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Transitions between 4-intersection values of planar regions

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Abstract

If $A(t)$ and $B(t)$ are subsets of the Euclidean plane which are continuously morphing, we investigate the question of whether they may morph directly from being disjoint to overlapping so that the boundary and interior of $A(t)$ both intersect the boundary and interior of $B(t)$ without first passing through a state in which only their boundaries intersect. More generally, we consider which 4-intersection values—binary 4-tuples specifying whether the boundary and interior of $A(t)$ intersect the boundary and interior of $B(t)$—are adjacent to which in the sense that one may morph into the other without passing through a third value. The answers depend on what forms the regions $A(t)$ and $B(t)$ are allowed to assume and on the definition of continuous morphing of the sets.

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1. Introduction

Given two sets A and B in the Euclidean plane, the *4-intersection value* associated with A and B is the binary 4-tuple

$$(\,\chi(\partial A \cap \partial B),\ \chi(A^\circ \cap B^\circ),\ \chi(\partial A \cap B^\circ),\ \chi(A^\circ \cap \partial B)\,)$$

where C° and ∂C denote the interior and boundary, respectively, of C, $\chi(C) = 0$ if $C = \varnothing$, and $\chi(C) = 1$ if $C \neq \varnothing$. The 4-intersection values are used

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in Geographic Information Systems to quantify the nature of the intersection of two regions A and B in the plane. For example, regions A and B may represent the habitats of a predator and its prey, the extent of a nature preserve and the moist regions of a desert, or the area protected by a military base and the area covered by cellular telephone service. Such regions are not static. As they dynamically change, their 4-intersection values, or simply values, may also change. We say two values V_1 and V_2 are *adjacent* if there exist dynamically changing sets $A(t)$ and $B(t)$ which pass from value V_1 to value V_2 without passing though any other intermediate values. Our goal is to determine which values are adjacent. The answer depends heavily on what restrictions are imposed. A common restriction in geographic applications is to assume the regions are *spatial regions*, that is, proper nonempty subsets of the plane which are regular closed and have connected interior. In particular, note that spatial regions must have positive area. Spatial regions behave relatively nicely, but they limit the situations which may be modeled. The habitats of species or cellular coverage areas may be disconnected regions. As an elliptical puddle of water dries up, it may shrink to the major axis (which is not regular closed) before disappearing entirely. To allow the modeling of such situations, we will not restrict our attention to spatial regions.

Throughout, we will consider functions $A, B : \mathbb{R} \to \mathcal{P}(\mathbb{R}^2)$ which give a subset of the Euclidean plane at each time t. We must stipulate how regions $A(t)$ and $B(t)$ are allowed to change. An elliptical puddle of water may dry up uniformly with the entire moist region disappearing at an instant. While this may be continuous in some measure (namely, the moisture content), the area is not continuously changing: it jumps from positive area to zero area discontinuously. Discontinuous areas may be useful for some models. For regions determined by electronic transmission coverage, such as WiFi accessibility, turning on a new transmitter will instantly and discontinuously increase the coverage area.

The 4-intersection values were introduced in [6], where they were applied to spatial regions. The work of [15] connects these concepts to relational algebras. 4-intersection values were applied to regions homeomorphic to 2-dimensional disks in [7] and to regions with holes in [5]. By comparing the exteriors (i.e., complements) of two regions along with the interiors and boundaries, there are $3^2 = 9$ possible matchings which could be empty or not, giving rise to the 9-intersection value model. This was introduced for spatial regions and has been studied for particular shapes in [12] and [13]. A variation of the 9-intersection model replacing the exterior with another set is studied in [2]. Our work is closest to that of [4], where the authors consider the adjacency graph for the 9-intersection values when restricted to specific transformations of spatial regions, such as scalings, translations, and rotations.

The restriction to spatial regions already limits the number of attainable intersection values to 8 (using 4-intersection or 9-intersection). For example, if A and B are spatial regions with $\partial A \cap B^\circ \neq \varnothing$, then $A^\circ \cap B^\circ \neq \varnothing$ since neighborhoods of every boundary point of A include interior points of A (see [7]).

Without restricting to spatial regions, all $2^4 = 16$ possible 4-intersection values are attainable, giving $\binom{16}{2} = 120$ possible adjacencies to consider. There are $2^9 = 512$ possible 9-intersection values, giving $\binom{512}{2} = 130,816$ possible adjacencies to consider. As this number would be unwieldy, we focus on the 4-intersection values. The techniques would be similar for 9-intersection values. We impose weaker restrictions on the allowed transformations than considered in [4]. Intersections of dynamically moving directed lines and regions have also been considered in [16] and [11].

Besides the 4- and 9-intersection models, geographers use a point-free approach called Regional Connection Calculus (RCC). In [9], it is shown that RCC is equivalent to considering regular closed, nonempty sets in a regular connected space. See [14] for further discussion of the interplay between these and other approaches.

A standard way to create a closed set in the plane which is not regular closed is to add a *whisker*, that is, a line segment protruding from the set. The closed ball centered at (h, k) of radius r is the set $\bar{B}((h, k), r) = \{(x, y) \in \mathbb{R}^2 : (x - h)^2 + (y - k)^2 \leq r\}$.

2. The 4-Intersection Values

If A and B are not required to be spatial regions, all 16 possible 4-intersection values may be realized. Table 1 systematically lists and labels the 16 possible intersection values, giving one possible realization of each. The table shows that with the exception of Value 5, each may be realized with compact sets A and B. The values which may be realized with spatial regions also include the name of the intersection value when applied to spatial regions.

If the 4-intersection value of $A(t)$ and $B(t)$ changes from V_1 to V_2 without assuming any other values, either there is a first instant of V_2 occurring or a last instant of V_1 occurring. We say the V_1 transforms to V_2 *instantly*, at the instant $t = a$, if $A(t)$ and $B(t)$ have the value V_1 for $t < a$ or $t > a$ and have the value V_2 at $t = a$. We say that V_1 transforms to V_2 *directly* if there exists a such that $A(t)$ and $B(t)$ have the value V_1 for $t = a$ and have the value V_2 for $t < a$ or for $t > a$. Thus, V_1 transforms to V_2 at an instant if and only if V_2 transforms to V_1 directly. If V_1 transforms to V_2 instantly or directly, we say that V_1 and V_2 are *adjacent*.

3. Continuous Area

In this section, we investigate possible adjacencies assuming that regions $A(t)$ and $B(t)$ are closed at all times and the functions $a(t), b(t)$, and $ab(t)$ giving the area of $A^\circ(t), B^\circ(t)$, and $A^\circ(t) \cap B^\circ(t)$, respectively, are continuous extended real-valued functions.

Continuity of area seems to be a natural requirement for morphing regions, although a simple example will show that this alone will not be adequate in many situations. If $A = \overline{B}((0,0), 2) \cup \overline{B}((5,0), 1)$ for $t \leq 0$ and $A = \overline{B}((0,0), 1) \cup \overline{B}((5,0), 2)$ for $t > 0$ and $B = [-3, 3] \times [-3, 3]$ for $t \leq 0$ and $B = [3, 9] \times [-2, 4]$

$(\partial A \cap \partial B, A^\circ \cap B^\circ, \partial A \cap B^\circ, A^\circ \cap \partial B)$

VALUE 1 $(0,0,0,0)$ Disjoint	VALUE 2 $(0,0,0,1)$	VALUE 3 $(0,0,1,0)$	VALUE 4 $(0,0,1,1)$
VALUE 5 $(0,1,0,0)$ $A = B = \mathbb{R}^2$	VALUE 6 $(0,1,0,1)$ Contains	VALUE 7 $(0,1,1,0)$ Inside	VALUE 8 $(0,1,1,1)$
VALUE 9 $(1,0,0,0)$ Meets	VALUE 10 $(1,0,0,1)$	VALUE 11 $(1,0,1,0)$	VALUE 12 $(1,0,1,1)$
VALUE 13 $(1,1,0,0)$ Equals	VALUE 14 $(1,1,0,1)$ Covers	VALUE 15 $(1,1,1,0)$ Covered By	VALUE 16 $(1,1,1,1)$ Overlaps

TABLE 1. The 16 values of 4-intersection values

for $t > 0$, then the areas of $A(t)$, $B(t)$, and $A(t) \cap B(t)$ are constant and thus continuous, though this hardly seems to be a good description of continuous morphing.

Still, we investigate the possible adjacencies under the weak assumption of continuous areas. We start with spatial regions in Example 3.1 and gradually weaken the assumptions on the spaces throughout the section.

For spatial regions A and B, Disjoint is adjacent to Overlaps. One may initially be tempted to believe that as disjoint regions A and B morph continuously from Value 1 $(0,0,0,0)$ = Disjoint to Value 16 $(1,1,1,1)$ = Overlaps, they should pass through Value 9 $(1,0,0,0)$ = Meets. We present two examples which show that this transformation need not pass through Value 9.

Example 3.1 (Disjoint is adjacent to Overlaps)**.** (a) Consider a static set $A = \{(x,y) \in \mathbb{R}^2 : x > 0, y \geq 1/x\}$ and a dynamic set $B(t) = \{(x,y) \in \mathbb{R}^2 : y \leq t\}$. Now A and B are Disjoint for $t \leq 0$ and Overlap for $t > 0$, showing that

Disjoint transforms directly to Overlap, and Overlap transforms instantly to Disjoint. This example depends on the choice of spatial regions which are not compact.

(b) To see that it is possible with compact spatial regions, let $A = [-1, 1] \times [1/2, 2]$ and $B = ([-1, 1] \times [-1, 0]) \cup (cl(-r, r) \times [0, 1])$. When $r \leq 0$, A and B are Disjoint rectangles. For $r > 0$, A and B Overlap.

For the next few results, we will assume that all regions have finite area. This will be satisfied if the regions are compact. In particular, the finite area assumption rules out Value 5 in which $A = B = \mathbb{R}^2$.

Proposition 3.2. *Suppose $A(t)$ and $B(t)$ are closed with connected interiors and positive finite areas and the areas $a(t)$, $b(t)$, and $ab(t)$ of $A(t), B(t)$, and $A(t) \cap B(t)$, respectively, are continuous. Then the possible adjacencies are those given in the graph of Figure 1.*

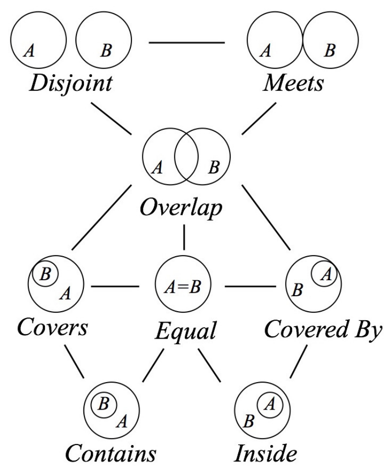

FIGURE 1. Adjacency graph for closed regions with connected interior and finite nonzero area, with $a(t), b(t)$, and $ab(t)$ continuous.

Proof. It is easy to see that the adjacencies shown are possible, with the surprising exception of Disjoint begin adjacent to Overlaps. Example 3.1(b) shows this adjacency under the hypotheses given. It remains to show that Disjoint and Meet are not adjacent to any of Covers, Equal, Covered By, Contains, or Inside, and that neither of Covers or Contains is adjacent to either Covered By or Inside.

Under the assumptions, Covers, Contains, Covered By, Inside, and Equals imply that the interior of one set is contained in the interior of the other. For example, if A covers B, and $B^\circ \not\subseteq A^\circ$, then A° and $X - A$ provide a separation of B°, contrary to B° being connected (cf. [6, Prop. 5.5]).

Suppose that Disjoint is adjacent to Covers, with A and B satisfying the Disjoint condition for $t < 0$ and Covers for $t > 0$. Now $ab(t) = 0$ for $t < 0$, so by continuity, $ab(0) = 0$. Since $B^\circ \subseteq A^\circ$ for $t > 0$, $f(t) = b(t) - ab(t) = 0$ for $t > 0$, so by continuity, $f(0) = 0$. Now $b(0) = f(0) + ab(0) = 0$, contrary to $b(t) > 0$. The same argument shows that Disjoint and Meet are not adjacent to any of Covers, Equal, Covered By, Contains, or Inside are similar.

To see that neither of Covers or Contains is adjacent to either Covered By or Inside, we will show that Covers is not adjacent to Covered By. The same argument works for the other proofs. Suppose A and B satisfy Covers for $t < 0$ and Covered By for $t > 0$, and do not assume a third value at $t = 0$. Now $b(t) < a(t)$ for $t < 0$ and $b(t) > a(t)$ for $t > 0$. The continuity conditions imply $a(0) = b(0)$, which is not possible if A and B satisfy Covers or Covered By. $\square$

We note that if $A(t)$ and $B(t)$ are spatial regions, then they are closed sets with connected interiors and positive areas.

The proof was clearly based on A and B having positive area, which we will now drop. This permits many more adjacencies.

Proposition 3.3. *Suppose $A(t)$ and $B(t)$ are closed with connected interiors and finite (possibly zero) areas and $a(t)$, $b(t)$, and $ab(t)$ are continuous. Then all values are adjacent except those shown in the non-adjacency graph of Figure 2.*

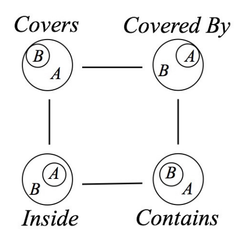

FIGURE 2. Non-adjacency graphs for closed regions with connected interior and finite area, with $a(t), b(t)$, and $ab(t)$ continuous.

Proof. The pairs Covers and Covered By, Covers and Inside, and Covered By and Contains are not adjacent (even under weaker hypotheses, omitting the connected interior condition) by Theorem 3.10 below. Under the hypotheses of the proposition, suppose $A(t)$ and $B(t)$ satisfy Contains for $t < 0$ and Inside for $t > 0$, and do not assume a third value at $t = 0$. Then $B^\circ(t) \subseteq A^\circ(t)$ and $a(t) \geq b(t)$ for $t < 0$ while $A^\circ(t) \subseteq B^\circ(t)$ and $a(t) \leq b(t)$ for $t > 0$. By continuity, $a(0) = b(0)$, so $A^\circ(0)$ and $B^\circ(0)$ have the same positive area and either $B^\circ(0) \subseteq A^\circ(0)$ or $A^\circ(0) \subseteq B^\circ(0)$. It follows that $A^\circ(0) = B^\circ(0) \neq \varnothing$, and thus (since they do not have Value 5 of $A = B = \mathbb{R}^2$) there exists $x \in \partial A^\circ(0) = \partial B^\circ(0)$. Now $\partial(C^\circ) \subseteq \partial C$ (see Proposition 3.5 below), so $x \in \partial A(0) \cap \partial B(0)$ and thus $A(t)$ and $B(t)$ assume a 4-intersection value of form $(1, y, z, w)$ at $t = 0$ and in particular, is neither Contains nor Inside at that instant.

To complete the proof that the non-adjacency graph in Figure 2 is complete, we present examples confirming adjacencies between all remaining states which were not shown in Figure 1.

Disjoint and Covers are adjacent. Let $A = \{(0,0)\} \cup (cl(-r,r) \times [-1,1])$ and $B = \{(0,1)\} \cup (cl(-r/2,r/2) \times [-1,1])$. For $r \leq 0$, $A = \{(0,0)\}$ and $B = \{(0,1)\}$ are disjoint. For $r > 0$, $A = [-r,r] \times [-1,1]$ covers $B = [-r/2,r/2] \times [-1,1]$. Interchanging A and B shows that *Disjoint and Covered By are adjacent.* Changing the singleton of B so that $B = \{(0,0)\}$ for $r \leq 0$ shows *Meets and Covers are adjacent*, and then interchanging A and B shows *Meets and Covered By are adjacent.*

Disjoint and Equals are adjacent. With A as above, let $B = \{(0,1)\} \cup (cl(-r,r) \times [0,1])$. For $r \leq 0$, $A = \{(0,0)\}$ and $B = \{(0,1)\}$ are disjoint. For $r > 0$, $A = B = [-r,r] \times [-1,1]$. Changing the singleton of B so that $B = \{(0,0)\}$ for $r \leq 0$ shows *Meets and Equals are adjacent.*

Disjoint and Contains are adjacent. With A as above, let $B = \{(0,1/2)\} \cup (cl(-r/2,r/2) \times [0,1/2])$. For $r \leq 0$, $A = \{(0,0)\}$ and $B = \{(0,1/2)\}$ are disjoint. For $r > 0$, $A = [-r,r] \times [-1,1]$ contains $B = [-r/2,r/2] \times [0,1/2]$. Interchanging A and B show that *Disjoint and Inside are adjacent.* Changing the singleton of B so that $B = \{0,0)$ for $r \leq 0$ shows *Meets and Contains* are adjacent, and then interchanging A and B shows *Meets and Inside* are adjacent.

Overlap and Contains are adjacent. Let $A = [-2,2] \times [-2,2]$ and $B = ([-1,1] \times [-1,1]) \cup (cl(-r,r) \times [0,4])$. A contains B for $r \leq 0$ and A and B overlap for $r > 0$. Interchanging A and B shows *Overlaps and Inside are adjacent.* $\square$

Next, we present several elementary results needed to prove the nonadjacencies of Theorem 3.10.

Recall that a topological space X is *locally connected* if for every $x \in X$ and every neighborhood U of x, there exists a connected neighborhood V of x with $V \subseteq U$. In particular, $\mathbb{R}^2$ is locally connected.

Proposition 3.4. *Suppose B is a subset of a locally connected topological space X. For $x \in B$, let B_x be the connected component of B which contains x. Then*

$$\bigcup_{x \in B} \partial B_x \subseteq \partial B = \partial\Big(\bigcup_{x \in B} B_x\Big),$$

and equality holds if and only if B is closed.

Proof. Suppose $z \in \partial B_x$ for some $x \in B$. Then every neighborhood of z intersects $B_x \subseteq B$, and every neighborhood of z intersects $X - B_x$. If there is a connected neighborhood V of z which does not intersect $X - B$, then $V \subseteq B$. Now $B_x \cup V$ is the union of two connected sets with a point z in common, so $C = B_x \cup V$ is a connected set. Furthermore, C is strictly larger than B_x, contrary to the fact that B_x was the largest connected subset of B containing

x. Thus, every neighborhood of z intersects B and $X - B$, so $z \in \partial B$. This proves the inclusion.

Suppose B is closed and $z \in \partial B$. Then $z \in clB \cap cl(X - B)$. Now $z \in B$ since $B = clB$, so $z \in B_z \subseteq \bigcup_{x\in B} B_x$. Also, since $B_z \subseteq B$, $z \in cl(X - B) \subseteq cl(X - B_z)$. Thus, $z \in clB_z \cap cl(X - B_z) = \partial B_z$. This shows that the inclusion is equality if B is closed.

To see that equality holds only if B is closed, suppose B is not closed. Then there exists a boundary point a of B which is not in B. Given $x \in B$, $a \in cl(X - B) \subseteq cl(X - B_x)$, so $a \in \partial B_x$ if and only if $a \in cl(B_x) = B_x \subseteq B$, which does not occur since $a \notin B$. Thus, for any point $a \in \partial B - B$, we have $a \notin \bigcup_{x\in B} B_x$, so equality fails. □

We observe that if B has a finite number of connected components, then as a finite union of closed sets, B is closed, and thus equality would hold in Proposition 3.4.

Proposition 3.5. *For any set B in a topological space X, $\partial(B^\circ) \subseteq \partial B$, and equality holds if and only if $B \subseteq cl(B^\circ)$. In particular, equality holds if B is open or is regular closed.*

Proof. If $x \in \partial(B^\circ) = cl(B^\circ) \cap cl(X - B^\circ)$, then $x \in cl(B^\circ) \subseteq cl(B)$, and $x \in cl(X - B^\circ) = X - B^\circ$. Thus, no neighborhood of x is contained in B, so every neighborhood of x intersects $X - B$, so $x \in cl(X - B) \cap cl(B) = \partial B$.

It remains to show that $\partial B \subseteq \partial(B^\circ)$ if and only if $B \subseteq cl(B^\circ)$. Suppose $B \subseteq cl(B^\circ)$ and $x \in \partial B = clB \cap cl(X - B)$. Now $x \in clB \subseteq clcl(B^\circ) = cl(B^\circ)$ and $x \in cl(X-B) \subseteq cl(X-B^\circ)$ (since $B^\circ \subseteq B$), so $x \in cl(B^\circ)\cap cl(X-B^\circ) = \partial(B^\circ)$. Conversely, suppose $B \nsubseteq cl(B^\circ)$ and choose an $x \in B-cl(B^\circ)$. Now $x \notin cl(B^\circ)$ implies $x \notin \partial(B^\circ) = cl(B^\circ) \cap cl(X - B^\circ)$. Now $x \notin cl(B^\circ)$ implies $x \notin B^\circ$, so every neighborhood of x intersects $X - B$, and thus $x \in cl(X - B)$. Since $x \in B \subseteq cl(B)$, we have $x \in cl(B) \cap cl(X - B) = \partial B$. Thus, $\partial B \nsubseteq \partial(B^\circ)$. □

For examples where $\partial B \nsubseteq \partial(B^\circ)$, take $B = \mathbb{Q}$ in $X = \mathbb{R}$ or $B = \bar{B}((0,0),1)\cup(\{0\} \times [0,3])$ in $\mathbb{R}^2$. From Proposition 3.5, we deduce the following.

Proposition 3.6. *If A and B are subsets of a topological space with $A^\circ \subseteq B^\circ$ and $\partial A \cap \partial B = \varnothing$, then $\partial(A^\circ) \subseteq B^\circ$.*

For the next proposition, we will use the following lemma.

Lemma 3.7 ([1, Lemma 6.16])**.** *Suppose A and B are subsets of a topological space, $\partial A\cap B = \varnothing$, and B is connected. Then either $B \subseteq A^\circ$ or $B\cap cl(A) = \varnothing$.*

Part (a) of the next result follows from Proposition 3.4 and Lemma 3.7, and part (b) follows from part (a), and Propositions 3.5 and 3.6.

Proposition 3.8. *Suppose A and B are subsets of a locally connected topological space X with $\partial A \cap B^\circ = \varnothing$, $\partial A \cap \partial B = \varnothing$, and $A^\circ \cap B^\circ \neq \varnothing$. Let A_x be the connected component of A which contains x, with B_x defined analogously. Then*

(a) *for any $x \in A^\circ \cap B^\circ$, $B_x \subseteq A^\circ_x$, and*

(b) *if* $\partial B_x \neq \varnothing$ *and* $\partial B_x \cap A_x^\circ = \varnothing$*, then* $B_x^\circ \cap A^\circ = \varnothing$*.*

Corollary 3.9. *If* A *and* B *are subsets of* $\mathbb{R}^2$ *with 4-intersection Value 5* $(0,1,0,0)$*, then* $A = B = \mathbb{R}^2$*.*

Theorem 3.10. *Suppose* $A(t)$ *and* $B(t)$ *are closed subsets of* $\mathbb{R}^2$ *and the area of the components* $A_x(t)$ *and* $B_x(t)$ *containing* x *as well as the areas of the intersections of the components* $A_x(t) \cap B_x(t)$ *are continuous extended real-valued functions of time. Then the transitions from*

Value 6 $(0,1,0,1)$ *to* *Value 7* $(0,1,1,0)$*,*
Value 6 $(0,1,0,1)$ *to* *Value 15* $(1,1,1,0)$*, and*
Value 7 $(0,1,1,0)$ *to* *Value 14* $(1,1,0,1)$

are not possible without passing through intermediate values.

Proof. Suppose Value 6 transforms to Value 7, with Value 6 occurring for time $t < a$ and Value 7 occurring for time $t > a$. Note that $A^\circ \cap B^\circ$ remains nonempty at all times, and for every $x \in A^\circ \cap B^\circ$, the components A_x and B_x of A and B, respectively, containing x must either have an intersection value of Value 6 or of Value 7, by Lemma 3.7. For $t < a$, we have $\partial B_x \cap A_x^\circ \neq \varnothing$ for those components A_x, B_x of $x \in A^\circ \cap B^\circ$ which have Value 6. When these $\partial B_x \cap A_x^\circ$ become empty at or after $t = a$, $B_x^\circ \cap A_x^\circ$ becomes empty by Proposition 3.8(b). Thus, the area of $A^\circ \cap B^\circ$ in components of Value 6 goes to zero at or after $t = a$. Since this area changes continuously, there is a first instant of no area, so this area in components of Value 6 is zero at time $t = a$. To maintain $A^\circ \cap B^\circ \neq \varnothing$ at time $t = a$, there must be nonzero area in $A^\circ \cap B^\circ$ in components of Value 7 at time $t = a$. However, the same argument applied to the area in components of Value 7 as time decreases to $t = a$, shows that there is no last instant of area in components of Value 7. Thus, if such area is nonzero at $t = a$, it was nonzero for some values $t < a$, when there was also nonzero area in components of Value 6, and the presence of nonzero area in components of Values 6 and 7 simultaneously for $t \leq a$ gives an intermediate Value 8 $(0,1,1,1)$.

The proof that Value 6 cannot transform to Value 15 is similar, and its dual shows that Value 7 cannot transform to Value 14. □

Note that in Theorem 3.10 we have weakened the assumptions so that $A(t)$ and $B(t)$ need only be closed sets with continuous extended real-valued areas and areas of intersections. In this generality, we may now again consider the exceptional Value 5, which by Corollary 3.9, is only realized as $A = B = \mathbb{R}^2$.

Proposition 3.11. *If* $A(t)$ *and* $B(t)$ *are closed subsets of the plane with* $a(t)$*,* $b(t)$ *and* $ab(t)$ *continuous extended real-valued functions, then Value 5 is not adjacent to Values 1–4 nor Values 9–12, and is adjacent to Values 6, 7, 8, and 13–16.*

Proof. Values 1–4 and 9–12 have $A^\circ \cap B^\circ = \varnothing$. Value 5 has the area of $A^\circ \cap B^\circ$ being infinite. This jump from 0 area to infinite area is not permitted by the assumption of continuously changing area.

Value 5 adjacencies are most easily seen moving to Value 5. Value 6 is realized by taking B to be the closed ball $\bar{B}((0,0),r) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq r\}$ and $A = \bar{B}((0,0),r+1)$. As r converges to infinity, A and B converge to $\mathbb{R}^2$, giving Value 5. Value 7 is dual to Value 6. Value 8 is realized by $A = \bar{B}((0,0),r+1) \cup \bar{B}((2r,0),1)$ and $B = \bar{B}((0,0),r) \cup \bar{B}((2r,0),2)$, Value 13 by $A = B = \bar{B}((0,0),r)$, Value 14 by $A = [-r,r+1]^2$ and $B = [-r,r]^2$, Value 15 by interchanging A and B in Value 14, and Value 16 by $A = (-\infty,r] \times [r,\infty)$ and $B = [-r,\infty) \times (-\infty,r]$. In each case, these values converges to Value 5, $A = B = \mathbb{R}^2$, as r goes to infinity. $\square$

Theorem 3.10 and Propostion 3.11 listed the non-adjacencies for closed sets under continuity of area conditions, shown in Figure 3. Indeed, these are the only non-adjacencies under these assumptions.

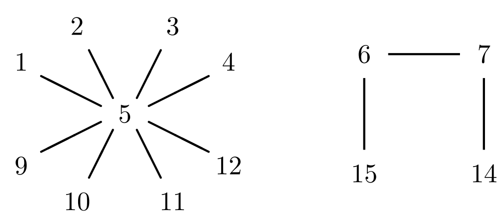

FIGURE 3. Non-adjacency graph for closed regions with $a(t), b(t)$, and $ab(t)$ continuous extended real-valued functions.

The following constructions may be used to show adjacencies.

Construction 1: *Create a whisker instantly. Delete an interior instantly.* Consider the set $C = [-r,r] \times [0,1]$ as r changes continuously with time. For $r < 0$, $C = \varnothing$, for $r = 0$, $C = \{0\} \times [0,1]$ is a whisker, and for $r > 0$, C is a regular closed set with nonempty interior. As r increases to zero, the whisker appears instantly when $r = 0$. As r decreases to zero, the area disappears at the instant $r = 0$. Note that the area of C changes continuously with r, and thus with time.

Construction 2: *Delete a whisker at an instant.* Consider the set $D = cl(-r,r) \times \{0\}$ as r changes continuously with time. If $r \leq 0$, $D = \varnothing$. If $r > 0$, $D = [-r,r] \times \{0\}$, a whisker. Letting r decrease to zero, the whisker will disappear instantly when $r = 0$. Note that the length of the whisker D changes continuously with r, and thus with time.

Construction 3: *Create interior and boundary directly and simultaneously,* without first creating a boundary. Consider the set $E = cl(-r,r) \times [0,1]$ as r changes continuously with time. If $r \leq 0$, $E = \varnothing$. If $r > 0$, $E = [-r,r] \times [0,1]$, a regular closed set with nonempty interior. Letting r decrease to zero, the interior and boundary both disappear simultaneously at the instant when $r = 0$. Reversing this operation, the area and boundary appear simultaneously and directly for $r > 0$. Note that the area of E changes continuously with r and thus with time.

We observe that since area is a continuous function, it can vanish at an instant, but it cannot appear at an instant—only directly. Constructions 1 and 2 allow us to create whiskers at an instant or delete whiskers at an instant, facilitating many transitions between 4-intersection values. If area needs to be created to cause A° to intersect B° or ∂B, then Construction 3 allows us to create area directly, and whiskers may be added or deleted directly by Constructions 1 or 2 in reverse.

While Construction 3 creates area directly and continuously, we cannot allow arbitrary use of it. For example, for $x \in \mathbb{R}$, let $E_x = cl(x - r, x + r) \times [0, 1]$. As r continuously increases, the area of E_x continuously increases, becoming nonnegative when $r > 0$. However, if we let $E = \bigcup\{E_x : x \in \mathbb{Q} \cap [0, 1]\}$, the area of E is not continuous. For $r \leq 0$, the area is zero, and for every $r > 0$, the area is greater than 1. The problem here is that we are adding area based on a dense set $\mathbb{Q} \cap [0, 1]$. Using any nowhere dense index set would produce a continuous area function, and in particular, using any finite number of sets created by Construction 3 will produce a continuous area function.

3.1. **Realizations of Adjacencies.** With the aid of these three constructions forwards and in reverse, it is not surprising that the remaining constructions are achievable. Specific examples showing the adjacencies are given below. By Construction $n^{\leftarrow}$ we mean Construction n in reverse.

Value 1 is adjacent to Values 2, 3, 4, 9, 10, 11, and 12 by adding a whisker to A, B, or both simultaneously using Construction 1. It is adjacent to Values 6, 7, 8, 13, 14, 15, and 16 by adding a new component of A, B, or both simultaneously using Construction 3. Values 1–4 are not adjacent to Value 5 for reasons given in the discussion of Value 5.

Value 2 is adjacent to Value 3 by instantly adding a whisker of A in the interior of B and instantly deleting the whiskers of B in the interior of A simultaneously, using Constructions 1 and 2. It is adjacent to Value 4 by adding a whisker to A in an existing second component of B° by Construction 1. Fattening the whisker of B in A° to have positive area by Construction $1^{\leftarrow}$ shows adjacency to Value 6. Deleting the whiskers of B in components of A° and simultaneously adding a component of B with boundary and interior in a component of A° gives Value 7, using Constructions $1^{\leftarrow}$ and 3. Value 8 is obtained by creating new components of A with area and boundary in B° and of B in A° directly and simultaneously using Construction 3 twice. Values 9 and 10 or obtained by deleting the whiskers of B in A° instantly and simultaneously adding a whisker to a component of A which intersects the boundary or boundary and interior of a component of B, using Constructions 1 and 2. Value 11 requires no construction: Simply slide the whisker of B in A° to intersect ∂A. Value 12 is obtained by adding new whiskers to A and B simultaneously using Construction 1. For Value 13, use Construction 3 to directly create a new region with area and boundary, which will become a new component of both A and B

while simultaneously deleting the whisker of B in A° using Construction $1^\leftarrow$. Using Construction 3 to directly add a component of B with positive area in A which shares a boundary with A gives Value 14. Value 15 is obtained by deleting the whisker of B in A° and directly adding a new component of B with positive area in A by which shares a boundary with A, using Constructions $1^\leftarrow$ and 3. For Value 16, slight modifications of Example 3.1 or the argument above Proposition 3.2 by adding a whisker of B in A gives the needed transition.

Value 3 is symmetric to Value 2 by interchanging A and B. Thus, it is adjacent to every other value except Value 5.

Value 4 is adjacent to Values 6, 7, and 8 by fattening a whisker (or two) to have positive area, using Construction $1^\leftarrow$. It is adjacent to Values 9, 10, and 11 by simply sliding a one-point interior whisker until it intersects the boundary, or a linear whisker until it intersects a parallel linear boundary. Value 12 is achieved by staring with a component of A° containing a linear whisker of B and a component of B° containing a linear whisker of B, and sliding the whiskers so that one endpoint of each simultaneously touches the boundary of its enclosing set. For Value 13, directly delete all whiskers of A in B° and all whiskers of B in A° by Construction $1^\leftarrow$ and simultaneously create a region with positive area and boundary by Construction 3 which will be a new component of both A and B. For Value 14, directly delete whiskers of A in B° by Construction $1^\leftarrow$ and simultaneously create a component of B with positive area and boundary by Construction 3 which is in A and intersects ∂A. Value 15 is symmetric to Value 14 by interchanging A and B, and Value 16 is achieved by performing the transition to Values 14 and 15 simultaneously in some components of A and B.

Value 5 adjacencies were given in Proposition 3.11.

Value 6 is not adjacent to Values 7 and 15 by Theorem 3.10. Value 6 transitions to Value 8 by an application of Construction 3, and to Value 9 by an application of Construction $1^\leftarrow$ simultaneous with the intersection of boundaries as components of A and B slide together. Or, with $A = [0,3]^2$ and $B_r = [1,2] \times [\frac{1}{r+1}, \frac{1}{r}]$, letting r approach infinity gives the transition from Value 9 to Value 6. Values 10, 11, and 12 are adjacent to Value 6 by simultaneous application of Construction $1^\leftarrow$ to delete that part of B in A° and Construction 1 to add whiskers. Values 13 and 14 are adjacent to Value 6 using spatial regions such as two nested squares, with the inner one expanding to equal or sliding to intersect the outer. Value 6 is realized by $A = [-2,2]^2$ and $B = [-1,1]^2 \cup [3,5]^2$; adding a whisker $[4,6] \times \{4\}$ to A by Construction 1 gives Value 16.

Value 7 is symmetric to Value 6.

Value 8 transforms to Value 9 (Value 12) by two copies of the transition from Value 6 to Value 9 (Value 10), with A and B interchanged in the second copy. With $A = [0,3]^2 \cup [5,6] \times [\frac{1}{r+1}, \frac{1}{r}]$ and $B_r = [1,2] \times [1-\frac{1}{r+1}, 1+\frac{1}{r}] \cup [4,7] \times [0,3]$, as r goes to infinity, A and B go from Value 8 to Value 10. Value 11 is symmetric to Value 10. With $A = \bar{B}((0,0),1) \cup \bar{B}((5,5),r)$ and $B = \bar{B}((0,0),r) \cup \bar{B}((5,5),1)$, A and B go from Value 8 to Value 13 as r increases to 1. With this A and B for

$r = 1/2$, adding a whisker $[0,5]\times\{0\}$ to A or B by Construction 1 gives Value 16. Value 8 is realized by $A = [0,3]^2 \cup ([5,6]\times\{y\})$ and $B = [1,2]^2 \cup ([4,7]\times[0,3])$ for $y \in (0,1]$ and transitions to Value 14 when y decreases to 0. Value 15 is symmetric to Value 14.

Value 9 is realized by $A = [0,3]^2 \cup ([3,4]\times\{1\}$ and $B = ([4,5]\times[0,3]) \cup ([3,4]\times\{2\})$. Extending one or both of the whiskers into the other set transitions to Values 10, 11, and 12. With $A = B = [0,1]\times[0,r]$, we have Value 9 for $r = 0$ and Value 13 for $r > 0$. With $A = [-2,2]\times[0,1]$ and $B = [-1,1]\times[0,r]$, we have Value 9 for $r = 0$ and Value 14 for $r > 0$; Value 15 is symmetric. Value 16 is easily obtained by translating the sets.

Value 10 transform to Values 11 and 12 by the deletion and addition of a whisker using Constructions 1 and 2, and to Values 13, 14, and 15 by deleting a whisker directly (Construction $1^{\leftarrow}$) and adding a region with positive area as a new component of both A and B, a component of B in A, or of A in B, directly by Construction 3. With $A = [-2,2]^2$ and $B = [1,3]\times[1,r]$, we have Value 10 for $r = 0$ and Value 16 for $r > 0$.

Value 11 is symmetric to Value 10.

Value 12 transforms to Values 13, 14, and 15 just as Value 10 does, with the simultaneous deletion of a whisker.

Value 13 is realized by $A = B = \bar{B}((0,0),r)$. By shrinking the radius on one of the sets, we transform to Values 14 and 15, and by shifting the center of one, to From 16.

Value 14 is realized by $A = [0,3]^2 \cup ([4,6]\times[0,3])$ and $B = ([1,2]\times\{1\}) \cup \cup([4,6]\times[0,3]) \cup ([7,10]\times[0,3])$. By deleting the whisker $[1,2]\times\{1\}$ of B in A° and adding a whisker $[8,9]\times\{1\}$ to A in B° by Constructions 1 and 2, we transform to Value 15. Value 16 is realizable by spatial regions by translating one region.

Value 15 is symmetric to Value 14.

4. Upper and Lower Semicontinuity

As seen in the previous sections, continuity of the areas of A, B, and $A \cap B$ is not always a good model for continuous morphing. Another way to model continuous deformation involves upper and lower semicontinuity.

Definition 4.1. A function $B : \mathbb{R} \to \mathcal{P}(\mathbb{R}^2)$ is *upper semicontinuous* (or *u.s.c.* or *upper Vietoris continuous*) at t if for every open set $M \subseteq \mathbb{R}^2$ with $B(t) \subseteq M$, there exists a neighborhood N of t with $B(t') \subseteq M$ for all $t' \in N$. A function $B : \mathbb{R} \to \mathcal{P}(\mathbb{R}^2)$ is *lower semicontinuous* (or *l.s.c.* or *lower Vietoris continuous*) at t if for every open set $M \subseteq \mathbb{R}^2$ with $B(t) \cap M \neq \varnothing$, there exists a neighborhood N of t with $B(t') \cap M \neq \varnothing$ for all $t' \in N$. A function which is both u.s.c. and l.s.c. at t is *Vietoris continuous* at t, or continuous with respect to the *Vietoris topology* at t. See [10, 8].

We note that the well-known Hausdorff distance between nonempty compact sets in a metric space X generates a topology on the collection $\mathcal{K}_0$ of nonempty compact subsets of X known as the Hausdorff topology. The Hausdorff topology agrees with the Vietoris topology on $\mathcal{K}_0$ (Corollary 4.2.3 of [10]).

Upper semicontinuity prevents B from expanding beyond a neighborhood of it quickly. Construction 3 of Section 3 was not u.s.c. However, neither u.s.c., l.s.c, nor u.s.c. and l.s.c. together imply continuity of area. For example, if $B(t) = [-2, 2]^2$ for $t \geq 0$ and $B(t) = [-1, 1]^2$ for $t < 0$, then $B(t)$ is u.s.c at every point, but the area is discontinuous at $t = 0$, and $B(t)$ is not l.s.c. at $t = 0$. With $A(t) = [-2, 2]^2$ for $t > 0$ and $A(t) = [-1, 1]^2$ for $t \leq 0$, A is l.s.c. everywhere but not u.s.c. at $t = 0$ where the area jumps discontinuously. Example 4.2(a) shows that u.s.c. and l.s.c. together do not imply continuity of area.

The non-compact sets of Example 3.1 are both changing upper- and lower-semicontinuously, so the additional assumption of u.s.c and l.s.c is not sufficient to prevent closed sets from morphing from disjoint to overlapping directly without first passing through the "meets" value. The next example shows that such a transition is still possible for compact sets.

Example 4.2 (Comb Spaces)**.** The following values are adjacent using closed-valued u.s.c. and l.s.c. functions $A(t)$ and $B(t)$.

(a) *Disjoint is adjacent to Overlaps.* Define $A(t)$ as follows:

$$\begin{aligned}
A(t) &= [0,1] \times \{0\} \text{ for } t \geq 1 \\
A(2^{-n}) &= A(1) \cup \{2^{-n}m\} \times [0,1] : m = 0, 1, \ldots, 2^n\} \text{ for } n \in \mathbb{N} \\
A(t) &= A(1) \cup \{2^{-n}m : m = 0, 1, \ldots, 2^n\} \times [0, 2^{n+1}(t - 2^{n+1})] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \in [2^{-n-1}, 2^{-n}] \\
A(t) &= [0,1]^2 \text{ for } t \leq 0.
\end{aligned}$$

Thus, at $t = 2^{-n}$, $A(t)$ is a comb with base $[0,1] \times \{0\}$ and teeth of height 1 at each $x = \frac{m}{2^n}$ ($m \in \{0, 1, 2, \ldots, 2^n\}$). In the time interval between $t = 2^{-n}$ and $t = 2^{-n-1}$, the new teeth grow continuously from the base at the midpoints between existing teeth until they reach height 1. It is easy to check that $A(t)$ is both u.s.c. and l.s.c, but has a discontinuous jump in area at $t = 0$. Furthermore, $A(t)$ is a compact set at every $t \in \mathbb{R}$.

Let $B(t)$ be the reflection of $A(t)$ over the line $y = \frac{7}{8}$ translated to the left by $s(t)$ where $s(t)$ is piecewise linear, $s(t) = 0$ for $t \leq 0$, $s(t) = \frac{1}{2}$ for $t \geq 1$, and $s(2^{-n}) = 2^{-n-1} =$ half the distance between existing teeth at time $t = 2^{-n}$. Then $A(t)$ and $B(t)$ are u.s.c and l.s.c. compact-valued functions with $A(t) \cap B(t) = \varnothing$ for $t > 0$, but $A(0) \cap B(0) = [0,1] \times [\frac{3}{4}, 1]$. In particular, $A(t)$ and $B(t)$ transform from disjoint to overlaps without passing through meets.

(b) *Disjoint is adjacent to Value 2* $(0,0,0,1)$. Let $A(t)$ be as above, and let $B'(t) = \{\frac{1}{2} - s(t)\} \times [\frac{3}{4}, \frac{7}{8}]$ be the segment of the center tooth of the comb $B(t)$ with $\frac{3}{4} \leq y \leq \frac{7}{8}$ for $t < 1$, and $B'(t) = \{\frac{1}{2}\} \times [\frac{3}{4}, \frac{7}{8}]$ for $t \geq 1$.

Example 4.2(b) seems to show that a whisker of B may instantly appear in the interior of A without introducing any other intersections among boundaries and interiors, even though A and B were disjoint and separated by open sets before that instant. This may seem to violate u.s.c. of B. However, the crux of our example is that the whisker of B is not created at that instant, but rather the interior of A engulfs the whisker at that instant.

The comb spaces in the example above are compact at each value of t, but are not always regular closed. This can be readily remedied by fattening each segment of the comb slightly. Alternately, our next example gives a variation of the comb space obtained by making the teeth triangular spikes and shows that even if the spaces are always regular closed (or indeed, spatial regions), disjoint is adjacent to overlaps under the u.s.c. and l.s.c. continuity conditions.

Example 4.3. By the *spike* centered at $x = a$ of width w and height h, we mean the closed triangular region $S(a, w, h)$ having vertices $(a - \frac{w}{2}, 0)$, (a, h), and $(a + \frac{w}{2}, 0)$. Define $A(t)$ as follows.

For $t = 1 - 2^{-n}$ $(n \in \mathbb{N})$, $A(t) = [0, 2] \times [\frac{-1}{2}, 0] \cup \{S(2^{-n}m, 2^{-(n+1)}, 1) : m \in \{1, 2, \ldots, 2^n\}\}$. Thus, $A(t)$ consists of a rectangular base together with 2^n spikes of width $2^{-(n+1)}$ and height 1. The combined area of the base and spikes is $1 + \frac{1}{4}$.

For $t \in (1 - 2^{-n}, 1 - 2^{-(n+1)})$, we will shrink the widths of the existing spikes by half linearly with time (so their total area decreases from $\frac{1}{4}$ to $\frac{1}{8}$) and create new spikes midway between existing spikes whose areas increase linearly with time from 0 to $\frac{1}{8}$ as their heights increase from 0 to 1. Specifically, for $t \in (1 - 2^{-n}, 1 - 2^{-(n+1)})$, let $t' = t - (1 - 2^{-n})$, so $t' \in (0, 2^{-(n+1)})$. Let $h(t') = 2^{n+1}t'$ be the linear function with $h(0) = 0$ and $h(2^{-(n+1)}) = 1$. Define $A(t') = [0, 2] \times [\frac{-1}{2}, 0] \cup \{S(2^{-n}m, \frac{h(t')}{2} 2^{-(n+1)}, 1) : m \in \{1, 2, \ldots, 2^n\}, m$ even$\} \cup \{S(2^{-n}m, \frac{t'}{2}, h(t') : m \in \{1, 2, \ldots, 2^n\}, m$ odd$\}$. Now for all $t \in (1 - 2^{-n}, 1 - 2^{-(n+1)})$, the area of $A(t)$ is $1 + \frac{1}{4}$.

For $t \le \frac{1}{2}$, put $A(t) = A(\frac{1}{2})$.

For $t \ge 1$, put $A(t) = ([0, 2] \times [-\frac{1}{2}, 0]) \cup ([0, 1] \times [0, 1])$.

Now $A(t)$ is u.s.c. and l.s.c., and $A(t)$ is regular closed and compact (indeed, is a compact spatial region) at every value of t. But, the area of $A(t)$ jumps discontinuously at $t = 1$.

With $B(t)$ defined in terms of $A(t)$ precisely as in the last paragraph of Example 4.2, the comments still apply, and $A(t)$ and $B(t)$ transform directly from disjoint to overlaps.

Below we use a spiral construction for a similar space-filling example.

Example 4.4 (Spirals)**.** Throughout, adjacencies refer to those achieved by u.s.c. and l.s.c. functions. These examples are also compact-valued.

(a) *Disjoint and Equals are adjacent.* For $t \in [0, 1)$, let $A(t) = \{(r, \theta) : r = (1 - t)\theta, \theta \in [0, \frac{1}{1-t}]\}$ and $B(t) = \{(r, \theta) : r = (1 - t)(\theta + \pi), \theta \in [0, \frac{1}{1-t} - \pi]\}$. For $t < 0$, put $A(t) = A(0)$ and $B(t) = B(0)$. For $t \ge 1$, put $A(t) = B(t) = \{(r, \theta) : r \le 1\}$. Now for $t \in [0, 1)$, $A(t)$ and $B(t)$ are disjoint Archimedean spirals with an increasing number of coils winding tighter around the origin and staying inside the unit circle. Now $A(t)$ and $B(t)$ are seen to be u.s.c. and l.s.c.

compact-valued functions with discontinuous area (at $t = 1$). Furthermore, $A(t)$ and $B(t)$ are disjoint for $t < 1$ and are equal for $t \geq 1$, showing that disjoint and equals are adjacent.

(b) *Disjoint and Contains are adjacent.* For $t \geq .9$, let $A(t)$ and $B(t)$ be as above and put $B'(t) = B(t) \cap \{(r, \theta) : r \leq \frac{1}{2}\}$ for $t \geq .9$. (Note that we take $t \geq .9$ only to assure that $B(t) \neq \varnothing$.) Now the unit disk $A(1)$ contains the disk $B'(1)$ of radius $\frac{1}{2}$, and $A(t)$ and $B'(t)$ are the desired functions.

(c) *Disjoint and Covers are adjacent.* For $t \geq .9$, let $A(t)$ and $B(t)$ be as above and put $B'(t) = B(t) \cap \{(r, \theta) : r \geq \frac{1}{2}\}$ for $t \geq .9$. Now the unit circle $A(1)$ covers the annulus $B'(1)$.

(d) *Disjoint and Value 2* $(0, 0, 0, 1)$ *are adjacent.* For $t \geq .9$, let $A(t)$ and $B(t)$ be as above. Now $B(t)$ restricted to the closed first quadrant $Q1$ contains many components. Let $B''(t)$ be the component of $B(t) \cap Q1$ closest to the origin, and $B''(0) = \{(0, 0)\}$. Now as time increases to 1, the spirals wind tighter and $B''(t)$ converges to the origin in a u.s.c., l.s.c. manner, as needed.

(e) *Disjoint and Value 5 are adjacent.* Recall that Value 5 only occurs if $A = B = \mathbb{R}^2$. For $t > 0$, let $A(t) = \{(r, \theta) : r = t\theta, \theta \geq 0\}$ and $B(t) = \{(r, \theta) : r = t(\theta + \pi), \theta \geq 0\}$. For $t \leq 0$, let $A(t) = B(t) = \mathbb{R}^2$. Now for $t > 0$, $A(t)$ and $B(t)$ are disjoint non-compact closed Archimedean spirals whose coils are becoming more tightly coiled as t decreases to 0. These functions have the required properties to prove the claim.

(f) *Disjoint and Overlap are adjacent.* This may be achieved by two disjoint copies of sets as in (c), with the second copy translated to remain disjoint and with the labels for A and B interchanged on that copy.

While the sets $A(t)$ and $B(t)$ of Example 4.4 are not regular closed sets for $t < 1$, , it is easy to see that these spiral curves may be fattened slightly to obtain regular closed sets illustrating the desired properties. Formally, the sets $A(t)$ for $t < 1$ may be replaced by their $\epsilon(t)$-fattening $A(t)_{\epsilon(t)} \bigcup \{\overline{B}(x, \epsilon(t)) : x \in A(t)\}$ and similarly $B(t)$ by $B(t)_{\epsilon(t)}$, for a function $\epsilon(t)$ decreasing to zero quickly enough to insure that $A(t)_{\epsilon(t)}$ and $B(t)_{\epsilon(t)}$ remain disjoint.

Indeed, such a modification of Example 4.4(f) shows that Disjoint and Overlaps are adjacent for compact, regular closed, nonempty, connected spaces even if $A^\circ(t)$ and $B^\circ(t)$ are u.s.c. and l.s.c.

Transitioning from one value to another requires the introduction or deletion of intersections between boundaries and interiors. We summarize some possible transitions below. Recall that transitioning from Value V_i to Value V_j at the instant $t = 0$ means that V_i exists for $t < 0$ (or $t > 0$) and V_j exists at $t = 0$. Note that the conditions on deleting intersections at an instant are more restrictive and have implications on other intersection values.

Proposition 4.5. *Assuming $A(t)$ and $B(t)$ are u.s.c. and l.s.c. closed-valued functions from $\mathbb{R}$ to $\mathcal{P}(\mathbb{R}^2)$, it is possible to introduce the following combinations of new intersections at an instant: (a) $\partial A \cap \partial B$, (b) $A^\circ \cap B^\circ$ and $\partial A \cap B^\circ$, (c) $\partial A \cap \partial B$ and $A^\circ \cap B^\circ$, (d) $\partial A \cap B^\circ$, and (e) $\partial A \cap \partial B$ and $A^\circ \cap B^\circ$ and $\partial A \cap B^\circ$.*

The following intersections may be deleted at an instant: (f) $A^\circ \cap B^\circ$, leaving $\partial A \cap \partial B$, (g) $\partial A \cap B^\circ$, leaving $\partial A \cap \partial B$, (h) $A^\circ \cap B^\circ$, leaving $A^\circ \cap \partial B$, (i) $\partial A \cap B^\circ$ and $A^\circ \cap B^\circ$, introducing $\partial A \cap \partial B$, and (j) $\partial A \cap B^\circ$, introducing $\partial A \cap \partial B$.

Proof. (a) Let $A(t) = [t, 1+t] \times [0,1]$ for $t > 0$, $A(t) = [0,1]^2$ for $t \leq 0$, and $B(t) = [-1, 0] \times [0, 1]$. (b), (c), (d), and (e) are shown by parts (b), (a), (d) and (c) of Example 4.4, respectively. (f) With $A(t)$ as in part (a), let $B(t) = [1,2] \times [-2,2]$. (g) Let $A(t) = [t, 1+t] \times \{0\}$ for $t > 0$, $A(t) = [0,1] \times \{0\}$ for $t \leq 0$, and $B(t) = [1,2] \times [-2,2]$. (h) Let $A(t) = [-2,2]^2$, $B(t) = [-1,1] \times [0,t]$ for $t > 0$ and $B(t) = [-1,1] \times \{0\}$ for $t \leq 0$. (i) Let $A(t) = [-2,2] \times [0,2]$, $B(t) = [-1,1] \times [t/2, t]$ for $t \in (0,1]$, and $B(t) = [-1,1] \times \{0\}$ for $t \leq 0$. (j) Let $A(t) = [-2,2] \times [0,2]$, $B(t) = [-1,1] \times \{t\}$ for $t \in (0,1]$, and $B(t) = [-1,1] \times \{0\}$ for $t \leq 0$. $\square$

Proposition 4.5(a) says that it is possible to transition from Value $(0, y, z, w)$ to $(1, y, z, w)$ at an instant. Proposition 4.5(b) shows that it is possible to transition from $(x, 0, 0, w)$ to $(x, 1, 1, w)$, or indeed if nonzero intersection values exist in other static components of A and B, it is possible to transform from (x, y, z, w) to $(x, 1, 1, w)$ at an instant. The dual of (b) obtained by interchanging the labels of A and B shows (x, y, z, w) to $(x, 1, 1, y)$. Similarly, (b) through (e) show transformations at an instant which toggle zeros to ones. Parts (f) through (j) describe some possible transitions toggling a one to a zero, but these have restrictions on other intersection values. Part (f) shows that $(1, 1, z, w)$ transitions to $(1, 0, z, w)$ at an instant; (g) and its dual show $(1, y, 1, w)$ is adjacent to $(1, y, 0, w)$ and $(1, y, z, 1)$ is adjacent to $(1, y, z, 0)$; (j) shows that $(0, y, 1, w)$ is adjacent to $(1, y, 0, w)$.

Corollary 4.6. *Using u.s.c. and l.s.c functions, Disjoint is adjacent to each of the other 15 values.*

Proof. Disjoint is the value $(0,0,0,0)$. Transitions to other values only add intersections. Using the parts of Example 4.4 in the proper combinations allows adding all possible intersections. Specifically, (a), (d) and the dual of (d) show, respectively, that zeros in the first, third, or fourth positions may be toggled to ones, while (c), (b), and the dual of (b) show that a zero in the second position may be toggled to one together with, respectively, zeros in the first, third, or fourth positions. Combining these, the only toggling from zeros to ones not accounted for is from $(0,0,0,0)$ to $(0,1,0,0)$. This is shown in Example 4.4(e). $\square$

If Value V_i transitions to V_j at an instant, then reversing time, the functions remain u.s.c. and l.s.c. and show that V_j transitions to V_i directly. As noted in the proof of Corollary 4.6, all combinations of toggling from zeros to ones are possible at an instant. Thus, transitions between values which only require deletions are all possible directly, except possibly transitions from the exceptional Value 5 $A = B = \mathbb{R}^2$.

Some transitions, such as from Covers to Covered By (Value 14 $(1,1,0,1)$ to Value 15 $(1,1,1,0)$), require simultaneous creation and deletion of certain intersection values. That is, both a zero and a one must be toggled. Some of these will be possible at an instant or directly using the results of Proposition 4.5, but some are not.

The next result shows some transitions are not possible at an instant.

Proposition 4.7. *If $A(t)$ and $B(t)$ are closed-valued, u.s.c., and $A(t) \cap B(t) \neq \varnothing$ for $t \in (-\epsilon, 0)$ for some $\epsilon > 0$, then $A(0) \cap B(0) \neq \varnothing$. Thus, none of the 15 other values can transition to Disjoint at an instant.*

Furthermore, if $A(t)$ and $B(t)$ are regular closed for all t and $A(t) \cap B(t) \neq \varnothing$ for $t \in (-\epsilon, 0)$ for some $\epsilon > 0$, then they may not transition to any value $(0, 0, z, w)$ at the instant $t = 0$.

Proof. Suppose to the contrary that for every $\epsilon > 0$ there exists a $t \in (-\epsilon, 0)$ with $A(t) \cap B(t) \neq \varnothing$, and $A(0) \cap B(0) = \varnothing$. Since $\mathbb{R}^2$ is normal, there exist disjoint open sets G_A and G_B with $A \subseteq G_A$, $B \subseteq G_B$. By u.s.c., $A(t) \subseteq G_A$ and $B(t) \subseteq G_B$ for all t in $(-\epsilon, 0)$, contradicting our the assumption. Furthermore, if the sets are regular closed, the only permissible value $(0, 0, z, w)$ is $(0, 0, 0, 0)$ (when $A(0)$ and $B(0)$ are disjoint) since for regular closed sets, the boundary of one intersecting the interior implies the interiors intersect. □

Recall that under the continuity of area restrictions of Theorem 3.10, Values 6 and 15 were not adjacent. They are adjacent using u.s.c. and l.s.c. functions. Indeed, let $A_1(t)$ and $B_1(t)$ be as in Example 4.4(b) for $t \geq .9$, $A_2(t) = [11, 12] \times [(1-t)/2, 1-t]$ for $t \in (.9, 1)$, $A_2(t) = [11, 12] \times \{0\}$ for $t \geq 1$, and $B_2(t) = [10, 13] \times [0, 1]$ for $t \geq .9$. Now $A(t) = A_1(t) \cup A_2(t)$ and $B(t) = B_1(t) \cup B_2(t)$ show that Value 6 transforms to Value 15 at the instant $t = 1$.

5. Continuous area, u.s.c, and l.s.c.

We have seen that the values Disjoint and Overlaps are, somewhat surprisingly, adjacent under our previous assumptions of (a) continuity of areas of $A(t), B(t)$, and the intersections of their components, or (b) u.s.c. and l.s.c. If we assume both sets of assumptions, then Disjoints is not adjacent to Overlaps.

Theorem 5.1. *Suppose $A(t)$ and $B(t)$ are u.s.c. functions with values being closed sets with finite areas, and the area of $A(t) \cap B(t)$ is a continuous function. Then $(0, 0, 0, 0)$ is not adjacent to $(x, 1, z, w)$.*

Proof. The basic idea is that u.s.c. prevents A° from hopping inside B° to introduce nonempty intersection of the interiors, and the continuity of the area prevents B° from engulfing A°. Indeed, Proposition 4.7 shows that $(x, 1, z, w)$ cannot transform to Disjoint at an instant. If Disjoint transformed to $(x, 1, z, w)$ at an instant, then there would exist $A(t)$, $B(t)$ with $A(t) \cap B(t) = \varnothing$ for $t < 0$ and $A^\circ(0) \cap B^\circ(0) \neq \varnothing$. Then if $a(t)$ is the area of $A(t) \cap B(t)$, we have $(-\infty, 0) \subseteq a^{-1}(\{0\})$ but $0 \notin a^{-1}(\{0\})$, so the inverse image of the closed set $\{0\}$ is not closed and thus $a(t)$ is not continuous. □

In conclusion, which 4-intersection values are adjacent to which depends heavily on the assumptions. In applications, there are many examples of dynamic regions which need not be connected or regular closed, but we have seen that allowing this generality, all transitions between 4-intersection values are possible except those given in Theorem 3.10 and Proposition 3.11 (see Figure 3). Restricting the regions to be spatial regions allows fewer adjacencies, but also fewer applications. The permissible adjacencies depend also on the types of dynamic morphing allowed. Continuity of area was a weak assumption and Vietoris continuity provides some better results, but still allowed the adjacency of Disjoint and Overlaps. Assuming continuity of area together with u.s.c. and l.s.c. provided one setting where Disjoints was not adjacent to Overlaps.

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