Izquierdo Sebastián, J.; Benítez López, J.; Berenguer, A.; Lago-Alonso, C. (2016). I Decide, Therefore I Am (Relevant!): A Project-Based Learning Experience in Linear Algebra. Computer Applications in Engineering Education. 24(3):481-492. doi:10.1002/cae.21725.

I decide, therefore I am (relevant!)

A project-based learning experience in linear algebra

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Abstract— We present a project-based learning experience in the context of linear algebra developed for the recently launched double major in Business Administration and Management/Engineering of Technology and Telecommunication Services at the Universitat Politècnica de València 2014-15. Decision-making is used to motivate students towards applying, understanding, and appreciating linear algebra in a diversity of projects. This experience introduces students to the analytic hierarchy process (AHP), a multi-attribute, decision-making technique that is rooted in linear algebra. Through a simulation scenario, each team of students develops a project about any real-world problem consisting of a decision-making process in the presence of multiple intangibles. At the same time, the algebraic fundamentals that make the process valid and consistent are clarified. The students showed great interest in the experience; and the results obtained confirmed that the activities helped them understand several complex concepts related to linear algebra, and fostered a significant interest in a subject traditionally considered frighteningly abstract. Finally, the students appreciated the stimulating insights provided by linear algebra that are crucial in decision-making. This multi-disciplinary experience enables the evaluation of several cross skills and competencies such as critical thinking and ethical leadership.

Keywords: mathematics for engineers, linear algebra, project-based learning, decision-making, cross competencies

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1. INTRODUCTION

Project-based learning (PBL) is a teaching methodology that seeks to engage students in the investigation of authentic problems [1]. The student is asked to design, implement, and evaluate a plan to solve a real or simulated professional-type task [2]. PBL constitutes a strategy based on study and personal experience since it is the student who, under instructor supervision, must execute the entire activity: i.e., propose theories; collect data; make calculations; run tests or trials; conduct demonstrations; verify solutions; and evaluate results. PBL is driven by student preferences and interests. Students are free to propose their own projects (unencumbered by outside influences) and investigate, discover, share, and lead in their own learning [1], [3]. Their motivation is enhanced by their feeling of self-efficacy [4].

"I decide, therefore I am (relevant!)"[5] is a specific and powerful slogan[6] intended to motivate and arouse student interest through emotional evocation [6]. This title underlines a fully open scenario closely linked to the future professional scientific and/or business activity of the students for whom the activity was devised.

This activity was designed for students enrolled in the core subject Mathematics II[7] of the recently launched double major in Business Administration and Management/Engineering of Technology and Telecommunication Services at the Universitat Politècnica de València (UPV) 2014-15[8]. The activity invited students to discover the close relationship between the analytic hierarchy process (AHP is a multi-criteria decision-making technique with a clear potential for use in the professional fields related to this double degree), and linear algebra (dealing with matrices, linear mappings, eigenvalues and eigenvectors, orthogonal projections, etc.). As a result, students in this target group attained an active motivation created by the nature of a scenario capable of including many different real-life situations and settings [8], [9]. The instructor, whose role is exclusively as task manager and supervisor of the project [10], establishes the scaffolding upon which the students acquire almost seamlessly a set of knowledge elements regarding linear algebra. This corresponds to cross competency DC13 (specific instrumental competency) following

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[5] Far from any philosophical relationship, following Kierkegaard [5], we use this *cogito*-like expression for its psychological appeal. However, in contrast with Kierkegaard's argument, we try to emphasize that 'existence' (the quality of having relevance in society) is a consequence of making (adequate) decisions.

[6] The *driving question* is the element around which project-based learning is organized [7]. In this case, we wanted to replace the driving question for a driving slogan, with exactly the same intention.

[7] Mathematics II is a semester subject integrated into two main parts: linear algebra and ordinary differential equations.

[8] Website: http: // www. upv.es/titulaciones/GDADETEL/menu_881737c.html

the taxonomy of cross competencies enforced in the UPV [11]. In addition, the instructor provides students with this theoretical previous knowledge and provokes student curiosity as a consequence of an 'untypical' algebraic evaluation task that involves their emotions. This is an inquiry based project (which is especially effective in science learning) [3].

The activity also helps develop student self-learning and stimulates creative thinking [12] in close relationship with other UPV-cross competencies such as: understanding and integration (DC1); implementation and practical thinking (DC2); analysis and solution of problems (DC3); design and project (DC5); critical thinking (DC9); and lifelong learning (DC11). A number of activities later described also help with other cross competencies such as: innovation, creativity and entrepreneurism (DC4); teamwork and leadership (DC6); ethical, environmental, and professional responsibility (DC7); effective communication (DC8); knowledge of contemporary issues (DC10); and planning and time management (DC12) [3].

The experiment has an attractive feature based on the fact that Teleco + ADE students are a diverse group – with students who have a preference for technical subjects, and others more interested in subjects related to economics and business administration. This effect has been observed by the authors from previous experience in similar cases (telecommunication students, for example, have profiles more oriented to mathematics and physics, or to computer programming and telematics). It is necessary to investigate the possible motivation techniques for students who are not particularly 'lovers' of mathematics, and counteract the current trend towards avoidance by students with scientific and technical majors [13]. This is of crucial importance for current multidisciplinary demands [14], [15].

Here we briefly list the objectives we pursued with the implementation of this PBL experience. The objectives correspond not only to the subject, linear algebra, but also to the problem of decision-making itself – especially in the presence of intangibles. In the following list, the objectives consider the student as the target:

- Identify problems of interest to be solved using AHP.
- Give examples of real-world situations in which the elements involved in decision-making are intangible.
- Explain the difficulties involved in the consideration of intangible elements (judging criteria).
- Communicate and explain the multi-attribute methodology called AHP for the treatment of intangibles.
- Recognize and learn how to present the mathematical foundations (included in linear algebra) that underlie AHP.

- Solve/simulate decision-making real-world problems of interest using AHP.
- Recognize the pervasiveness of mathematics in various disciplines (corresponding to the problems addressed), thus re-launching the mainstreaming of mathematics as a tool.
- Internalize the importance of a core subject, such as linear algebra, and understand the need for core subjects in the chosen major (which consequently boost student motivation and are of special interest to freshmen).

Despite the apparent specificity of this academic context, the authors argue that the project herein described can be perfectly extrapolated to other contexts and environments. Since AHP has a very wide range of applications, the presented PBL can be easily customized to many scenarios in other engineering majors. In particular, lectures in linear algebra can directly implement the experience in their own teaching activity. We will show that the experience we describe is a clearly multidisciplinary PBL activity with a vocation that openly transcends the mere transmission of knowledge.

In the rest of this paper, we present the project scenario, describe the development of the proposal, present the results, and offer some conclusions. An appendix includes essential technical elements. A section of references closes the paper.

2. THE SCENARIO

Originally, the implementation of the experience is the result of a convergence of the teaching and research activities of the first two authors of this paper. The teaching aspects stem from subjects taught at UPV in which linear algebra was a substantial part. Secondly, both authors have been working jointly on a line of research on multi-attribute decision-making for more than five years and which has resulted in a relevant and on-going scientific production.[9] The convergence of these two aspects was triggered by the creation of the double major. The third co-author (a lecturer on the subject) and the fourth (an expert in teaching methodologies) were invited to implement the experience. It is worth mentioning that the third author is a recent student who is now a telecommunications engineer. We believe that we have thus formed a team that combines knowledge of the scientific basis for decision making and extensive experience in the teaching of the subject, with recent information about the needs of students

[9] Some of the relevant references for this paper can be found elsewhere in this document.

in telecommunications and expertise in learning methodologies. This team has been able to adequately prepare the experiment – evidencing that such balanced partnerships better succeed in motivating students [16].

In this section we present the conceptual scenario and leave for Appendix A concise presentation of its building elements (consisting of the necessary details from both AHP and linear algebra). This technical material has been deliberately placed in the appendix so as not to distract from the main description of the experience.

2.1 Conceptual scenario

Our first activity was to prepare a base document for the students as a study guide for their projects. The following excerpt from the introduction of the study guide presents the scenario. We note that it is not a typical statement of a problem, so characteristic in academia, but is an approach to an open setting that will enable students to occupy a professional role within an assigned mission.

> In real life we constantly have to make decisions: what career to study, who to vote for, what car to buy, where to go on holidays? On other levels (managerial, commercial, entrepreneurial, political, etc.) many important decisions must be made continuously. For example, how should an entrepreneur with limited resources invest? Clearly, it is important to prioritize investments and decide the percentage to be invested in each option: it is not the same to invest 60% of the money in advertising, 30% in infrastructure, and 10% in improving the working conditions, as to invest 80%, 15% and 5% respectively. The respective outcomes will most probably be very different.
>
> The following list is just a small sample of situations in which it is important to know which is the best choice, and/or what is the best ordering for a number of options or alternatives.
>
> - Where to place franchise outlets
> - Select staff for a company
> - Decide what kind of products a company should manufacture
> - Establish the broadcast programing of a private television station
> - What types of crops to plant
> - Decide the diplomatic decisions of a country
> - Calibrate the amount of funds to invest in various shares and securities
>
> In making a decision you must choose among a number of alternatives. When the stakes derived from a decision are important, suitably prioritizing the alternatives is crucial. However, assessing the priorities on which the decision is based can become very complex for several reasons. We may have a large number of alternatives. Very often, the goals we seek

are in conflict with each other. However, we focus here on the fact that the elements on which a decision must be based may not be objective or measurable, i.e., may not be derived from specific numerical values. Instead, they may spring from subjective, qualitative, intangible aspects. And yet, a decision must be made! Quantifying the subjective, the intangible, is essential if decision-making is to be transparent and not proceed from personal whims.

In this activity we focus primarily on this feature: how to decide in the presence of elements (criteria) that are not directly measurable, i.e., in the presence of intangibles.

AHP (analytic hierarchy process) is a mathematical tool developed by Thomas Saaty [17] in the 1980s and aimed at achieving the best choice from a range of views or opinions provided by the actors involved in a given decision-making process. Such views are simply pairwise comparisons of the elements involved, just opinions or feelings as to what extent one item is preferred or considered more important than another. The pairwise comparisons frequently result in many values (numbers) easily produced in general by actors. However, proper subsequent handling of all these numerical values is crucial if we are to produce robust and accurate priorities.

After this introduction, the connection with matrices is straightforward, and the study guide provided to the students clarifies examples, algebraic concepts and results, and a comprehensive approach to AHP[10].

2.2 The core activity and the scenario

A number of complementary individual assignments (described in the next section) constitute the entire activity. However, the core activity consisted in choosing, by groups of some five students, a decision-making problem of real interest, using intangible elements taken from either personal, scientific, or business fields. These problems should include: a clearly defined goal, five to seven relevant (intangible) criteria, and two or three realistic alternatives.

Since this scenario has the ability to support a plethora of real situations, we aim to create an opportunity for the starting point of the learning process. We want to foment the need to learn in the students. Activities and curricula content will be introduced later. The experience is therefore within PBL as students are actively participating in a project involving their hearts and minds, and provides relevance to the real world in the process of learning [18].

[10] The study guide, available from the authors, provides suitable examples, presents the basics of AHP, and adds a number of results for the student to use in their decision-making project. For the sake of completeness, we have included the most prominent ingredients in Appendix A.

3. PROJECT DEVELOPMENT

The first step was the development of an appropriate document to present the AHP methodology in an intelligible way for students, without forgetting the strict connection with the algebraic elements that are part of the instrumental competencies of the subject. Such a document was made available to the students as a resource on the UPV website for Mathematics II. In addition, in a specific session, students were presented with a summary of the project together with a specific timetable for the implementation of the activities as outlined. Following the presentation, the implementation of the PBL activity followed these steps.

1. Study by the student of the base document. DC1 - understanding and integration can be evaluated in this step through the following elements.
   a. The basics for decision making, using easy-to-understand examples ranging from the very simple to the more complex in various areas of interest.
   b. The foundations of the AHP methodology, which include:
      i. the three-level hierarchical structure of AHP with the goal/objective on the first level, the criteria one level down, and the candidate alternatives to achieve the goal on the lowest level;
      ii. pairwise comparisons of criteria with respect to the objective and the alternatives with respect to the various criteria;
      iii. building of the corresponding comparison matrices;
      iv. studying matrix consistency;
      v. improvement, if necessary, of consistency in coordination with the appropriate actor or actors;
      vi. calculation of the vectors of the priorities; and
      vii. adequate aggregation of the priorities of the hierarchy.
   c. References to the required algebraic results to support decision-making under AHP; including aspects related to:
      i. matrices and their operations, as the most basic elements;
      ii. various linear mappings that play a key role in the rigor of the methodology;
      iii. spectral theory to obtain eigenvalues and eigenvectors, specifically the so-called Perron vector of a matrix, which provides the vector of the priorities;

iv. elements of geometry and approximation theory, which enable us to obtain and clearly visualize the necessary consistency improvements in the presence of inconsistent judgments.

v. also the relationship with the laboratory associated to the subject in terms of methods found in MatLab or Octave.

Notably, these aspects exhaustively cover the algebraic elements of Mathematics II.

d. Several illustrative and relevant examples of real interest are included. To suitably describe the examples it is necessary to:

i. adequately identify the hierarchical structure of the problem;

ii. adequately identify the relevant actors for making pairwise comparisons;

iii. perform the necessary feedback to harmonize the provided opinions with the need for consistency;

iv. perform suitable weighting in those processes that involve several actors with different viewpoints; etc.

e. Appropriate literature on the subject. In this case, this consisted of one of the most complete sources on AHP [19], and three works by the authors [20], [21], [22], which are directly relevant to the study guide.

2. Individual evaluation of each student to ensure that they acquired the knowledge and learning results sought. This assessment was based on the following elements: (cross competencies relevant to each element are provided at the end of each item).

a. Individual submission of the following deliverables uploaded onto the Mathematics II website. These deliverables, weighing 40% of the evaluation of the student grade in this activity, can also be used at the teacher's discretion to adjust the collective evaluation in the subsequent group work.

i. Deliverable 1. Personal version of AHP that the student summarizes in a maximum of two thousand words. Students were asked for conciseness, professional presentation, and rigor. In most cases, it was the first 'research' work for the students. This promotes individual responsibility and gives the students a feeling of being involved in a project on a professional level (DC1 - understanding and integration).

ii. Deliverable 2. Exercises proposed in the study guide. This is a collection of exercises contextualized within the document that connect AHP with the algebraic issues within the contents of the subject. Students learn linear algebra in contact with reality. This is therefore an innovative educational method, perfectly in line with current teaching guidelines (DC13 - specific instrumental competency).

iii. Deliverable 3. Detailed identification of the algebraic elements used, stating links between the algebraic concepts used in the AHP project and the specific contents of the agenda and instrumental competencies of the subject (DC13 - specific instrumental competency).

iv. Deliverable 4. Three examples in different fields. This was used to assess the level of maturity of students when dealing with the problem. We tried to encourage a critical view of reality based on problems from their own environment. This is very important for the professional future of the students. Developing these skills is essential since the profile of their major indicates that students are likely to work in a technology company (DC3 - analysis and problem solution; and, DC10 - knowledge of contemporary issues).

b. A specific multiple choice test on the study guide (Deliverable 5). This test had a weight of 20% of the activity score. This activity helped motivate the consolidated learning of the AHP technique, which is essential for developing the project with the necessary resources and quality. It also served as a checkpoint for teachers to enhance or clarify aspects of the subject.

3. Work groups composed of four to six students performed the following activities:

a. Selection of a decision-making problem (DC4 - innovation, creativity and entrepreneurship) using intangible elements of real-world interest taken from either personal, scientific, or business fields. Personal aspects as well as scientific and entrepreneurial aspects are considered (thus helping students to grow as individuals, and covering the two above-mentioned legs of their double major). As a result, students are made aware of the versatility and usefulness of decision-making, and experience how their work helps them grow in personal, scientific, and managerial areas.

The problem should include: a goal, five to seven criteria, and two or three alternatives. The stated number of criteria and alternatives guarantee a reasonable problem size, so the work to be done is neither unduly burdensome nor trivial. Subjected to these constraints, students must decide the problem to solve and be aware of its dimension and the resources necessary to solve it. Unconsciously, they are developing the ability of resource management – a cross competency of paramount importance (DC12).

A scheduled checkpoint in which each group had to present the selected problem, properly justify its interest, and match the specified guidelines before the entire class was established. One member of the group was appointed secretary and the session minutes included the establishment and list of members of the work group, the title of the project, and a short description of the goal, criteria, and alternatives (this is the Deliverable 6, a necessary activity but without an associated weight in the final mark). The minutes can be considered a draft of the first part of Deliverable 7 discussed below. Several cross competencies, including the above mentioned, may be applied at this stage.

b. Group development of the work must formalize the appropriate decision-making, by:

   i. Building the pairwise comparison matrices for the development of the process through the necessary fieldwork: i.e., consulting adequate experts/actors; and acquiring the opinions of the experts from the appropriate niche identified by students. In most cases the work group will have to learn about a new field. They will have to move into unchartered territory as would-be entrepreneurs looking for their own niche (DC11 - continuous professional development). To obtain data, we suggest that students use two types of techniques, a written survey or direct interview, leaving aside more complex options such as the facilitation meetings organized in the DELPHI type method [23].

   ii. Organization of the obtained information and necessary calculations to test the consistency of the comparison matrices using the MatLab (or Octave) computing environment (which is completely familiar to students at this stage of the course). This compilation work shows students that what they learned in the practical lab classes coincides with the project needs. The laboratory results demonstrated that they could now face an authentic real-world problem and not merely theoretical examples. Frequently, usefulness is justified using fake concepts, such as the algorithm itself or the abstract importance associated with theory. The relative simplicity of AHP offers the perfect level of difficulty such that freshman can appreciate the benefits of having a suitable software tool for a real-world application. By relieving the students from burdensome calculations, the software tool introduces them to the field of data analysis, an area that is essential for 21st century engineering and finance professionals [24], [25].

   iii. Use, if necessary, of the linearization technique [20], [22], fully described in the study guide, to improve comparison matrix consistency, via MatLab or Octave. As in the previous point, this work poses no difficulty for students, and similar comments apply.

   iv. Performing, interacting, re-evaluating and adjusting the feedback [21] with the adequate actors to validate the comparison matrices and reconcile the need for consistency within the views of the actors.

Negotiation must be performed in such a way that it does not introduce any bias in the actor (DC1 - understanding and integration; DC9 - critical thinking; DC8 - effective communication; and DC7 - ethics and professional responsibility). This point also stimulates the interaction of the students with their social environment – a powerful tool in this era of social networks.

v. Completion of the calculations by properly aggregating priorities, again via MatLab or Octave. Once more, as in ii. and iii., this point poses no additional difficulty for students.

vi. Documentation of the process – which consists of:

- detailed description of the problem;
- thorough analysis of its hierarchical structure expressing the objective, criteria, and alternatives;
- discussion regarding the intangibility of the criteria;
- identification of the actors involved and the methodology used for obtaining pairwise comparisons;
- justification of the consistency of the comparison matrices;
- description of the performed consistency improvements;
- description of the performed feedback processes;
- aggregation of priorities; and,
- presentation of the aggregated vector of priorities, supported by an adequate discussion of the results and a consistent conclusion.

This encourages entrepreneurial vision and leadership as a result of observing again the problem in perspective (DC9 - critical thinking; and DC2 - application and practical thinking).

c. <u>Deliverables 7 and 8</u>: Public presentation of the work performed on the Mathematics II website (Deliverable 7) and in the classroom (Deliverable 8) by groups. This presentation corresponds to 40% of the mark, which is weighted using the individual mark previously obtained. This weighting is intended to promote uniform, active participation of members within groups. As future professionals who deal with projects, two fundamental aspects are tested here: the capacity to develop a document transmitting the details of the project, and to effectively communicate and summarize project information to an audience that will assess its quality/suitability (DC8 - effective communication; and DC6 - teamwork and leadership).

4. Evaluation of the projects and the individual contributions in the previous activities using appropriate monitoring, and consideration of the preceding elements discussed above by the instructors. Key aspects of PBL and cooperative learning are taken into account in the evaluation, such as:
   - analysis of the considered project;
   - adequacy of the learning objectives;
   - reflections and decisions made during the project development;
   - observation of the cooperative character within the group work;
   - observed student capacity to modify their communicative behavior in the search for information and development of the study.

4. RESULTS

This section presents the most relevant results connected with each phase of the PBL experience we have described.

4.1 The study guide

The study guide is a base document in PDF format that is available on the Mathematics II website. We have prepared a fluid and easy-to-read text that skips excessively technical elements not within the scope of this course (such as the proofs of some particularly cumbersome theorems which would have blurred the true objective of the activity). Specific examples have proven to be attractive, easy to understand, and motivating for the project development.

After the initial psychological boost necessary to overcome freshmen inertia, students recognized and appreciated the experience. The study was performed within a set timetable and evaluated with the following results.

4.2 Individual evaluation

This corresponds to Deliverables 1 to 5 as described above. Regarding the first four jointly, the average score was 3.58 (out of 4), with a $\sigma = 0.31$; 80% of 25 students, after their own fashion, were perfectly capable of using the AHP tool (Deliverable 1). The exercises proposed in the study guide were successfully resolved by almost all of students (Deliverable 2). Except for the impact of linear mappings, the relations between the algebraic content of Mathematics II and AHP (Deliverable 3) were recognized without difficulty. Most students were unable to recognize linear mappings as the main elements in theorem demonstration mechanisms. The instructors were able to pinpoint this learning gap. This led the instructors to reinforce the knowledge students needed on linear mappings in the theoretical lectures. Finally, real-world examples that can be solved with AHP (Deliverable 4) were of exceptional quality. In addition, the average grade in the multiple choice test, which constitutes Deliverable 5, and in which a percentage of incorrect answers was subtracted from the final score, was 8.5 (out of 10), with a $\sigma = 2.2$; 60% of students answered all the questions correctly.

4.3 Formation of work groups

Group formation was left in the hands of the students. Contrary to other experiences we found that self-group determination had none of the drawbacks that are sometimes mentioned. Only in the case of one individual did the tutor have to intervene and negotiate with a group of four to accept him as their fifth member. Eventually, the class divided itself into five groups of five each. The intimate nature of the class enabled us to ensure that the groups were homogeneous and balanced.

4.4 Selection of the problem

In theory this could have been a problematic step. It is not easy to expect freshmen to have the maturity to choose issues of interest that are well-connected to the real world. The guidance from the instructor, clearly defining the scaffolding upon which the problems were to hang, was obviously helpful. However, surprisingly, each group had drawn up a list of problems with several items other than the examples provided by the instructor. Thus, the control point for the selection and validation of the problem was straightforward, if perhaps hampered by the need for the students to restrict themselves to choosing only one of the problems they had suggested. Evidently, the activity succeeded in helping students develop skills of autonomy and maturity when dealing with problems. All the problems were sufficiently realistic and reasonable for the project. The final choice was therefore a pseudo-random decision or directed by the particular interest to a given problem raised by one or more members of the group (suggesting active participation and enhanced motivation). Two of the groups chose their projects based upon the anticipated ease of finding suitable actors/experts. The fact that students had complete freedom in choosing the problem generated enthusiasm and stimulated interactive discussion. We still remember the fun they had and the intense hustle and bustle in the room that day. They were thrilled with the activity.

Specifically, we include below some of the problems chosen (objective/goal), along with an example of a criterion, and examples of the actor(s) selected for the corresponding problem (extracted from Deliverable 6):

- Decision on the best location for a wind-energy farm

    - Criterion: visual impact generated by the farm
    - Actor: expert from a leading company in the renewable energy sector
- Distribution of resources to boost production of various ranges of a firm's products
    - Criterion: ecofriendliness of each product
    - Actor: staff member of the coordination and logistics department of a multinational company
- Decision by a young professional soccer player in choosing between several offers
    - Criterion: growth prospects as a soccer player
    - Actors: the player and a sport radio journalist
- Decision on how to boost sales in a new store
    - Criterion: assessment of sector competitiveness
    - Actors: expert in business management from the UPV and a distributor
- Decision between several tourist destinations
    - Criterion: destination authenticity
    - Actor: staff member of a travel agency

The various alternatives (omitted here for brevity) are perfectly feasible in all cases. The selected criteria are entirely suitable and most clearly correspond to the category of intangibles. Finally, the selected actors made the problems transcend the banality of an academic exercise, and clearly positioned these problems close to authentic real-world problems.

4.5 Development of the group project

As we had predicted, the MatLab/Octave environment created a positive and difficulty-free experience for the students who suffered no difficulty with the project, and students were aware of the close relationship between the lab sessions and the activity needs (another point of cross competency). Our having proposed the use of this method, enabled the students to assess the value of higher education. This improves their receptivity to new concepts in future abstract subjects, i.e., mathematics, physics, and theoretical developments in engineering or financial fields. Although the introduced concepts may seem very theoretical, students grasped the idea that this could lead to very useful practical applications. It should be highlighted that we are modifying the attitudes of engineering freshmen and finance majors from an initial rejection of theoretical bases to acceptance. We are preconditioning students to be open to a solid base of learning through an experiment that does not require great effort because they are already highly motivated.

4.6 The presentation of project results

Here we make a general appraisal of the quality of the student presentations – both the written report (Deliverable 7) and the public presentation (Deliverable 8).

In short, among the specific results that we have achieved:

- High level of motivation of the students on a business-related course for an issue closely related to their possible future activities, i.e., decision-making in the presence of intangibles.
- Selection of problems of interest in several areas that have revealed a surprising level of maturity, even in freshmen, when they participate in an adequate learning methodology.
- Learning and application of linear algebra elements related to Mathematics II. After completing their project, students remember what they have learned and retain it longer than with traditional teaching. A hands-on component demonstrates the long-lasting effectiveness of this style of teaching. As a result, students that gain knowledge through PBL are better able to apply what they know and can better cope with new situations.
- Approach to real-world problems (including fieldwork) from a subject that many describe as typically theoretical. In the 21st century workplace, success requires more than just basic knowledge and skills. In PBL, students not only understand the content more deeply, but also learn to take responsibility and build trust, solve problems, work in teams, communicate ideas, and be innovative and creative.
- Measurable assessment of the cross competencies defined by the UPV. Current standards emphasize the effective application of knowledge and the development of 21st century skills such as collaboration, critical thinking, and communication in a variety of media. PBL provides an effective way to address such standards.

Finally, PBL allows teachers to work more closely with active and engaged students doing meaningful and quality work. In our experience, we have rediscovered the joy of learning with students. However, we could not fail to note that the management of the class for the instructors involves a good deal of work and we believe that methodologies of this type work best with groups no larger than 25.

5. Conclusions

In this paper we present a teaching experience in the core course Mathematics II of the recently launched double major in Business Administration and Management/Engineering of Technology and Telecommunication Services at the UPV 2014-15.

Our experience was based on PBL, which can be described as a teaching methodology that involves students working in a sustained and collaborative manner in a research investigation taken from the real world. Students develop projects that are organized around a driving question (or slogan), and participate in the work (selection, planning, research, and production of a project) with the aim of finding a solution to the problem [7]. We believe that our work is fully consistent with this definition. We have designed an activity that permits students to become truly aware of human behavior in the context of decision-making. To add to the level of interest, we propose the use of analytic hierarchy process (AHP) – a technique rooted in linear algebra that includes intangibles in the decision-making process. The results indicate that the experience has been successful as students have exhibited high levels of motivation and most UPV-cross competencies have been encouraged, tested, and evaluated.

This symbiotic relationship between AHP and linear algebra is a pioneering experience in a theoretical subject such as linear algebra. This success may foster future research in seeking similar projects in linear algebra[11], as well as other branches of mathematics and other subjects. For example, students may be invited to develop a complete computational tool implementing the main aspects of AHP within a subject of their syllabus devoted to computer programing. In this activity the concept of competition [29] may be introduced, so that the results may be assessed by a selected group of users. In this way, students who have followed the experience described in this paper have reinforced their knowledge of AHP (contributing to DC11 - long-term or lifelong learning) and this is beneficial for their professional career [30]. Following a more general line, decision-making may also be considered in project-based learning or inquiry-based learning through competition motivation [31] for the development of (for example) more efficient apps. This would gather several subjects in the activity, thus becoming highly multidisciplinary, bringing together instructors from different subjects, increasing motivation [32], and generating integrated multidisciplinary education – and so creating enhanced possibilities for future success [14], [15]. As a third line, it is worthwhile assessing the value of this methodology through the organization of suitable test groups. The authors

[11] The authors are planning extensions of the presented approach based on such aspects as dynamic AHP [26], [27], and participatory decision-making through AHP [28].

think that this type of experiment is feasible as long as the student/instructor ratio is not too large (as this could impede the success of the experience). However, opportunities for trying similar experiences with larger groups of students, similar to the one described in [33], may be worth exploring using collaborative tools such as blogs, wikis, and RSS feeds to reinforce group collaboration.

Despite the theoretical and fundamental character traditionally assigned to branches of mathematics, such as linear algebra, with this project we show that it is possible to engage students in fundamental and theoretical subjects by providing them with suitable frameworks where the application is clear, appealing, and interesting. Specifically, we note the following:

- The experiment helped reveal the difficulty students experienced with linear mappings; so we can conclude that, in general, this approach may be used as a tool for learning assessment.
- Leaving group formation in the hands of students was highly positive. This made them feel more self-sufficient, and the feeling of being in a mandatory class exercise became less present, as the students themselves explained. Moreover, the fact of joining a team of their choice created extra motivation, as preferences in the choice of the problem are more similar. Also the team feeling was more stimulating than in teams built in a more artificial manner.
- The degree of connection with the student is inversely proportional to the number of students, and this can affect motivation. This information was obtained from group feedback. It became obvious that the relationship between instructors and students in this group was quite different than that of other groups. This created a better working environment and enhanced motivation.

With the completion of their projects students clearly grasp the nature of mathematics, especially linear algebra and several of its fundamental elements, as a tool to be used effectively when tackling real-world problems.

APPENDIX A: SCENARIO SCAFFOLDING ELEMENTS AND THEIR ALGEBRAIC CLAMPS

This appendix includes seven items. The content of the first six was provided to the students in a completely different format and, in addition, was supplemented with abundant illustrative examples. The version we offer here is included for the sake of completeness and provides the necessary references for further explanation. The seventh issue is a MatLab/Octave code implementing the tools described in items A.3 to A.5. We provide this code for the reader to assess the real difficulty of the tools and, consequently, the difficulty confronted by the students. We clearly state that we did not provide this code to the students, who were able to develop similar computational materials to successfully solve the projects they had devised.

A.1 The hierarchy

The AHP technique introduced by Saaty is his pioneering work [17] and is intended (in its simplest form) to achieve a concise and clearly stated objective through a number of clearly identified alternatives taking into consideration a set of mainly intangible criteria[12]. These three key elements of the problem may be placed in three descending levels organized in a hierarchical structure, as shown in Figure 1.

Fig. 1 Hierarchical structure of AHP

A.2 Pairwise comparisons

Once the problem is clearly stated, knowledge about the problem must be identified. To this end, one or more experts (stakeholders, actors in general) in the process were identified and they provided their opinions on the problem elements using simple pairwise comparisons (PWCs). Saaty devised a very simple way to assign numerical values to the judgments given by an actor, making it possible to measure how each element contributes to the hierarchical level that is immediately above it in the AHP hierarchy (Fig. 1). Use is made of a specific scale (using integers) for these comparisons in terms of preference or importance – ranging from indifference (equal importance of two elements) associated with the value 1, to extreme importance of one element over a second with an associated value of 9.

Table 1. Saaty's 9-point scale [19]

The inverses of these values, ranging from 1 to 1/9, show a reciprocal kaleidoscope of 'less important' shades. We use here the 9-point scale developed by Saaty [17] with the possibility of including intermediate numerical (decimal) values in the scale to model hesitation between two adjacent judgments [34].

[12] Although objective criteria may be used, the rationale behind AHP resides in its ability to cope with non-objective, intangible issues. When all the criteria are quantifiable there are various ways to more simply make decisions.

A.3 PWC matrices and their properties

In the first step, the expert makes comparisons between pairs of criteria. He or she discusses the importance of one criterion with respect to a second one using Saaty's 9-point scale. Based on this scale of values, a comparison matrix of criteria is built. If there are n criteria on the table, by comparing each element against another, a square matrix of order n, $A = [a_{ij}]$, is compiled. The element a_{ij} represents the comparison between element i and element j. Subsequently, a similar exercise comparing alternatives for each criterion is performed, thus building comparison matrices of alternatives. The properties for these matrices are similar and we now consider them indistinguishably.

A comparison matrix $A = (a_{ij})$ must be positive and reciprocal, i.e., it must satisfy $a_{ij} > 0$ and $a_{ji}\, a_{ij} = 1$, respectively.

A.4 Consistency and the priority vector

In addition to the above properties, A is said to be consistent if $a_{ik} = a_{ij}a_{jk}$, for all i, j, k. This key property means that if element i is a_{ij} times more important than element j, and element j is a_{jk} times more important than k, consistently, then i should be a_{ik} times more important than k.

The principal eigenvalue of a consistent comparison matrix and its associated eigenvector (Perron vector) provides information for complex decision-making: the normalized Perron eigenvector provides the priority vector sought [19], [35]. Generally, however, A is not consistent. The hypothesis that the estimates of these values are small perturbations of the 'correct' values also guarantees small perturbation to the eigenvalues, see, for example, [36]. For non-consistent matrices, the problem to be solved is the eigenvalue problem $A\mathbf{w} = \lambda_{\max}\mathbf{w}$, where $\lambda_{\max}$ is the single largest eigenvalue of A that provides the Perron eigenvector as an estimate of the vector of priorities.

As a measure of inconsistency, Saaty proposes using the consistency index $CI = (\lambda_{\max} - n)/(n - 1)$, and the consistency ratio $CR = CI/RI$, where RI is an average consistency [19]. If $CR < 0.1$, the estimate is accepted, otherwise, a new comparison matrix is requested until $CR < 0.1$.

Table 2. Random index [19]

A.5 Consistency improvement and tradeoff between synthetic consistency and expert judgment

If consistency is unacceptable, it should be improved. Several alternatives, mostly based on various optimization techniques, have been proposed in the literature to help improve consistency, including [35], [37], [38]. Benítez et al.

[20] have developed a linearization technique that provides the closest consistent matrix to a given non-consistent matrix, X, by using the following scheme:

$$X \xrightarrow{L} L(X) \xrightarrow{p_n} p_n(L(X)) \xrightarrow{E} X^c,$$

where $L(X)$ is the matrix of the natural logarithms of the elements of X, $p_n(L(X))$ is the orthogonal projection of $L(X)$ onto the linear space

$$L_n = \{L(A) \text{ such that } A \text{ is an } n\times n \text{ positive consistent matrix}\},$$

and E is the mapping that associates a matrix $X = (x_{ij})$ with the matrix whose (i,j) entry is $\exp(x_{ij})$. X^c is the sought matrix. Recently, a simple formula to obtain this orthogonal projection (see Theorme 3 of [22]) has been produced. If X is an $n\times n$ pairwise comparison matrix, then

$$p_n(L(X)) = \frac{1}{n}\left[\left(L(X)U_n\right) - \left(L(X)U_n\right)^{\mathrm{T}}\right],$$

where $U_n = 1_n 1_n^{\mathrm{T}}$, 1_n being the n-vector $(1 \ldots 1)^{\mathrm{T}}$, and $^{\mathrm{T}}$ denotes matrix transposition.

To overcome inconsistency in AHP while still taking into account expert know-how, Benítez et al. [21] propose a model to balance the latter with the former, as follows:

If the expert matrix is not acceptably consistent, the linearization technique can be used to build a consistent matrix. The new matrix, however, may be considered by the expert to partially reflect their opinions and perhaps they will choose to modify some of the matrix entries. Changing one or more entries of the matrix (while, of course, preserving reciprocity) will produce a new (most probably) inconsistent matrix. If consistency is acceptable this last matrix may be eventually accepted. Otherwise, the linearization process can push once again towards consistency, but may need expert approval once again. This process can be undergone iteratively in an attempt to reach a reasonable trade-off between consistency and expert know-how compliance (see Figure 3).

Fig. 2. Flowchart for the consistency improvement process

When dealing with criteria, a priority vector **w** of length n (size $n\times 1$) is eventually obtained. In a similar way, when dealing with, say m alternatives, n priority vectors, $\mathbf{v}_j$, of length m (size $m\times 1$) are obtained for each criterion.

A.6 Aggregation of priorities

A score is computed for a given alternative by multiplying its priority value with respect to each criterion by the priority of this criterion (with respect to the objective), and summing through all the criteria. The scores for the alternatives are given in the coordinates of a vector,

$$\mathbf{z} = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)\cdot\mathbf{w},$$

obtained by multiplying the matrix built by columns with the alternative priority vectors, $\mathbf{v}_j$, with the criteria priority vector $\mathbf{w}$. The largest coordinate of $\mathbf{w}$ will be associated with 'the best alternative' and the lowest with 'the worst alternative', with the importance of other alternatives ranked accordingly.

A.7 MatLab/Octave code

This self-explanatory code includes the main tools needed for the project.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [B v] = ahp(A)
% Use: [B v] = ahp(A)
% A -> The comparison matrix
% If A is consistent -> B=A
% If consistency of A is acceptable (Saaty) -> B=A
% If consistency of A is not acceptable -> B is
% the consistent matrix closer to A
% v -> B's priority vector
[n m]= size(A);
if n ~= m
   error("Matrix must be square")
end
if n > 15
   error("Matrix size should not exceed 15")
end
aux1 = sum(sum(A>0));
if aux1 ~= n^2
   error("Matrix should be positive")
```

```
end
aux2 = norm(A - (1./A)');
if aux2 > 0.01    % Avoid roundoff errors
   error("Matrix should be reciprocal")
end
[X D] = eig(A);   % eigenvectors and eigenvalues of A
D = diag(D);      % eigenvalues of A
[lambda position] = max(d); % Perron eigenvalue
if abs(lambda-n) < 0.01   % Avoid roundoff errors
   disp("Already consistent matrix")
   B = A;
   v = X(:,position)/sum(X(:,position)); % priority (Perron) vector
else
   disp("Non consistent matrix")
   ri = [0 0 0.52 0.89 ...
   1.11 1.25 1.35 1.40 1.45 ...
   1.49 1.52 1.54 1.56 1.58 1.59]; % Saaty’s random index
   ci = (lambda-n)/(n-1);          % consistency index
   if ci/ri(n) < 0.1  % Saaty’s criterion of acceptable consistency
      disp("Acceptable consistent matrix")
      B=A;
      v=X(:,position)/sum(X(:,position)); % priority (Perron) vector
   else
      disp("Matrix consistency is not acceptable")
      M = log(A);        % skew-hermitian matrix of logs of A
      MU = M*ones(n,n);
      P = (MU-MU')/n;    % projection by simple formula
      B = exp(P);        % closest consistent matrix
      [X D] = eig(B);    % eigenvectors and eigenvalues of B
      D = diag(D);       % eigenvalues of B
      [lambda position] = max(d);  % Perron eigenvalue
      v = X(:,position)/sum(X(:,position)); % priority vector
```

```
    end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

ACKNOWLEDGEMENT

The use of English has been revised with the help of Alba Toscano and John Rawlins.

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