Dynamically-tunable transformation thermodynamics

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Abstract. Recently, the introduction of transformation thermodynamics has provided a way to design thermal media that alter the flow of heat according to any *spatial* deformation, enabling the construction of novel devices such as thermal cloaks or concentrators. However, in its current version, this technique only allows static deformations of space. Here, we develop a *space-time* theory of transformation thermodynamics that incorporates the possibility of performing time-varying deformations. This extra freedom greatly widens the range of achievable effects, providing an additional degree of control for heat management applications. As an example, we design a reconfigurable thermal cloak that can be opened and closed dynamically, therefore being able to gradually adjust the temperature distribution of a given region.

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1. Introduction

Transformation thermodynamics is a tool that allows us to design devices able to mold the flow of heat according to a coordinate transformation [1]. Using this technique, it has been possible to experimentally demonstrate, for instance, a thermal cloak based on artificial (meta)materials [2]. While the current approach is limited to static spatial transformations, the ability to modify the transformation as a function of time would allow for the construction of devices with time-varying functionalities as in the optical and acoustic cases [3, 4, 5, 6, 7, 8], greatly widening the range of addressable effects and applications. This possibility implies the use of space-time mappings, which, in general, involve more complex materials for their implementation. For example, in the case of optics, such materials are linked to magneto-electric couplings, which can often be related to moving media [9]. Likewise, it has been recently demonstrated that this kind of media are fundamental for the space-time version of transformation acoustics [5, 6]. Therefore, moving media seem to be a common key feature of space-time transformation techniques, so it is natural to conjecture that it will also be required in the field of thermodynamics. In this case, moving media are associated with convective heat transfer. The behavior of the convection-diffusion equation under purely spatial transformations and for time-independent parameters was studied in [10, 11]. However, no space-time analysis has been performed so far and, therefore, the possibility of building dynamically-tunable transformational thermal devices has not been considered yet. This is the goal of the present work.

2. Space-time transformation thermodynamics

According to the previous reasoning, our starting point will be the equation that models the evolution of temperature T as a result of both heat diffusion and convection. As we will see below, considering time-dependent thermal parameters will be essential to build the sought space-time transformation theory. For this reason, it is important to look carefully at the derivation of the convection-diffusion equation, as the full version with space and time parameter dependence might be different from the usual version of the equation, in which the parameters are taken to be time-independent.

To derive the mentioned equation, we start from the energy balance for a continuous medium in the absence of work done by body or surface forces [12]

$$\rho\frac{Du}{Dt} + \nabla\cdot\mathbf{q} = 0, \tag{1}$$

where u is the internal energy per unit mass, $\mathbf{q}$ the heat flux, ρ the density, and D/Dt the material derivative operator defined as $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla$, $\mathbf{v}$ being the velocity of the material point. A priori, there is no restriction on the time or space dependence of ρ and $\mathbf{v}$. On the other hand, according to Fourier's law, $\mathbf{q}$ can be expressed as $\mathbf{q} = -K\nabla T$, where K is the thermal conductivity. In principle, the spatio-temporal dependence of K is not restricted either. Inserting Fourier's law in the energy equation and writing the internal energy per unit mass as $u = cT$, with c the specific heat capacity, we finally obtain

$$a\frac{DT}{Dt} - \nabla\cdot(K\nabla T) = 0, \tag{2}$$

where $a = \rho c$. Note that we have assumed that $Dc/Dt = 0$. This is the only restriction regarding the coordinate dependence of the thermal parameters. Equation (2) will be our basic equation, which we now rewrite for convenience using Einstein's notation and in a specific system of coordinates adapted to the problem at hand

$$a\left(\partial_t T + v^i \partial_i T\right) - \frac{1}{\sqrt{\gamma}}\partial_i\left(\sqrt{\gamma} K \gamma^{ij} \partial_j T\right) = 0, \quad (3)$$

with $\partial_i = \partial/\partial x^i$. For example, in the problem later to be presented these coordinates will be cylindrical coordinates $x^1 = r$, $x^2 = \theta$, and $x^3 = z$, and the γ^{ij} the flat Euclidean metric written in these coordinates; γ represents as usual its determinant. It can be demonstrated that, for a general space-time transformation, the previous equation does not retain its form. A similar situation is found in acoustics, where an analogue transformation method can solve the problem [5, 13]. However, as we will show below, there is a fundamental difference here; while in the acoustic case the equations are form-variant under almost any space-time mixing transformation [6], the heat equation turns out to be form-invariant as long as the new time variable does not depend on the original space variables. Actually, this will be enough to address the majority of situations in which a dynamically-tunable device is required, avoiding the need for an analogue method. Such transformations can be expressed as

$$\bar{x}^i = f_i(x^1, x^2, x^3, t), \quad (4)$$
$$\bar{t} = f_0(t). \quad (5)$$

The origin of the difference between the acoustic and thermodynamic cases lies in the distinct nature of their respective equations. Specifically, sound propagation is modeled by a hyperbolic differential equation containing a second-order time derivative, while heat transfer is governed by a parabolic differential equation in which only a first-order time derivative appears.

In order to prove the mentioned form invariance, let us express equation (3) in terms of the new variables $\bar{x}^i$ and $\bar{t}$. Defining

$$\Lambda_i^{\bar{i}} \equiv \partial_i \bar{x}^{\bar{i}}, \qquad \Lambda_{\bar{i}}^{i} \equiv \partial_{\bar{i}} x^i \quad (6)$$
$$V^{\bar{i}} \equiv \partial_t \bar{x}^{\bar{i}}, \qquad \phi \equiv \partial_t \bar{t}, \quad (7)$$

where $\partial_{\bar{i}} = \partial/\partial \bar{x}^{\bar{i}}$, we obtain the following identities

$$\begin{aligned}
\partial_i &= \Lambda_i^{\bar{i}} \partial_{\bar{i}}, & (8)\\
\partial_t &= V^{\bar{i}} \partial_{\bar{i}} + \phi\, \partial_{\bar{t}}, & (9)\\
v^i \partial_i T &= \Lambda_{\bar{i}}^{i} \bar{v}^{\bar{i}} \Lambda_i^{\bar{j}} \partial_{\bar{j}} T = \bar{v}^{\bar{i}} \partial_{\bar{i}} T, & (10)\\
\partial_i\left(\sqrt{\gamma} K \gamma^{ij} \partial_j T\right) &= \Lambda_i^{\bar{i}} \partial_{\bar{i}}\left(\sqrt{\gamma} K \gamma^{ij} \Lambda_j^{\bar{j}} \partial_{\bar{j}} T\right) \\
&= \frac{\sqrt{\gamma}}{\sqrt{\bar{\gamma}}} \partial_{\bar{i}}\left(\sqrt{\bar{\gamma}} K \bar{\gamma}^{\bar{i}\bar{j}} \partial_{\bar{j}} T\right), & (11)
\end{aligned}$$

where $\bar{v}^{\bar{i}}$ are the components of the velocity vector in the new coordinate system and $\bar{\gamma}_{ij}$ is the spatial metric in such a system. Therefore, equation (3) becomes

$$\begin{aligned}
a\sqrt{\bar{\gamma}}\phi\, \partial_{\bar{t}} T + a\sqrt{\bar{\gamma}}\left(V^{\bar{i}} + \bar{v}^{\bar{i}}\right)\partial_{\bar{i}} T \\
- \partial_{\bar{i}}\left(K\sqrt{\bar{\gamma}}\bar{\gamma}^{\bar{i}\bar{j}}\partial_{\bar{j}} T\right) = 0 \quad (12)
\end{aligned}$$

Clearly, equation (12) still has the form of the original heat equation. Therefore, we can follow the standard procedure and interpret equation (12) as the heat equation for a new medium in the original system. To this end, it will be necessary to consider this new medium to posses an anisotropic thermal conductivity characterized by the tensor $\tilde{K}^{ij}$, as well as a specific heat capacity $\tilde{c}$, a density $\tilde{\rho}$ (with $\tilde{a} = \tilde{c}\tilde{\rho}$) and a velocity $\tilde{\mathbf{v}}$. This corresponds to equation (3) with the replacements $K\gamma^{ij} \to \tilde{K}^{ij}$, $a \to \tilde{a}$, and $v^i \to \tilde{v}^i$. Such an equation will be mathematically identical to the transformed equation, equation (12), when the following relations hold

$$\tilde{a} = \frac{a}{\sqrt{\gamma}}\left[\sqrt{\bar{\gamma}}\phi\right]_{\bar{x}^i, \bar{t} \to x^i, t}, \quad (13)$$

$$\tilde{v}^i = \left[\frac{V^i + \bar{v}^i}{\phi}\right]_{\bar{x}^i, \bar{t} \to x^i, t}, \quad (14)$$

$$\tilde{K}^{ij} = \frac{K}{\sqrt{\gamma}} \left[\sqrt{\bar{\gamma}}\bar{\gamma}^{ij}\right]_{\bar{x}^i, \bar{t} \to x^i, t}, \quad (15)$$

Here $\bar{x}^i, \bar{t} \to x^i, t$ means relabeling $\bar{x}^i, \bar{t}$ to x^i, t (in the particular case below identify the new coordinates again as cylindrical coordinates).

3. Dynamically-tunable heat cloak

To exemplify the theory developed above and in analogy with the time cloaks studied in optics and acoustics [3, 5], we consider a transformation that gradually opens and closes a hole in space, as depicted in figure 1. Nevertheless, instead of transforming only one spatial dimension as in the previous works on time cloaking, we consider here a time-dependent transformation of two-dimensional space. For that purpose, we assume that the problem is z-invariant, which implies that the components of the metric with $i = 3$ or $j = 3$ are not relevant. In fact, this effectively reduces the number of spatial dimensions to two, allowing us to work in the xy plane. Specifically, we propose the following time-dependent transformation of the disk with radius r_{B}, expressed in cylindrical coordinates, to achieve a dynamically-adjustable cloaking effect:

$$\bar{r} = r_{\mathrm{A}}(t) + \frac{r_{\mathrm{B}} - r_{\mathrm{A}}(t)}{r_{\mathrm{B}}} r, \quad (16)$$

$$\bar{\theta} = \theta, \quad (17)$$

$$\bar{t} = t. \quad (18)$$

That is, a hole of variable radius $r_{\mathrm{A}}(t)$ is opened in the xy plane. Note that the transformation is continuous at $r = r_{\mathrm{B}}$. According to equations 13-15, the parameters required to implement this transformation are (assuming a non-moving initial medium):

$$\tilde{K}^{ij} = \mathrm{diag}\left(\frac{r - r_{\mathrm{A}}(t)}{r}, \frac{r}{r - r_{\mathrm{A}}(t)}\right), \quad (19)$$

$$\tilde{a} = \frac{r - r_{\mathrm{A}}(t)}{r} \frac{r_{\mathrm{B}}^2}{(r_{\mathrm{B}} - r_{\mathrm{A}}(t))^2} a, \quad (20)$$

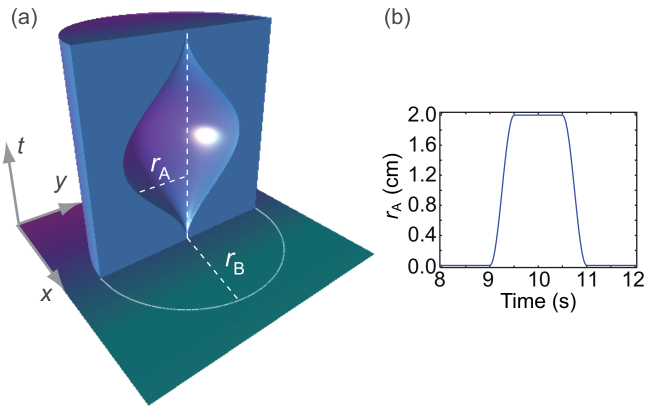

Figure 1. (a) Time cloaking in two spatial dimensions. The disk of radius r_{B} is compressed to the annulus given by $r_{\mathrm{A}} \leq r \leq r_{\mathrm{B}}$. The value of r_{A} varies with time, starting from zero and eventually returning to zero again. In three-dimensional space-time (two spatial dimensions plus time), this transformation creates a hole within the cylinder $r \leq r_{\mathrm{B}}$ whose axis lies along the t axis. Half of such cylinder and hole are shown in the figure. (b) Function $r_{\mathrm{A}}(t)$ employed in the simulations.

$$\tilde{\mathbf{v}} = \partial_t r_{\mathrm{A}}(t) \frac{r_{\mathrm{B}} - r}{r_{\mathrm{B}} - r_{\mathrm{A}}(t)} \hat{r}, \quad (21)$$

where $\hat{r}$ is the unit vector along the r direction. To verify the previous relations, we numerically analyze (via COMSOL Multiphysics) the particular example shown in figure 2(a). It consists of a rectangular region thermally insulated at the upper and lower ends and whose left (right) boundary is assumed to be at a constant temperature of T_{L} (T_{R}), with $T_{\mathrm{L}} > T_{\mathrm{R}}$ (see figure 2). We take the initial temperature value to be T_{R} in the whole domain and the background material to be copper [$K = 394$ W/(Km), $a = 3.49$ MJ/(Km^3)]. This configuration results in a similar problem to that studied in [2]. For a homogeneous copper background, a time-dependent temperature gradient in the x direction is obtained [figure 2(a)]. If we now change the system parameters according to equations 19-21, a disk of variable radius r_{A} and homogeneous temperature is created. In particular, the value of

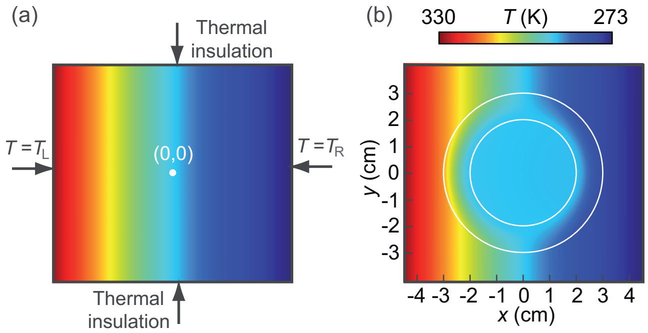

Figure 2. Simulated temperature distribution at $t = 10$ s for two different scenarios. (a) Homogeneous copper sheet. (b) Time-varying thermal cloak (copper background). In all simulations, $T_L = 330$ K, $T_R = 273$ K, and $r_B = 3$ cm. An animation showing the temporal evolution of the cloak is provided as Media 1.

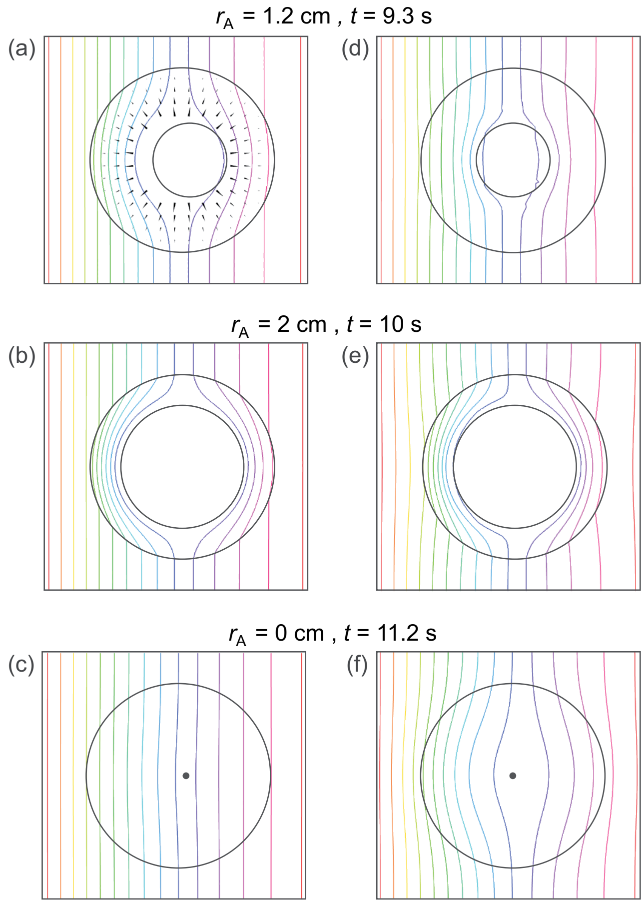

Figure 3. Simulated isotherms at different instants. The value of r_A at each instant is also shown. (a-c) Cloaking with flow. The arrows indicate the velocity field at $t = 9.3$ s (no flow is required at the other shown instants). (d-f) Cloaking without flow.

temperature within the disk is that of the point $(x, y) = (0, 0)$ in the untransformed problem. In the simulations, r_A is given by the function in figure 1(b). The temperature distribution at a time at which the radius is maximum is depicted in figure 2(b). The proposed device could be used to dynamically adjust the temperature of a given region by taking a point with the desired value as the origin of the cloak, which will be expanded to the cloaked area. Afterwards, if a different temperature is required, we can close the hole and open a second one starting from a different point, or even open several holes simultaneously.

4. Practical implementation

From the implementation point of view, synthesizing the required flow is probably the most technologically-challenging aspect. Fortunately, in some cases, the contribution of the flow term in equation (12) might not be determinant, provided that it is sufficiently low in comparison with the other ones. In fact, taking $\tilde{\mathbf{v}} = 0$ in the previous example does not entail a serious degradation of the cloaking performance. To see this, in figure 3 we have depicted the isotherms for the cases with and without flow at different instants. Clearly, a perfect cloaking effect is achieved via the full-parameter device, with no temperature modification outside the cloak. On the other hand, the simplified no-flow cloak alters the outer temperature distribution, even when the cloak is already closed, and introduces a slight deviation from the desired temperature transformation. Nevertheless, an approximate cloaking effect is still attained, which might be acceptable in many situations.

Another issue is the synthesis of an anisotropic time-varying thermal conductivity.

Composites made up of the proper combination of isotropic materials offer a feasible solution. Moreover, the homogenization theory developed for time-independent spatially periodic media can be extended straightforwardly to time-dependent thermal properties. For example, following a standard two-scale approach [14], it can be shown that, in the no-flow case, the effective time-dependent parameters are obtained just by including the time dependence in the usual expressions for the homogenized static parameters (e.g., those obtained in [1]). For instance, the effective parameters of a periodic 2D cylindrical multilayer structure made up of two different homogeneous isotropic materials characterized by parameters $a_{\mathrm{A}}(t)$, $K_{\mathrm{A}}(t)$ and $a_{\mathrm{B}}(t)$, $K_{\mathrm{B}}(t)$ are

$$K_{\mathrm{eff}}^{r} = \frac{d_{\mathrm{A}} + d_{\mathrm{B}}}{d_{\mathrm{A}} K_{\mathrm{A}}^{-1}(t) + d_{\mathrm{B}} K_{\mathrm{B}}^{-1}(t)}, \tag{22}$$

$$K_{\mathrm{eff}}^{\theta} = \frac{d_{\mathrm{A}} K_{\mathrm{A}}(t) + d_{\mathrm{B}} K_{\mathrm{B}}(t)}{d_{\mathrm{A}} + d_{\mathrm{B}}}, \tag{23}$$

$$a_{\mathrm{eff}} = \frac{d_{\mathrm{A}} a_{\mathrm{A}}(t) + d_{\mathrm{B}} a_{\mathrm{B}}(t)}{d_{\mathrm{A}} + d_{\mathrm{B}}}. \tag{24}$$

where d_{A} (d_{B}) is the thickness of the layers of material A (B), which may depend on time as well. This kind of composite could be used to implement the approximate version of the time-varying cloak described above [that is, to synthesize the medium given by equations (19)-(20)], as well as other dynamical devices with cylindrical symmetry, such as reconfigurable concentrators. In addition, note that a diffusion equation for photons with the same structure as equation (2) (with $\mathbf{v} = 0$) has been recently used to demonstrate a cloak for diffusive light [15]. Therefore, the theory developed here could also be applied in that case, at least the simplified version with no flow, to create dynamically-tunable optical cloaks.

Finally, it is worth mentioning that, to obtain a full space-time thermodynamics transformation theory (i.e., including transformations in which the new time variable also depends on the old spatial variables), a completely relativistic (form-invariant) equation should be employed. In the literature one can find the so-called hyperbolic heat equation [16] and its associated second sound effects [17], as well as the relativistic heat equation [18]. However, these equations present some controversial aspects and their experimental validity is still under study [19]. Furthermore, considering a moving medium gives rise to additional complications in this case [20].

5. Conclusion

In conclusion, building on the heat convection-diffusion equation, we have developed a space-time transformation thermodynamics theory that provides an additional degree of control for heat management applications. As an example, we have designed a reconfigurable thermal cloak that can be dynamically opened and closed by using an inhomogeneous moving medium. In addition, we have shown that a simplified no-flow version of this device provides an approximate cloaking effect. The recent advances in the dynamic tuning of the thermal properties of different materials, for instance, through electrochemical [21] or voltage [22] modulation, could provide a suitable platform for the implementation of the proposed thermal space-time transformation media.

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