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Metric spaces and textures

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Abstract

Textures are point-set setting for fuzzy sets, and they provide a framework for the complement-free mathematical concepts. Further dimetric on textures is a generalization of classical metric spaces. The aim of this paper is to give some properties of dimetric texture space by using categorical approach. We prove that the category of classical metric spaces is isomorphic to a full subcategory of dimetric texture spaces, and give a natural transformation from metric topologies to dimetric ditopologies. Further, it is presented a relation between dimetric texture spaces and quasi-pseudo metric spaces in the sense of J. F. Kelly.

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1. Introduction

Texture theory is point-set setting for fuzzy sets and hence, some properties of fuzzy lattices (i.e. Hutton algebra) can be discussed based on textures [2, 3, 4, 5]. Ditopologies on textures unify the fuzzy topologies and classical topologies without the set complementation [6, 7]. Recent works on textures show that they are also useful model for rough set theory [8] and semi-separation axioms [10]. On the other hand, it was given various types of completeness for di-uniform texture spaces [13]. As an expanded of classical metric spaces, the dimetric notion on texture spaces was firstly defined in [11]. In this paper, we give the categorical properties of dimetric texture spaces, and present some relation between classical metric spaces and dimetric texture spaces.

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This section is devoted to some fundamental definitions and results of the texture theory from [2, 3, 4, 5, 6].

Definition 1.1. Let U be a set and $\mathcal{U} \subseteq \mathcal{P}(U)$. Then $\mathcal{U}$ is called *a texturing* of U if

(T1) $\varnothing \in \mathcal{U}$ and $U \in \mathcal{U}$,
(T2) $\mathcal{U}$ is a complete and completely distributive lattice such that arbitrary meets coincide with intersections, and finite joins with unions,
(T3) $\mathcal{U}$ is point-seperating.

Then the pair $(U, \mathcal{U})$ is called a *texture space* or *texture*.

For $u \in U$, the *p-sets* and the *q-sets* are defined by

$$P_u = \bigcap\{A \in \mathcal{U} \mid u \in A\}, \quad Q_u = \bigvee\{A \in \mathcal{U} \mid u \notin A\}, \quad \text{respectively.}$$

A texture $(U, \mathcal{U})$ is said to be *plain* if $P_u \nsubseteq Q_u$, $\forall u \in U$.

A set $A \in \mathcal{U} \setminus \{\varnothing\}$ is called a *molecule* if $A \subseteq B \cup C$, $B, C \in \mathcal{U}$ implies $A \subseteq B$ or $A \subseteq C$. The texture $(U, \mathcal{U})$ is called *simple* if the sets P_u, $u \in U$ are the only molecules in $\mathcal{U}$.

Example 1.2. (1) For any set U, $(U, \mathcal{P}(U))$ is the *discrete texture* with the usual set structure of U. Clearly, $P_u = \{u\}$ and $Q_u = U \setminus \{u\}$ for all $u \in U$, so $(U, \mathcal{P}(U))$ is both plain and simple.

(2) $\mathcal{I} = \{[0, t] \mid t \in [0, 1]\} \cup \{[0, t) \mid t \in [0, 1]\}$ is a texturing on $\mathbb{I} = [0, 1]$. Then $(\mathbb{I}, \mathcal{I})$ is said to be *unit interval texture*. For $t \in I$, $P_t = [0, t]$ and $Q_t = [0, t)$. Clearly, $(I, \mathcal{I})$ is plain but not simple since the sets Q_u, $0 < u \leq 1$, are also molecules.

(3) For textures $(U, \mathcal{U})$ and $(V, \mathcal{V})$, $\mathcal{U} \otimes \mathcal{V}$ is product texturing of $U \times V$ [5]. Note that the product texturing $\mathcal{U} \otimes \mathcal{V}$ of $U \times V$ consists of arbitrary intersections of sets of the form $(A \times V) \cup (U \times B)$, $A \in \mathcal{U}$ and $B \in \mathcal{V}$. Here, for $(u, v) \in U \times V$

$P_{(u,v)} = P_u \times P_v$ and $Q_{(u,v)} = (Q_u \times V) \cup (U \times Q_v)$.

Ditopology: A pair (τ, κ) of subsets of $\mathcal{U}$ is called a *ditopology* on a texture $(U, \mathcal{U})$ where the *open sets* family τ and the *closed sets* family κ satisfy

$$\begin{aligned} U, \varnothing \in \tau, &\qquad U, \varnothing \in \kappa \\ G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau, &\qquad K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa \\ G_i \in \tau, i \in I \implies \bigvee_{i \in I} G_i \in \tau, &\qquad K_i \in \kappa, i \in I \implies \bigcap_{i \in I} K_i \in \kappa. \end{aligned}$$

Direlation: Let $(U, \mathcal{U})$, $(V, \mathcal{V})$ be texture spaces. Now we consider the product texture $\mathcal{P}(U) \otimes \mathcal{V}$ of the texture spaces $(U, \mathcal{P}(U))$ and $(V, \mathcal{V})$. In this texture, p-sets and the q-sets are denoted by $\overline{P}_{(u,v)}$ and $\overline{Q}_{(u,v)}$, respectively. Clearly,

$\overline{P}_{(u,v)} = \{u\} \times P_v$ and $\overline{Q}_{(u,v)} = (U \setminus \{u\} \times V) \cup (U \times Q_v)$ where $u \in U$ and $v \in V$. According to:

(1) $r \in \mathcal{P}(U) \otimes \mathcal{V}$ is called a *relation* from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ if it satisfies
  R1 $r \not\subseteq \overline{Q}_{(u,v)}, P_{u'} \not\subseteq Q_u \implies r \not\subseteq \overline{Q}_{(u',v)}$.
  R2 $r \not\subseteq \overline{Q}_{(u,v)} \implies \exists u' \in U$ such that $P_u \not\subseteq Q_{u'}$ and $r \not\subseteq \overline{Q}_{(u',v)}$.
(2) $R \in \mathcal{P}(U) \otimes \mathcal{V}$ is called a *corelation* from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ if it satisfies
  CR1 $\overline{P}_{(u,v)} \not\subseteq R, P_u \not\subseteq Q_{u'} \implies \overline{P}_{(u',v)} \not\subseteq R$.
  CR2 $\overline{P}_{(u,v)} \not\subseteq R \implies \exists u' \in U$ such that $P_{u'} \not\subseteq Q_u$ and $\overline{P}_{(u',v)} \not\subseteq R$.
(3) If r is a relation and R is a corelation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ then the pair (r, R) is called a direlation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$.

A pair (i, I) is said to be identity direlation on $(U, \mathcal{U})$ where $i = \bigvee\{\overline{P}_{(u,u)} \mid u \in U\}$ and $I = \bigcap\{\overline{Q}_{(u,u)} \mid U \not\subseteq Q_u\}$.

Recall that [5] we write $(p, P) \sqsubseteq (q, Q)$ if $p \subseteq q$ and $Q \subseteq P$ where (p, P) and (q, Q) are direlations.

Let (p, P) and (q, Q) be direlations from $(U, \mathcal{U})$to $(V, \mathcal{V})$. Then the greatest lower bound of (p, P) and (q, Q) is denoted by $(p, P) \sqcap (q, Q)$, and it is defined by $(p, P) \sqcap (q, Q) = (p \sqcap q, P \sqcup Q)$ where

$$p \sqcap q = \bigvee\{\overline{P}_{(u,v)} \mid \exists z \in U \text{ with } P_u \not\subseteq Q_z, \text{ and } p, q \not\subseteq \overline{Q}_{(z,v)}\},$$
$$P \sqcup Q = \bigcap\{\overline{Q}_{(u,v)} \mid \exists z \in U \text{ with } P_z \not\subseteq Q_u, \text{ and } \overline{P}_{(z,v)} \not\subseteq P, Q\}.$$

Inverses of a direlation: If (r, R) is a direlation then the inverse direlation of $(r, R)^{\leftarrow}$ is a direlation from $(V, \mathcal{V})$ to $(U, \mathcal{U})$, and it is defined by $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ where

$$r^{\leftarrow} = \bigcap\{\overline{Q}_{(v,u)} \mid r \not\subseteq \overline{Q}_{(u,v)}\} \text{ and } R^{\leftarrow} = \bigvee\{\overline{P}_{(v,u)} \mid \overline{P}_{(u,v)} \not\subseteq R\}$$

The A-sections and the B-presections under a direlation (r, R) are defined as

$$r^{\rightarrow} A = \bigcap\{Q_v \mid \forall u, r \not\subseteq \overline{Q}_{(u,v)} \Longrightarrow A \subseteq Q_u\},$$
$$R^{\rightarrow} A = \bigvee\{P_v \mid \forall u, \overline{P}_{(u,v)} \not\subseteq R \Longrightarrow P_u \subseteq A\},$$
$$r^{\leftarrow} B = \bigvee\{P_u \mid \forall v, r \not\subseteq \overline{Q}_{(u,v)} \Longrightarrow P_v \subseteq B\},$$
$$R^{\leftarrow} B = \bigcap\{Q_u \mid \forall v, \overline{P}_{(u,v)} \not\subseteq R \Longrightarrow B \subseteq Q_v\}.$$

The composition of direlations: Let (p, P) be a direlation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$, and (q, Q) be a direlation on $(V, \mathcal{V})$ to $(W, \mathcal{W})$. The composition

$(q,Q)\circ(p,P)$ of (p,P) and (q,Q) is a direlation from $(U,\mathcal{U})$ to $(W,\mathcal{W})$ and it is defined by $(q,Q)\circ(p,P)=(q\circ p, Q\circ P)$ where

$$q\circ p=\bigvee\{\overline{P}_{(u,w)} \mid \exists\, v\in V \text{ with } p\not\subseteq \overline{Q}_{(u,v)} \text{ and } q\not\subseteq \overline{Q}_{(v,w)}\},$$
$$Q\circ P=\bigcap\{\overline{Q}_{(u,w)} \mid \exists\, v\in V \text{ with } \overline{P}_{(u,v)}\not\subseteq P \text{ and } \overline{P}_{(v,w)}\not\subseteq Q\}.$$

Difunction: A direlation from $(U,\mathcal{U})$ to $(V,\mathcal{V})$ is called a *difunction* if it satisfies the conditions:

(DF1) For $u,u'\in U$, $P_u\not\subseteq Q_{u'} \implies \exists\, v\in V$ with $f\not\subseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u',v)}\not\subseteq F$.

(DF2) For $v,v'\in V$ and $u\in U$, $f\not\subseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u,v')}\not\subseteq F \implies P_{v'}\not\subseteq Q_v$.

Obviously, identity direlation (i,I) on $(U,\mathcal{U})$ is a difunction and it is said to be *identity difunction*.

It is well known that [5] the category **dfTex** of textures and difunctions is the main category of texture theory.

Definition 1.3. Let $(f,F):(U,\mathcal{U})\to(V,\mathcal{V})$ be a difunction. If (f,F) satisfies the condition

SUR. For $v,v'\in V$, $P_v\not\subseteq Q_{v'} \implies \exists\, u\in U$ with $f\not\subseteq \overline{Q}_{(u,v')}$ and $\overline{P}_{(u,v)}\not\subseteq F$.

then it is called *surjective*.

Similarly, (f,F) satisfies the condition

INJ. For $u,u'\in U$ and $v\in V$, $f\not\subseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u',v)}\not\subseteq F \implies P_u\not\subseteq Q_{u'}$.

then it is called *injective*.

If (f,F) is both injective and surjective then it is called *bijective*.

Note 1.4. In general, difunctions are not directly related to ordinary (point) functions between the base sets. We note that [5, Lemma 3.4] if $(U,\mathcal{U})$, $(V,\mathcal{V})$ are textures and a point function $\varphi:U\to V$ satisfies the condition

$$\text{(a)}\quad P_u\not\subseteq Q_{u'} \implies P_{\varphi(u)}\not\subseteq Q_{\varphi(u')}$$

then the equalities

$$f_\varphi=\bigvee\{\overline{P}_{(u,v)} \mid \exists z\in U \text{ satisfying } P_u\not\subseteq Q_z \text{ and } P_{\varphi(z)}\not\subseteq Q_v\},$$
$$F_\varphi=\bigcap\{\overline{Q}_{(u,v)} \mid \exists z\in U \text{ satisfying } P_z\not\subseteq Q_u \text{ and } P_v\not\subseteq Q_{\varphi(z)}\},$$

define a difunction (f_φ,F_φ) on $(U,\mathcal{U})$ to $(V,\mathcal{V})$. For $B\in\mathcal{V}$, $F_\varphi^{\leftarrow}B=\varphi^{\leftarrow}B=f_\varphi^{\leftarrow}B$, where $\varphi^{\leftarrow}B=\bigvee\{P_u \mid P_{\varphi(u')}\subseteq B\ \forall\, u'\in U \text{ with } P_u\not\subseteq Q_{u'}\}$.

Furthermore, the function $\varphi=\varphi_{(f,F)}:U\to V$ corresponding as above to the difunction $(f,F):(U,\mathcal{U})\to(V,\mathcal{V})$, with $(V,\mathcal{V})$ plain, has the property (a) and in addition the property:

(b) $P_{\varphi(u)}\not\subseteq B$, $B\in\mathcal{V} \implies \exists\, u'\in U$ with $P_u\not\subseteq Q_{u'}$ for which $P_{\varphi(u')}\not\subseteq B$.

Conversely, if $\varphi:U\to V$ is any function satisfying (a) and (b) then there exists a unique difunction $(f_\varphi,F_\varphi):(U,\mathcal{U})\to(V,\mathcal{V})$ satisfying $\varphi=\varphi_{(f_\varphi,F_\varphi)}$.

On the other hand, if we consider simple textures it is obtained the same class of point functions.

The category of textures and point functions which satisfy the conditions (a)-(b) between the base sets is denoted by **fTex**.

Bicontinuous Difunction: A difunction $(f, F) : (U, \mathcal{U}, \tau_U, \kappa_U) \to (V, \mathcal{V}, \tau_V, \kappa_V)$ is called continuous (cocontinuous) if $B \in \tau_V$ $(B \in \kappa_V) \Longrightarrow F^{\leftarrow}(B) \in \tau_U$ $(f^{\leftarrow}(B) \in \kappa_U)$. A difunction (f, F) is called bicontinuous if it is both continuous and cocontinuous.

The category of ditopological texture spaces and bicontinuous difunctions was denoted by **dfDitop** in [6].

2. Some categories of dimetrics on texture spaces

The notion of dimetric on texture space was firstly introduced in [11]. In this section, we will give some properties of dimetric texture spaces, and we present a link between classical metrics and dimetrics with categorical approach.

Definition 2.1. Let $(U, \mathcal{U})$ be a texture, $\overline{\rho}, \underline{\rho} : U \times U \to [0, \infty)$ two point function. Then $\rho = (\overline{\rho}, \underline{\rho})$ is called a pseudo dimetric on $(U, \mathcal{U})$ if

M1 $\overline{\rho}(u, z) \leq \overline{\rho}(u, v) + \overline{\rho}(v, z)$,
M2 $P_u \not\subseteq Q_v \Longrightarrow \overline{\rho}(u, v) = 0$,
DM $\overline{\rho}(u, v) = \underline{\rho}(v, u)$,
CM1 $\underline{\rho}(u, z) \leq \underline{\rho}(u, v) + \underline{\rho}(v, z)$,
CM2 $P_v \not\subseteq Q_u \Longrightarrow \underline{\rho}(u, v) = 0$.

for all $u, v, z \in U$. In this case $\overline{\rho}$ is called pseudo metric, $\underline{\rho}$ the pseudo cometric of ρ.

If ρ is a pseudo dimetric which satisfies the conditions

M3 $P_u \not\subseteq Q_v, \overline{\rho}(v, y) = 0, P_y \not\subseteq Q_z \Longrightarrow P_u \not\subseteq Q_z \ \forall u, v, y, z \in U$,
CM3 $P_v \not\subseteq Q_u, \underline{\rho}(u, y) = 0, P_z \not\subseteq Q_y \Longrightarrow P_z \not\subseteq Q_u \ \forall u, v, y, z \in U$

it is called a dimetric.

If $\rho = (\overline{\rho}, \underline{\rho})$ is (pseudo) dimetric on $(U, \mathcal{U})$ then $(U, \mathcal{U}, \rho)$ is called (pseudo) dimetric texture space.

Let $(U, \mathcal{U}, \rho)$ be a (pseudo) dimetric texture space. It was shown in [11, Proposition 6.3] that $\beta_\rho = \{N_\epsilon^\rho(u) \mid u \in U^\flat, \epsilon > 0\}$ is a base and $\gamma_\rho = \{M_\epsilon^\rho(u) \mid u \in U^\flat, \epsilon > 0\}$ a cobase for a ditopology (τ_ρ, κ_ρ) on $(U, \mathcal{U})$ where

$$N_\epsilon^\rho(u) = \bigvee\{P_z \mid \exists v \in U, \text{ with, } P_u \not\subseteq Q_v, \overline{\rho}(v, z) < \epsilon\},$$

$$M_\epsilon^\rho(u) = \bigcap\{Q_z \mid \exists v \in U, \text{ with, } P_v \not\subseteq Q_u, \underline{\rho}(v, z) < \epsilon\}.$$

In this case $(U, \mathcal{U}, \tau_\rho, \kappa_\rho)$ is said to be (pseudo) dimetric ditopological texture space.

Definition 2.2. Let $(U, \mathcal{U}, \rho)$ be a (pseudo) dimetric texture space. Then $G \in \mathcal{U}$ is called

(1) open if for every $G \not\subseteq Q_u$, then there exists $\epsilon > 0$ such that $N^{\rho}_{\epsilon}(u) \subseteq G$,
(2) closed if for every $P_u \not\subseteq G$, then there exists $\epsilon > 0$ such that $G \subseteq M^{\rho}_{\epsilon}(u)$.

We set $\mathcal{O}_\rho = \{G \in \mathcal{U} \mid G \text{ is open in } (U, \mathcal{U}, \rho)\}$ and $\mathcal{C}_\rho = \{K \in \mathcal{U} \mid K \text{ is closed in } (U, \mathcal{U}, \rho)\}$.

Proposition 2.3. *Let $(U, \mathcal{U}, \rho)$ be a (pseudo) dimetric texture space. For $u \in U$ and $\epsilon > 0$,*

(i) *$N^{\rho}_{\epsilon}(u)$ is open in $(U, \mathcal{U}, \rho)$,*
(ii) *$M^{\rho}_{\epsilon}(u)$ is closed in $(U, \mathcal{U}, \rho)$.*

Proof. We prove (i), and the second result is dual. Let $N^{\rho}_{\epsilon}(u) \not\subseteq Q_v$ for some $v \in U$. By the definition of $N^{\rho}_{\epsilon}(u)$, there exists $y, z \in U$ such that $P_y \not\subseteq Q_v$ and $P_u \not\subseteq Q_z$, $\overline{\rho}(z, y) < \epsilon$. We set $\delta = \epsilon - \overline{\rho}(z, y)$. Now we show that $N^{\rho}_{\delta}(v) \subseteq N^{\rho}_{\epsilon}(u)$. We suppose $N^{\rho}_{\delta}(v) \not\subseteq N^{\rho}_{\epsilon}(u)$. Then $N^{\rho}_{\delta}(v) \not\subseteq Q_r$ and $P_r \not\subseteq N^{\rho}_{\epsilon}(u)$ for some $r \in U$. By the first inclusion, there exists $m, n \in U$ such that $P_m \not\subseteq Q_r$, $P_v \not\subseteq Q_n$ and $\overline{\rho}(n, m) < \delta$. Now we observe that $\overline{\rho}(z, y) + \overline{\rho}(n, m) < \epsilon$ and

$$\overline{\rho}(z, r) \le \overline{\rho}(z, y) + \overline{\rho}(y, v) + \overline{\rho}(v, n) + \overline{\rho}(n, m) \le \epsilon$$

by the condition $(M2)$. Since $P_u \not\subseteq Q_z$ and $\overline{\rho}(z, r) \le \epsilon$, we have the contradiction $P_r \subseteq N^{\rho}_{\epsilon}(u)$. □

Definition 2.4. Let $(U_j, \mathcal{U}_j, \rho_j)$, $j = 1, 2$ be (pseudo) dimetric texture spaces and (f, F) be a difunction from $(U_1, \mathcal{U}_1)$ to $(U_2, \mathcal{U}_2)$. Then (f, F) is called

(1) $\rho_1 - \rho_2$ continuous if $\overline{P}_{(u,v)} \not\subseteq F$ then $N^{\rho_1}_{\delta}(u) \subseteq F^{\leftarrow}(N^{\rho_2}_{\epsilon}(v))$, $\forall \epsilon > 0$ $\exists \delta > 0$,
(2) $\rho_1 - \rho_2$ cocontinuous if $f \not\subseteq \overline{Q}_{(u,v)}$ then $f^{\leftarrow}(M^{\rho_2}_{\epsilon}(v)) \subseteq M^{\rho_1}_{\delta}(u)$, $\forall \epsilon > 0$ $\exists \delta > 0$,
(3) $\rho_1 - \rho_2$ bicontinuous if it is continuous and cocontinuous.

Proposition 2.5. *Let (f, F) be a difunction from $(U_1, \mathcal{U}_1, \rho_1)$ to $(U_2, \mathcal{U}_2, \rho_2)$.*

(i) *(f, F) is continuous $\iff F^{\leftarrow}(G) \in \mathcal{O}_{\rho_1}$, $\forall G \in \mathcal{O}_{\rho_2}$.*
(ii) *(f, F) is cocontinuous $\iff f^{\leftarrow}(K) \in \mathcal{C}_{\rho_1}$, $\forall K \in \mathcal{C}_{\rho_2}$.*

Proof. We prove (i), and the second result is dual.

$(\Longrightarrow:)$ Let (f, F) be a continuous difunction. Take $G \in \mathcal{O}_{\rho_2}$. We show that $F^{\leftarrow}(G)$ is open in $(U_1, \mathcal{U}_1, \rho_1)$. Let $F^{\leftarrow}(G) \not\subseteq Q_u$ for some $u \in U$. By the definition of inverse image, there exists $v \in V$ such that $\overline{P}_{(u,v)} \not\subseteq F$ and $G \not\subseteq Q_v$. Since G is open, we have $N^{\rho}_{\epsilon}(v) \subseteq G$ for $\epsilon > 0$. By the definition of continuity, there exists $\delta > 0$ such that $N^{\rho_1}_{\delta}(u) \subseteq F^{\leftarrow}(N^{\rho_2}_{\epsilon}(v))$. Then $N^{\rho_1}_{\delta}(u) \subseteq F^{\leftarrow}(G)$, and so $F^{\leftarrow}(G) \in \mathcal{O}_{\rho_1}$.

$(\Longleftarrow:)$ Let $\overline{P}_{(u,v)} \not\subseteq F$. We consider $N^{\rho_2}_{\epsilon}(v)$ for some $\epsilon > 0$. Since $N^{\rho_2}_{\epsilon}(v) \in \mathcal{O}_{\rho_2}$, we have $F^{\leftarrow}(N^{\rho_2}_{\epsilon}(v)) \in \mathcal{O}_{\rho_1}$ by assumption. Since $P_v \subseteq N^{\rho_2}_{\epsilon}(v)$ and $P_v \not\subseteq F^{\rightarrow}(Q_u)$, $N^{\rho_2}_{\epsilon}(v) \not\subseteq F^{\rightarrow}(Q_v)$. Hence, we have $F^{\leftarrow}(N^{\rho_2}_{\epsilon}(v)) \not\subseteq Q_u$. Then there exists $\delta > 0$ such that $N^{\rho_1}_{\delta}(u) \subseteq F^{\leftarrow}(N^{\rho_2}_{\epsilon}(v))$. □

Corollary 2.6. *Let $(U_j, \mathcal{U}_j, \rho_j)$, $j = 1, 2$ be (pseudo) dimetric texture spaces and (f, F) be a difunction from $(U_1, \mathcal{U}_1)$ to $(U_2, \mathcal{U}_2)$.*

(1) *(f, F) is $\rho_1 - \rho_2$ continuous $\Longleftrightarrow$ (f, F) is $\tau_{\rho_1} - \tau_{\rho_2}$ continuous.*
(2) *(f, F) is $\rho_1 - \rho_2$ cocontinuous $\Longleftrightarrow$ (f, F) is $\kappa_{\rho_1} - \kappa_{\rho_2}$ cocontinuous.*

Proof. We prove (1), leaving the dual proof of (2) to the interested reader.
($\Longrightarrow$:) Let (f, F) be $\rho_1 - \rho_2$ continuous. Let $G \in \tau_{\rho_2}$. To prove $F^{\leftarrow}(G) \in \tau_{\rho_1}$, we take $F^{\leftarrow}(G) \not\subseteq Q_u$ for some $u \in U_1$. By definition of inverse relation, there exists $v \in U_2$ such that $\overline{P}_{(u,v)} \not\subseteq F$ and $G \not\subseteq Q_v$. Since $G \in \tau_{\rho_2}$, we have $N_\epsilon^{\rho_2}(v) \subseteq G$ for some $\epsilon > 0$. Then $F^{\leftarrow}\big(N_\epsilon^{\rho_2}(v)\big) \subseteq F^{\leftarrow}(G)$. From the assumption, we have $\delta > 0$ such that $N_\delta^{\rho_1}(u) \subseteq F^{\leftarrow}\big(N_\epsilon^{\rho_2}(v)\big) \subseteq F^{\leftarrow}(G)$. Thus, $F^{\leftarrow}(G) \in \tau_{\rho_1}$.
($\Longleftarrow$:) Let $\overline{P}_{(u,v)} \not\subseteq F$. We consider $N_\epsilon^{\rho_2}(v)$ for some $\epsilon > 0$. Since $P_v \subseteq N_\epsilon^{\rho_2}(v)$, we have $N_\epsilon^{\rho_2}(v) \not\subseteq F^{\rightarrow}(Q_u)$. Hence, $F^{\leftarrow}\big(N_\epsilon^{\rho_2}(v)\big) \not\subseteq Q_u$, and since $F^{\leftarrow}\big(N_\epsilon^{\rho_2}(v)\big)$ is open in $(U_1, \mathcal{U}_1, \rho_1)$, we have $N_\epsilon^{\rho_1}(u) \subseteq F^{\leftarrow}\big(N_\delta^{\rho_2}(v)\big)$ for some $\delta > 0$. Thus, (f, F) is $\rho_1 - \rho_2$ continuous. $\square$

Theorem 2.7. *(Pseudo) dimetric texture spaces and bicontinuous difunctions form a category.*

Proof. Since bicontinuity between ditopological texture spaces is preserved under composition of difunction [6], and identity difunction on $(S, \mathcal{S}, \rho)$ is $\rho - \rho$ bicontinuous, and the identity difunctions are identities for composition and composition is associative [5, Proposition 2.17(3)], (pseudo) di-metric texture spaces and bicontinuous difunctions form a category. $\square$

Definition 2.8. The category whose objects are (pseudo) di-metrics texture spaces and whose morphisms are bicontinuous difunctions will be denoted by (**dfDiMP**) **dfDiM**.

Clearly, **dfDiM** is a full subcategory of **dfDiMP**.

If we take as objects di-metric on a simple texture we obtain the full subcategory **dfSDiM** and inclusion functor $\mathfrak{S}$: **dfSDiM** $\hookrightarrow$ **dfDiM**.
Also we obtain the full subcategory **dfPDiM** and inclusion functor $\mathfrak{P}$: **dfPDiM** $\hookrightarrow$ **dfDiM** by taking as objects di-metrics on a plain texture.

In the same way we can use **dfPSDiM** to denote the category whose objects are di-metrics on a plain simple texture, and whose morphisms are bicontinuous difunctions.

Now, we define $\mathcal{G}$: **dfDiM** $\to$ **dfDitop** by

$$\mathcal{G}((U, \mathcal{U}, \rho) \xrightarrow{(f,F)} (V, \mathcal{V}, \mu)) = (U, \mathcal{U}, \tau_\rho, \kappa_\rho) \xrightarrow{(f,F)} (V, \mathcal{V}, \tau_\mu, \kappa_\mu).$$

Obviously, $\mathcal{G}$ is a full concrete functor from Corollary 2.6. Likewise, the same functor may set up from **dfDiMP** to **dfDitop**.

We recall [11] that a ditopology on $(U, \mathcal{U})$ is called (pseudo) dimetrizable if it is the (pseudo) dimetric ditopology of some (pseudo) dimetric on $(U, \mathcal{U})$. We denote by $\mathbf{dfDitop}_{dm}$ the category of dimetrizable ditopological texture space and bicontinuous difunction. Clearly it is full subcategory of the category **dfDitop**.

Proposition 2.9. *The categories* $\mathbf{dfDitop}_{dm}$ *and* **dfDiM** *are equivalent.*

Proof. Consider the functor $\mathcal{G} : \mathbf{dfDiM} \to \mathbf{dfDitop}_{dm}$ which is defined above. It can be easily seen that $\mathcal{G}$ is a full and faithfull functor, since the hom-set restriction function of $\mathcal{G}$ is onto and injective. Now we take a dimetrizable ditopological texture space $(U, \mathcal{U}, \tau, \kappa)$ such that $\tau = \tau_\rho$ and $\kappa = \kappa_\rho$, where ρ is a dimetric on $(U, \mathcal{U})$. Clearly, the identity difunction $(i, I) : (U, \mathcal{U}, \rho) \to \mathcal{G}(U, \mathcal{U}, \rho)$ is an isomorphism in the category $\mathbf{dfDitop}_{dm}$. Hence, $\mathcal{G}$ is isomorphism-closed, and so the proof is completed. □

Corollary 2.10. *The category* **dfDiMP** *is equivalent to the category of pseudo dimetrizable completely biregular* [7] *ditopological texture spaces and bicontinuous difunctions.*

Proof. Let $(U, \mathcal{U}, \rho)$ be a pseudo dimetric space. Then the dimetric ditopology $(U, \mathcal{U}, \tau_\rho, \kappa_\rho)$ is completely biregular by [11, Corollary 6.5]. Consequently, the functor $\mathcal{G}$ which is the above proposition is given an equivalence between the categories **dfDiMP** and the category of pseudo metrizable completely biregular ditopological texture spaces. □

On the other hand, since every pseudo dimetric ditopology is T_0 [11, Corollary 6.5], so the category **dfDiMP** is equivalent to the category of pseudo metrizable T_0 ditopological texture spaces and bicontinuous difunctions.

Now we give some properties of morphisms in the category **dfDiM**. Note that it takes consideration the reference [1] for some concepts of category theory

Proposition 2.11. *Let* (f, F) *be a morphism from* $(U, \mathcal{U}, \rho)$ *to* $(V, \mathcal{V}, \mu)$ *in the category* **dfDiM** *(***dfDiMP***).*

1. *If* (f, F) *is a section then it is injective.*
2. *If* (f, F) *is injective morphism then it is a monomorphism.*
3. *If* (f, F) *is retraction then it is surjective.*
4. *If* (f, F) *is surjective morphism then it is an epimorphism.*
5. (f, F) *is an isomorphism if and only if it is bijective and the inverse difunction* $(f, F)^{\leftarrow}$ *is bicontinuous difunction.*

Proof. The proof of $(1)-(4)$ can be obtained easily in the category **dfTex** by [5, Proposition 3.14]. We show that the result (5). Note that, (f, F) is bijective if and only if it is an isomorphism in **dfTex**. Since (f, F) is bijective, its inverse $(f, F)^{\leftarrow}$ is a morphism in **dfTex** such that $(f, F)^{\leftarrow} \circ (f, F) = (i_U, I_U)$, $(f, F) \circ (f, F)^{\leftarrow} = (i_V, I_V)$. Consequently, (f, F) is $\rho - \mu$ bicontinuous iff $(f, F)^{\leftarrow}$ is $\mu - \rho$ bicontinuous. □

Now let (U, d) be a classical (pseudo) metric space. Then $\rho = (d, d)$ is a (pseudo) dimetric on the discrete texture space $\big(U, \mathcal{P}(U)\big)$. As a result, a subset of U is open (closed) in the metric space (U, d) if and only if it is open (closed) in the dimetric texture space $(U, \mathcal{P}(U), \rho)$.

On the other hand, recall that [5] if (f, F) is a difunction from $(U, \mathcal{P}(U))$ to $(V, \mathcal{P}(V))$, then f and F are point functions from U to V where $F = (U \times V) \setminus f = f'$.

The category of metric spaces and continuous functions between metric spaces is denoted by **Met**.

According to:

Theorem 2.12. *The category* **Met** *is isomorphic to the full subcategory of* **dfDiM***.*

Proof. We consider a full subcategory **D-dfDiM** of **dfDiM** whose objects are dimetric texture spaces on discrete textures and morphisms are bicontinuous difunctions. Now we prove that the mapping $\mathfrak{T}$:**Met**→ **D-dfDiM** is a functor such that

$$\mathfrak{T}(U, d) = (U, \mathcal{P}(U), \rho) \text{ and } \mathfrak{T}(f) = (f, f')$$

where f is a morphism in **Met**. Note that (f, f') is a bicontinuous difunction in **D-dfDiM** if and only if f is a continuous point function in **Met**. It can be easily seen that if i is identity function on U then (i, I) is identity difunction on $(U, \mathcal{P}(U))$ where $I = (U \times U) \setminus i$. Since $f' \circ g' = (f \circ g)'$, we have $\mathfrak{T}(f \circ g) = \mathfrak{T}(f) \circ \mathfrak{T}(g)$. Hence, $\mathfrak{T}$ is a functor. Furthermore, $\mathfrak{T}$ is bijective on objects, and the hom-set restriction of $\mathfrak{T}$ is injective and onto. Consequently, $\mathfrak{T}$ is clearly an isomorphism functor. □

By using same arguments, the category **PMet** of pseudo metric spaces and continuous functions is isomorphic to the full subcategory of **dfDiMP**.

Now suppose that (U, d) is a classical metric space and $(U, \mathcal{T}_d)$ is the metric topological space. Then the pair $(\mathcal{T}_d, \mathcal{T}_d^c)$ is a ditopology on $(U, \mathcal{P}(U))$. On the other hand, we consider the dimetric ditopological texture space $(U, \mathcal{P}(U), \tau_\rho, \kappa_\rho)$ where $\rho = (d, d)$. Now we consider the functors $\mathcal{M}$: **Met** → **dfDitop** and $\mathcal{N}$: **Met** → **dfDitop** which are defined by

$$\mathcal{M}((U, d) \xrightarrow{f} (V, e)) = (U, \mathcal{P}(U), \mathcal{T}_d, \mathcal{T}_d^c) \xrightarrow{(f, f')} (V, \mathcal{P}(V), \mathcal{T}_e, \mathcal{T}_e^c),$$

$$\mathcal{N}((U, d) \xrightarrow{f} (V, e)) = (U, \mathcal{P}(U), \tau_\rho, \kappa_\rho) \xrightarrow{(f, f')} (V, \mathcal{P}(V), \tau_\mu, \kappa_\mu)$$

where $\rho = (d, d)$ and $\mu = (e, e)$. According to:

Proposition 2.13. *Let $\tau : \mathcal{M} \to \mathcal{N}$ be a function such that assigns to each* **Met***-object (X, d) a* **dfDitop***-morphism $\tau_{(X,d)} = (i, I) : \mathcal{M}(X, d) \to \mathcal{N}(X, d)$. Then τ is a natural transformation.*

Proof. We prove that naturality condition holds. Let $f : (U, d) \to (V, e)$ be a **Met**-morphism. From Theorem 2.12, $(f, f') : (U, \mathcal{P}(U), \rho) \to (V, \mathcal{P}(V), \mu)$ is a **dfDiM**-morphism. Further, it is a **dfDitop**-morphism by Corollary 2.6.

$$\begin{array}{ccc} (U,\mathcal{P}(U),\mathcal{T}_d,\mathcal{T}_d^c) & \xrightarrow{\tau_{(U,d)}=(i,I)} & (U,\mathcal{P}(U),\tau_\rho,\kappa_\rho) \\ \Big\downarrow{\scriptstyle (f,f')} & & \Big\downarrow{\scriptstyle (f,f')} \\ (V,\mathcal{P}(V),\mathcal{T}_e,\mathcal{T}_e^c) & \xrightarrow[\tau_{(V,e)}=(i,I)]{} & (V,\mathcal{P}(V),\tau_\mu,\kappa_\mu) \end{array}$$

On the other hand, the identity difunction $\tau_{(U,d)} = (i, I) : (U, \mathcal{P}(U), \mathcal{T}_d, \mathcal{T}_d^c) \to (U, \mathcal{P}(U), \tau_\rho, \kappa_\rho)$ is bicontinuous on $(U, \mathcal{P}(U))$, and so it is a **dfDitop**-morphism. Clearly the above diagram is commutative, and the proof is completed. □

3. Point functions between dimetric texture spaces

As we have noted earlier, however, it is possible to represent difunctions by ordinary point functions in certain situations. The construct **fDitop**, where the objects are ditopological texture spaces and the morphisms bicontinuous point functions satisfying (a) and (b) which is given Note 1.4, and we will to define a similar construct of (pseudo) dimetric texture spaces.

Definition 3.1. Let $(U, \mathcal{U}, \rho)$ and $(V, \mathcal{V}, \mu)$ be (pseudo) dimetric texture spaces, and φ on U to V a point function satisfy the condition (a). Then φ is called

(1) continuous if $\varphi^{\leftarrow}(P_v) \not\subseteq Q_u$ implies $N_\delta^\rho(u) \subseteq \varphi^{\leftarrow}(N_\epsilon^\mu(v))$, $\forall \epsilon > 0$ $\exists \delta > 0$.
(2) cocontinuous if $P_u \not\subseteq \varphi^{\leftarrow}(Q_v)$ implies $\varphi^{\leftarrow}(M_\epsilon^\mu(v)) \subseteq M_\delta^\rho(u)$, $\forall \epsilon > 0$ $\exists \delta > 0$.
(3) bicontinuous if it is continuous and cocontinuous.

Proposition 3.2. *Let φ be a point function satisfy the condition (a) from $(U_1, \mathcal{U}_1, \rho_1)$ to $(U_2, \mathcal{U}_2, \rho_2)$.*

(i) *φ is continuous* $\Longleftrightarrow \varphi^{\leftarrow}(G) \in \mathcal{O}_{\rho_1}$, $\forall G \in \mathcal{O}_{\rho_2}$.
(ii) *φ is cocontinuous* $\Longleftrightarrow \varphi^{\leftarrow}(K) \in \mathcal{C}_{\rho_1}$, $\forall K \in \mathcal{C}_{\rho_2}$.

Proof. Let φ be a point function satisfy the condition (a) and (f_φ, F_φ) be the corresponding difunction. Then $\varphi^{\leftarrow}(B) = F_\varphi^{\leftarrow}(B) = f_\varphi^{\leftarrow}(B)$ for all $B \in \mathcal{U}_2$. Now we take $G \in \mathcal{O}_{\rho_2}$. We show that $\varphi^{\leftarrow}(G)$ is open in $(U_1, \mathcal{U}_1, \rho_1)$. Let $\varphi^{\leftarrow}(G) \not\subseteq Q_u$ for some $u \in U$. By the definition of inverse image, there exists $v \in V$ such that $\overline{P}_{(u,v)} \not\subseteq F$ and $G \not\subseteq Q_v$. Since G is open, we have $N_\epsilon^{\rho_2}(v) \subseteq G$ for $\epsilon > 0$. By the definition of continuity, there exists $\delta > 0$ such that $N_\delta^{\rho_1}(u) \subseteq F^{\leftarrow}(N_\epsilon^{\rho_2}(v))$. Then $N_\delta^{\rho_1}(u) \subseteq F^{\leftarrow}(G)$, and so $F^{\leftarrow}(G) \in \mathcal{O}_{\rho_1}$.

($\Longleftarrow$:) Let $\overline{P}_{(u,v)} \not\subseteq F$. We consider $N_\epsilon^{\rho_2}(v)$ for some $\epsilon > 0$. Since $N_\epsilon^{\rho_2}(v) \in \mathcal{O}_{\rho_2}$, we have $F^{\leftarrow}(N_\epsilon^{\rho_2}(v)) \in \mathcal{O}_{\rho_1}$ by assumption. Since $P_v \subseteq N_\epsilon^{\rho_2}(v)$ and $P_v \not\subseteq F^{\rightarrow}(Q_u)$, $N_\epsilon^{\rho_2}(v) \not\subseteq F^{\rightarrow}(Q_v)$. Hence, we have $F^{\leftarrow}(N_\epsilon^{\rho_2}(v)) \not\subseteq Q_u$. Then there exists $\delta > 0$ such that $N_\delta^{\rho_1}(u) \subseteq F^{\leftarrow}(N_\epsilon^{\rho_2}(v))$. □

Corollary 3.3. *Suppose that* $\varphi : (U_1, \mathcal{U}_1) \to (U_2, \mathcal{U}_2)$ *is a point function satisfy the condition (a) and that* ρ_k *is a (pseudo) dimetric on* $(U_k, \mathcal{U}_k)$*,* $k = 1, 2$*. Then*

(1) φ *is bicontinuous if and only if* (f_φ, F_φ) *is bicontinuous.*
(2) φ *is* $\rho_1 - \rho_2$ *bicontinuous if and only if* φ *is* $(\tau_{\rho_1}, \kappa_{\rho_1})$*–*$(\tau_{\rho_2}, \kappa_{\rho_2})$ *bicontinuous where* $(\tau_{\rho_j}, \kappa_{\rho_j})$*,* $j = 1, 2$ *is dimetric ditopological texture space.*

Proof. Since $\varphi^{\leftarrow}(B) = F_\varphi^{\leftarrow}(B) = f_\varphi^{\leftarrow}(B)$ for all $B \in \mathcal{U}_2$, the proof is automatically obtained by Corollary 2.6. □

The category whose objects are dimetrics and whose morphisms are bicontinuous point functions satisfying the conditions (a) and (b) will be denoted by **fDiM**.

Proposition 3.4. *Let* f *be a morphism from* $(U, \mathcal{U}, \rho)$ *to* $(V, \mathcal{V}, \mu)$ *in the category* **fDiM**.

(1) *If* f *is a section then it is an* **fDiM***-embedding.*
(2) *If* f *is injective morphism then it is a monomorphism.*
(3) *If* f *is a retraction then it is a* **fDiM***-quotient.*
(4) *If* f *is a surjective morphism then it is an epimorphism.*
(5) f *is an isomorphism if and only if it is a textural isomorphism and its inverse is bicontinuous.*

Proof. Since the category **fDiM** is a construct, the first four results are automatically obtained.

Recall that f is a textural isomorphism from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ if it is a bijective point function from U to V satisfying $A \in \mathcal{U} \implies f(A) \in \mathcal{V}$ such that $A \to f(A)$ is a bijective from $\mathcal{U}$ to $\mathcal{V}$. Hence, this is equivalent to requiring that f be bijective with inverse g, and $A \in \mathcal{U} \implies f(A) \in \mathcal{V}$ and $B \in \mathcal{V} \implies g(B) \in \mathcal{U}$.

By [5, Proposition 3.15], f is textural isomorphism if and only if f is isomorphism in **fTex**. □

We define $\mathfrak{D} :$ **fDiM** $\to$ **dfDiM** by

$$\mathfrak{D}((U, \mathcal{U}, \rho) \xrightarrow{\varphi} (V, \mathcal{V}, \mu)) = (U, \mathcal{U}, \rho) \xrightarrow{(f_\varphi, F_\varphi)} (V, \mathcal{V}, \mu).$$

Theorem 3.5. $\mathfrak{D}$: **fDiM** $\to$ **dfDiM** *defined above is a functor. The restriction* $\mathfrak{D}_p$: **fPDiM** $\to$ **dfPDiM** *is an isomorphism with inverse* $\mathfrak{V}_p$: **dfPDiM** $\to$ **fPDiM** *given by*

$$\mathfrak{V}_p((U, \mathcal{U}, \rho) \xrightarrow{(f,F)} (V, \mathcal{V}, \mu)) = (U, \mathcal{U}, \rho) \xrightarrow{\varphi(f,F)} (V, \mathcal{V}, \mu).$$

Likewise we have isomorphism between **fSDiM** *and* **dfSDiM**.

Proof. It is easy to show that $\mathfrak{D}(\iota_U) = (i_U, I_U)$. Now let $(U, \mathcal{U})$, $(V, \mathcal{V})$, $(Z, \mathcal{Z})$ be textures, $\varphi : U \to V$, $\psi : V \to Z$ point functions satisfying (a) and (b). We have $(f_{\psi\circ\varphi}, F_{\psi\circ\varphi}) = (f_\psi, F_\psi) \circ (f_\varphi, F_\varphi)$ by [5, Theorem 3.10]. We can also say that a point function is (texturally) bicontinuous if and only if the corresponding difunction is bicontinuous. Thus $\mathfrak{D} : \mathbf{fDiM} \to \mathbf{dfDiM}$ is a functor. If we restrict to $\mathfrak{D}_p : \mathbf{fPDiM} \to \mathbf{dfPDiM}$ we again obtain a functor. Now let us define $\mathfrak{V}_p : \mathbf{dfPDiM} \to \mathbf{fPDiM}$ by $\mathfrak{V}_p(U, \mathcal{U}, \rho) = (U, \mathcal{U}, \rho)$ and $\mathfrak{V}_p(f, F) = \varphi_{(f,F)}$ which is also a functor and the inverse of $\mathfrak{D}_p$. This means that $\mathfrak{D}_p$ is an isomorphism. The other isomorphisms can be proved similarly. □

We recall that a quasi-pseudo metric on a set U in the sense of J. C. Kelly [9] is a non-negative real-valued function $\rho(,)$ on the product $U \times U$ such that

(1) $\rho(u, u) = 0$, $(u \in U)$
(2) $\rho(u, z) \leq \rho(u, v) + \rho(v, z)$, $(u, v, z \in U)$

Now let $\rho(,)$ be a quasi-pseudo metric on a set U, and let $q(,)$ be defined by $q(u, v) = \rho(v, u)$. Then it is a trivial matter to verify that $q(u, v)$ is a quasi-pseudo metric on U. In this case, $\rho(,)$ and $q(,)$ are called conjugate, and denote the set U with this structure (U, ρ, q).

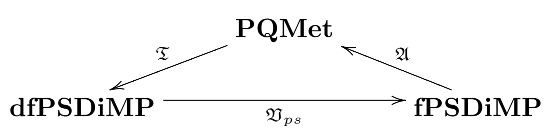
Now let (U_1, ρ_1, q_1) and (U_2, ρ_2, q_2) be quasi-pseudo metric spaces. A function $f : U_1 \to U_2$ is pairwise continuous if and only if f is ρ_1–ρ_2 continuous and q_1–q_2 continuous. So, quasi-pseudo metric spaces and pairwise continuous functions form a category, and we will denote this category **PQMet**.

Obviously, **Met** is a full subcategory of **PQMet**.

Now let $(U, \mathcal{U}, \rho)$ be a dimetric space with $(U, \mathcal{U})$ plain. Then $u = v \Longrightarrow \overline{\rho}(u, v) = 0$ and $\underline{\rho}(u, v) = 0$, by the dimetric condition (M2). So, $(U, \overline{\rho}, \underline{\rho})$ is pseudo-quasi metric space in the usual sense. Thus we have a forgetful functor $\mathfrak{A} : \mathbf{fPSDiMP} \to \mathbf{PQMet}$, if we set $\mathfrak{A}(U, \mathcal{U}, \rho) = (U, \overline{\rho}, \underline{\rho})$ and $\mathfrak{A}(\varphi) = \varphi$.

Likewise, the functor $\mathfrak{T} : \mathbf{Met} \to \mathbf{dfDiM}$ becomes a functor $\mathfrak{T} : \mathbf{PQMet} \to \mathbf{dfDiMP}$ on setting $\mathfrak{T}(U, p, q) = (U, \mathcal{P}(U), (p, q))$ and $\mathfrak{T}(\varphi) = \varphi$.

Now we consider the following diagram.



Then:

Theorem 3.6. *$\mathfrak{A}$ is an adjoint of $\mathfrak{V}_{ps} \circ \mathfrak{T}$ and $\mathfrak{T}$ a co-adjoint of $\mathfrak{A} \circ \mathfrak{V}_{ps}$.*

Proof. Take $(U, p, q) \in \mathrm{Ob}\,(\mathbf{PQMet})$. We show that $(\iota_U, (U, \mathcal{P}(U), (p, q)))$ is an $\mathfrak{A}$-universal arrow. It is clearly an $\mathfrak{A}$-structured arrow, so take an object $(U, \mathcal{U}, \mu)$ in **fPSDiMP** and $\varphi \in \mathbf{PQMet}((U, p, q), (U, \overline{\mu}, \underline{\mu}))$. Then, by [5, Theorem 3.12], we know that $\varphi \in \mathrm{Mor}\,\mathbf{fPSTex}$, and that it is the unique such morphism satisfying $\mathfrak{A}(\varphi) \circ \iota_U = \varphi$, so it remains to verify that $\varphi : (U, \mathcal{P}(U), (p, q)) \to (U, \mathcal{U}, \mu)$ is bicontinuous. However, for every open set G in $(U, \mathcal{U}, \mu)$, we have $\varphi^{\leftarrow}(G) = \varphi^{-1}[G]$, by [5, Lemma 3.9], and $\varphi^{-1}[G]$ is open in (U, p, q) since φ is p–$\overline{\mu}$ continuous. Likewise, for every closed set K in $(U, \mathcal{U}, \mu)$ we have $\varphi^{\leftarrow}(K) = U \setminus \varphi^{-1}[U \setminus K]$ is closed in (U, p, q) since φ is q–$\underline{\mu}$ continuous. □

4. Dimetrics and Direlational Uniformity

In this section, we will give a relation between dimetrics and direlational uniformity by using categorical approach. Firstly, we recall some basic definitons and results for direlational uniformity from [11].

Let us denote by $\mathcal{DR}$ the family of direlations on $(U, \mathcal{U})$.

Direlational Uniformity: Let $(U, \mathcal{U})$ be a texture space and $\mathcal{D}$ a family of direlations on $(U, \mathcal{U})$. Then $\mathcal{D}$ is called *direlational uniformity* on $(U, \mathcal{U}$ if it satisfies the following conditions:

(1) If $(r, R) \in \mathcal{D}$ implies $(i, I) \sqsubseteq (r, R)$.

(2) If $(r, R) \in \mathcal{D}$, $(e, E) \in \mathcal{DR}$ and $(r, R) \sqsubseteq (e, E)$ then $(e, E) \in \mathcal{D}$.

(3) If $(r, R), (e, E) \in \mathcal{D}$ implies $(d, D) \sqcap (e, E) \in \mathcal{D}$.

(4) If $(r, R) \in \mathcal{D}$ then there exists $(e, E) \in \mathcal{D}$ such that $(e, E) \circ (e, E) \sqsubseteq (r, R)$.

(5) If $(r, R) \in \mathcal{D}$ then there exists $(c, C) \in \mathcal{U}$ such that $(c, C)^{\leftarrow} \sqsubseteq (r, R)$.

Then the triple $(U, \mathcal{U}, \mathcal{D})$ is said to be *direlational uniform texture*.

It will be noted that this definition is formally the same as the the usual definition of a diagonal uniformity, and the notions of base and subbase may be defined in the obvious way. Further, if $\bigsqcap \mathcal{D} = (i, I)$ then $\mathcal{D}$ is said to be separated.

Inverse of a direlation under a difunction: Let (f, F) be a difunction from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ and (r, R) be a direlation on $(V, \mathcal{V})$. Then

$$(f,F)^{-1}(r) = \bigvee\{\overline{P}_{(u_1,u_2)} \mid \exists\, P_{u_1} \not\subseteq Q_{u_1'} \text{ so that } \overline{P}_{(u_1',v_1)} \not\subseteq F, f \not\subseteq \overline{Q}_{(u_2,v_2)} \implies \overline{P}_{(v_1,v_2)} \subseteq r\}$$

$$(f,F)^{-1}(R) = \bigcap\{\overline{Q}_{(u_1,u_2)} \mid \exists\, P_{u_1'} \not\subseteq Q_{u_1} \text{ so that} f \not\subseteq \overline{Q}_{(u_1',v_1)}, \overline{P}_{(u_2,v_2)} \not\subseteq F, \implies R \subseteq \overline{Q}_{(v_1,v_2)}\}$$

$$(f,F)^{-1}(r,R) = ((f,F)^{-1}(r), (f,F)^{-1}(R)).$$

Uniformly bicontinuos difunction: Let $(U, \mathcal{U}, \mathcal{D})$ and $(V, \mathcal{V}, \mathcal{E})$ be direlational uniform texture space and (f, F) be a difunction from $(U, \mathcal{U})$ to $(V, \mathcal{V})$. Then (f, F) is called $\mathcal{D}$–$\mathcal{E}$ *uniformly bicontinuous* if $(r, R) \in \mathcal{E} \Longrightarrow (f, F)^{-1}(r, R) \in \mathcal{D}$.

Recall that [12] the category whose objects are direlational uniformities and whose morphisms are uniformly bicontinuous difunctions was denoted by **dfDiU**.

Now let us verify that a pseudo dimetric also defines a direlational uniformity.

Theorem 4.1. *Let ρ be a pseudo dimetric on $(U, \mathcal{U})$.*

i) *For $\epsilon > 0$ let*

$$r_\epsilon = \bigvee\{\overline{P}_{(u,v)} \mid \exists z \in U, P_u \not\subseteq Q_z \text{ and } \overline{\rho}(z,v) < \epsilon\}$$

$$R_\epsilon = \bigcap\{\overline{Q}_{(u,v)} \mid \exists z \in U, P_z \not\subseteq Q_u \text{ and } \underline{\rho}(z,v) < \epsilon\}$$

Then the family $\{(r_\epsilon, R_\epsilon) \mid \epsilon > 0\}$ is a base for a direlational uniformity $\mathcal{D}_\rho$ on $(U, \mathcal{U})$.

ii) *The uniform ditopology [11] of $\mathcal{U}_\rho$ coincides with the pseudo metric ditopology of ρ.*

A direlational uniformity $\mathcal{D}$ on $(U, \mathcal{U})$ is called (pseudo) dimetrizable if there exists a (pseudo) dimetric ρ with $\mathcal{D} = \mathcal{D}_\rho$.

Lemma 4.2. *Let $(U_j, \mathcal{U}_j, \rho_j)$, $j = 1, 2$ be (pseudo) dimetrics and (f, F) be a difunction from $(U_1, \mathcal{U}_1)$ to $(U_2, \mathcal{U}_2)$. Then (f, F) is $\rho_1 - \rho_2$ bicontinuous if and only if (f, F) is $\mathcal{D}_{\rho_1} - \mathcal{D}_{\rho_2}$ uniformly bicontinuous.*

Proof. Let (f, F) be a $\rho_1 - \rho_2$ bicontinuous difunction from $(U_1, \mathcal{U}_1, \rho_1)$ to $(U_2, \mathcal{U}_2, \rho_2)$. From Corollary 2.6, (f, F) is also bicontinuous from $(U_1, \mathcal{U}_1, \tau_{\rho_1}, \kappa_{\rho_1})$ to $(U_2, \mathcal{U}_2, \tau_{\rho_2}, \kappa_{\rho_2})$ where $(\tau_{\rho_j}, \kappa_{\rho_j})$ is (pseudo) dimetric ditopology on $(U_j, \mathcal{U}_j)$, $j = 1, 2$. On the other hand, the uniform ditopology of $\mathcal{D}_{\rho_j}$ coincides with the (pseudo) dimetric ditopology of ρ_j, $j = 1, 2$. Further, (f, F) is also uniformly bicontinuous by [11, Proposition 5.13]. □

Now we define $\mathcal{G} : \mathbf{dfDiM} \to \mathbf{dfDiU}$ by

$$\mathcal{G}((U, \mathcal{U}, \rho) \xrightarrow{(f,F)} (V, \mathcal{V}, \mu)) = (U, \mathcal{U}, \mathcal{D}_\rho) \xrightarrow{(f,F)} (V, \mathcal{V}, \mathcal{D}_\mu).$$

Obviously, $\mathcal{G}$ is a full concrete functor from Lemma 4.2.

We denote by $\mathbf{dfDiU}_{dm}$ the category of dimetrizable direlational uniform textures and uniformly bicontinuous difunctions.

Proposition 4.3. *The categories $\mathbf{dfDiU}_{dm}$ and $\mathbf{dfDiM}$ are equivalent.*

Proof. It is easy to show that the functor $\mathcal{G} : \mathbf{dfDiM} \to \mathbf{dfDiU}_{dm}$ which is defined above is full and faitfull. Now we take an object $(U, \mathcal{U}, \mathcal{D})$ in $\mathbf{dfDiU}_{dm}$. Since it is a metrizable direlational uniform space, then there exists a dimetric ρ on $(U, \mathcal{U})$ such that $\mathcal{U} = \mathcal{U}_\rho$. Because of the identity difunction $(i, I) : (U, \mathcal{U}, \rho) \to \mathcal{G}(U, \mathcal{U}, \rho)$ is an isomorphism in the category $\mathbf{dfDiU}_{dm}$, the functor $\mathcal{G}$ is isomorphism-closed. Hence, the proof is completed. □

Recall that [11] a direlational uniformity $\mathcal{U}$ is (pseudo) dimetrizable if and only if it has a countable base. If the category of direlational uniformities with countable bases and uniformly bicontinuous difunctions denote by $\mathbf{dfDiU_{cb}}$ then we have next result automatically from Proposition 4.3:

Corollary 4.4. *The categories $\mathbf{dfDiU_{cb}}$ and $\mathbf{dfDiM}$ are equivalent.*

A direlational uniformity $\mathcal{D}$ is dimetrizable if and only if it is separated [11]. We denote the category of separated direlational uniformities and uniformly bicontinuous difunctions by $\mathbf{dfDiU_s}$. From Proposition 4.3, we have:

Corollary 4.5. $\mathbf{dfDiU_s}$ *is equivalent to the category* **dfDiM**.

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