < \omega$  choose  $G_k \in \text{SSGP}(k+1) \setminus \text{SSGP}(k)$ , and set  $G := \prod_{k < \omega} G_k$ . Then (a) holds with  $n_k = k+1$ . If there is  $n < \omega$  such that  $G \in \text{SSGP}(n)$  then since the projection  $\pi_n : G \rightarrow G_n$  is continuous we would from Theorem 3.15(b) have the contradiction  $G_n \in \text{SSGP}(n)$ ; thus (b) holds.  $\square$

*Remark 4.8.* Any group  $G$  satisfying the conditions of Corollary 4.7 is necessarily an m.a.p. group. This is the case since each  $G_k \in \text{SSGP}(n_k) \subseteq \text{m.a.p.}$  and since the m.a.p. property is preserved by products.

While Theorem 3.13 furnishes an ample supply of well-behaved abelian groups which admit no SSGP( $n$ ) topology, we have found that an SSGP topology can be constructed for many of the standard building blocks of infinite abelian groups. We now verify Theorem 3.18(c), that is, we give a construction of an SSGP topology for groups of the form  $G := \bigoplus_{p_i} \mathbb{Z}_{p_i}$  (with  $(p_i)$  a sequence of distinct primes); this illustrates the method used throughout the second-listed co-author's thesis [19].

Using additive notation, write  $0 = 0_G$  and let  $\{e_i : i = 1, 2, \dots\}$  be the canonical basis for  $G$ , so that  $p_1e_1 = p_2e_2 = \dots = p_i e_i \dots = 0$ . We define a provisional norm  $\nu$ , just as in the description preceding Lemma 4.5. This will generate a norm  $\|\cdot\|$  via Definition 4.4 in such a way that in the generated topology every neighborhood of 0 contains sufficiently many subgroups to generate a dense subgroup of  $G$ . Suppose we can show that  $G$  is Hausdorff and that each  $U \in \mathcal{N}(0)$  contains a family of subgroups  $\mathcal{H}$  such that  $G/\overline{H}$  is torsion of bounded order, where  $H := \langle \cup \mathcal{H} \rangle$ . Then if also  $G/\overline{H}$  has no proper open subgroup, we have from Theorem 3.24 that  $G/\overline{H} \in \text{SSGP}$ , so that  $G \in \text{SSGP}(2)$ . Our plan is to choose a norm so that  $G/H$ , and thus  $G/\overline{H}$ , is finite. Then if  $G$  contains no proper open subgroup, it is necessarily the case that  $\overline{H} = G$ . We will, then, define a norm  $\|\cdot\|$  so that

- (1) Every neighborhood of 0 contains a set of subgroups of  $G$  whose union generates a subgroup  $H$  such that  $G/H$  is finite;
- (2)  $G$  has no proper open subgroups, or equivalently, every neighborhood of 0 generates  $G$ ; and
- (3)  $G$  is Hausdorff.

To that end, define  $\nu(me_n) = \frac{1}{n}$  for every  $m < \omega$  such that  $m \not\equiv 0 \pmod{p_n}$ . We then obtain (1) because the neighborhood of 0 defined by  $\|g\| < \frac{1}{n}$  contains subgroups which generate  $H := p_1 p_2 \dots p_{n-1} G$ . Thus,  $G/H$  is finite, as desired.

To satisfy (2) define  $\hat{e}_n := \sum_{i=1}^n e_i$  for  $n < \omega$ , and define  $\nu(\hat{e}_n) := \frac{1}{n}$  for each  $n < \omega$ . What then remains (the most difficult piece) is to show that  $G$  with this topology is Hausdorff. We will then have the following result.

**Theorem 4.9.** *Let  $G = \bigoplus_{i < \omega} \mathbb{Z}_{p_i}$  where  $p_1 < p_2 < p_3 < \dots$  are primes. Let  $S = \{me_n : n < \omega, 0 < m < p_n\} \cup \{\hat{e}_n : n \in \mathbb{N}\}$ , with  $e_n, \hat{e}_n$  defined as above. Let  $\nu(me_n) = \frac{1}{n}$  for  $0 < m < p_n$ , and let  $\nu(\hat{e}_n) = \frac{1}{n}$ . Then the norm defined by*

$$\|g\| = \inf \left\{ \sum_{i=1}^n |\alpha_i| \nu(s_i) : g = \alpha_1 s_1 + \dots + \alpha_n s_n, s_i \in S, \alpha_i \in \mathbb{Z}, n < \omega \right\}$$

*generates an SSGP topology on  $G$ .*

*Proof.* As noted above, our construction for the norm  $\|\cdot\|$  guarantees that every  $\epsilon$ -neighborhood  $U$  of 0 generates  $G$  and also contains subgroups whose union generates an  $H$  such that  $G/H$  is finite. Then, as also noted, if  $G$  is Hausdorff we are done.

Suppose  $0 \neq g \in G$  and  $n$  is the largest nonzero coordinate index for  $g$ . We show that  $\|g\| \geq \frac{1}{n}$ . For convenience we extend the domain of  $\nu$  to all formal

finite sums of elements from  $S$  with coefficients from  $\mathbb{Z}$ :

$$\text{For } \varphi = \sum_{i=M}^N (a_i e_i + b_i \widehat{e}_i), \quad \text{let } \nu(\varphi) = \sum_{i=M}^N (\eta_i + |b_i|) \frac{1}{i}$$

where each  $\eta_i$  is either 0 or 1, according as to whether or not  $a_i \equiv 0 \pmod{p_i}$ .

In addition, we will assume

- (1)  $g = \text{val}(\varphi)$ , so the formal sum  $\varphi$  evaluates to  $g \in G$ ;
- (2) each  $e_i$  and each  $\widehat{e}_i$  appears at most once in any formal sum; and
- (3)  $0 \leq a_i < p_i$  for each  $i$ ,

item (2) being justified by the fact that we are ultimately interested in the norm, which minimizes  $\nu(\varphi)$ .

Let  $\mathcal{F}(M, N)$  be the set of such formal sums where  $M$  is the smallest coordinate index for a nonzero coefficient  $a_M$  or  $b_M$  and where  $N$  is the largest such index. (Here for  $b_i$ , “nonzero” indicates that  $b_i$  is not a multiple of  $p_1 p_2 \dots p_i$ .)

We want to show that  $\nu(\varphi) \geq \frac{1}{n}$  where  $g = \text{val}(\varphi)$ , where  $\varphi = \sum_{i=M}^N (a_i e_i + b_i \widehat{e}_i)$  and where either  $a_M$  or  $b_M$  is nonzero and either  $a_N$  or  $b_N$  is nonzero. In other words,  $\varphi \in \mathcal{F}(M, N)$ . Clearly  $\nu(\varphi) \geq \frac{1}{n}$  when  $M \leq n$ .

Suppose that  $N = M = n + 1$ . Then, since the  $n + 1$  component of  $g$  is 0 we have that  $a_{n+1} + b_{n+1} \equiv 0 \pmod{p_{n+1}}$ . Both coefficients are 0 only if  $g = 0$ , so either both are nonzero or  $a_{n+1} = 0$  and  $b_{n+1} = m p_{n+1}$  for some  $m \neq 0$ . In the first case we have  $\nu(\varphi) \geq \frac{2}{n+1} > \frac{1}{n}$  and in the second case we have  $\nu(\varphi) \geq \frac{p_{n+1}}{n+1} > 1 \geq \frac{1}{n}$ .

Suppose instead that  $M = N > n + 1$ . In this case, we know that the  $(N - 1)$  component of  $g$  is 0. Then, since  $g \neq 0$  can be written as  $\varphi = a_N e_N + b_N \widehat{e}_N$ , we have  $b_N = m p_{N-1}$  for some  $m \neq 0$ . But then we have  $\nu(\varphi) \geq \frac{p_{N-1}}{N} \geq 1 \geq \frac{1}{n}$ .

Finally, we fix  $M$  and use induction on  $N$ . Assume that we have already shown that  $\nu(\varphi) \geq \frac{1}{n}$  when  $\varphi \in \mathcal{F}(M, Q)$  for  $n < M \leq Q \leq N - 1$ , and suppose that  $\varphi \in \mathcal{F}(M, N)$ . We treat three cases separately.

- (a) Case 1.  $|b_{N-1} + b_N| \geq p_{N-1}$ . Then

$$\nu(\varphi) \geq \frac{|b_{N-1}|}{N-1} + \frac{|b_N|}{N} \geq \frac{p_{N-1}}{N} \geq 1 \geq \frac{1}{n}.$$

- (b) Case 2.  $b_{N-1} + b_N = 0$ . Recalling that all coordinates of  $g$  vanish after the  $n^{\text{th}}$ , we note that  $(a_N + b_N)e_N = 0$ , and so

$$b_{N-1} \widehat{e}_{N-1} + b_N \widehat{e}_N + a_N e_N = 0.$$

This means that we can delete these terms from  $\varphi$  without affecting its value, and with that done, our induction assumption can be applied.

- (c) Case 3.  $|b_{N-1} + b_N| < p_{N-1}$  and  $b_{N-1} + b_N \neq 0$ . Again, from  $(a_N + b_N)e_N = 0$  we obtain the equality

$$b_{N-1} \widehat{e}_{N-1} + b_N \widehat{e}_N + a_N e_N = (b_{N-1} + b_N) \widehat{e}_{N-1}.$$

Let  $\varphi'$  be the formal sum obtained from  $\varphi$  by replacing the three terms on the left with the one on the right, and compare the provisional

norms:

$$\nu(\varphi') - \nu(\varphi) = \frac{|b_{N-1} + b_N|}{N-1} - \left( \frac{|b_{N-1}|}{N-1} + \frac{|b_N|}{N} + \frac{1}{N} \right) \leq \frac{|b_N|}{N(N-1)} - \frac{1}{N}.$$

We see that this difference is negative or zero as long as  $|b_N| \leq N - 1$ . Then, since  $\varphi' \in \mathcal{F}(M, N - 1)$ , our induction assumption applies. If, on the contrary,  $|b_N| \geq N$ , we already have  $\nu(\varphi) \geq 1 \geq \frac{1}{n}$ .

We conclude in each case that  $\nu(\varphi) \geq \frac{1}{n}$ , so  $G$  is Hausdorff, as desired.  $\square$

### 5. CONCLUDING COMMENTS

Here we discuss briefly some other classes of groups which are closely related to the classes SSGP( $n$ ) and the class m.a.p.

*Remark 5.1.* In the dissertation [19], the second-listed co-author found it convenient to introduce the class of *weak* SSGP groups (briefly, the WSSGP groups), that is, those topological groups  $G = (G, \mathcal{T})$  which contain no proper open subgroup and have the property that for every  $U \in \mathcal{N}(1_G)$  there is a family  $\mathcal{H}$  of subgroups of  $G$  such that  $\bigcup \mathcal{H} \subseteq (U)$ ,  $H = \overline{\langle \mathcal{H} \rangle}$  is normal in  $G$ , and  $G/H$  is torsion of bounded order. Subsequent analysis (as in Theorem 3.24 above) along with the definitions of the classes SSGP( $n$ ) has revealed the class-theoretic inclusions  $\text{SSGP}(1) \subseteq \text{WSSGP} \subseteq \text{SSGP}(2)$ . A consequence of Theorem 3.24 is that the Markov-Graev-Remus examples (as in Remark 3.29) are not just WSSGP but are, in fact, SSGP. The same is true of Prodonov's group  $G$ :  $G \in \text{SSGP} = \text{SSGP}(1)$  (Theorem 4.3). From these facts we conclude that the class of WSSGP groups contributes little additional useful information to the present inquiry, and we have chosen to suppress its systematic discussion in this paper.

In Theorems 3.9 and 3.13 we have identified several classes of groups which do not admit an SSGP topology. That suggests the following natural question.

**Question 5.2.** What are the (abelian) groups which admit an SSGP topology?

Our work also leaves open this intriguing question:

**Question 5.3.** Does every abelian group which for some  $n > 1$  admits an SSGP( $n$ ) topology also admit an SSGP topology?

There is another important and much-studied class of groups related to the class of m.a.p. groups, namely the class of groups whose every continuous action on a compact space has a fixed point, the so-called *fixed point on compacta* groups (hereafter, the f.p.c. groups); for basic facts, some recent developments and a bit of history, see for example [18], [32] and [11]. (The reader will recall that a continuous action of a topological group  $G$  on a space  $X$  is a continuous map  $\phi : G \times X \rightarrow X$  such that (a)  $\phi(g, \cdot) : X \rightarrow X$  is a bijection for each  $g \in G$  with  $\phi(e_G, \cdot) = \text{id}_X$  and (b)  $\phi(g, \phi(h, x)) = \phi(gh, x)$  for all  $g, h \in G$  and  $x \in X$ . A *fixed point* for the configuration  $(G, X, \phi)$  is a point  $x \in X$  such that  $\phi(g, x) = x$  for all  $g \in G$ .) It is easy to see that every f.p.c. group is a m.a.p.



group (we write simply  $\text{f.p.c.} \subseteq \text{m.a.p.}$ ): Given a nontrivial homomorphism  $h$  from a (non-m.a.p.) group  $G$  into a compact group  $K$ , the continuous map  $\phi : G \times K \rightarrow K$  given by  $\phi(g, x) := h(g) \cdot x \in K$  is a  $G$ -action on  $K$  with no fixed point. The class-theoretic inclusion  $\text{f.p.c.} \subseteq \text{m.a.p.}$  is proper since, as we remarked in Discussion 2.1(a), the group  $SL(2, \mathbb{R})$  is an m.a.p. group even in its discrete topology, while Veech [39](2.2.1) has shown that every locally compact group, m.a.p. or not, has a continuous action on a compact space such that each non-identity element of the group moves every element of the compact space. (This is called a “free action”.) Whether or not every abelian f.p.c. group is a m.a.p. group, however, is a difficult long-standing open question in abelian topological group theory raised in 1998 by Glasner [18]:

**Question 5.4.** Do the f.p.c. abelian groups constitute a *proper* subclass of the m.a.p. abelian groups?

Even the characterization of abelian m.a.p. groups and abelian f.p.c. groups by different “big set” conditions (see [31] and [10]) did not settle Question 5.4. Unfortunately, and contrary to our hopes, our own work with the SSGP property also has so far not shed light on this question. It is known [18], however, that there are f.p.c. topologies for  $\mathbb{Z}$ , so the class-theoretic inclusion  $\text{f.p.c.} \subseteq \text{SSGP}$  fails. We have not successfully addressed the issue of the reversed inclusion, so we list it as another question to be resolved:

**Question 5.5.** Do the SSGP groups constitute a subclass of the f.p.c groups? What about the abelian case?

NOTE ADDED IN PROOF

Shortly after this paper had been completed in final form and accepted for publication, we received a preprint of [9] from its authors. They build substantially on our results, reformulating and possibly generalizing SSGP(n) with the use of some pleasing algebraic characterizations, and extending the concept to families  $\text{SSGP}(\alpha)$  for ordinals  $\alpha$ . In the process they have provided a positive solution to our Question 5.3, and they have made considerable progress in answering Question 5.2 for abelian groups.

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