COMBINED MATRICES OF SIGN REGULAR MATRICES *

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Abstract. The combined matrix of a nonsingular matrix A is the Hadamard (entry wise) product $C(A) = A \circ (A^{-1})^T$. Since each row and column sum of C(A) is equal to one, the combined matrix is doubly stochastic when it is nonnegative. In this work, we study the nonnegativity of the combined matrix of sign regular matrices, based upon their signature. In particular, a few coordinates of the signature ε of A play a crucial role in determining whether or not $C(A)$ is nonnegative.

Key words. Sign regular matrix. Combined matrix. Hadamard product. Nonnegative matrix.

AMS subject classifications. 15A48, 15A57

1. Introduction. The combined matrix of a given real matrix A, denoted by $C(A)$, has been studied in [4], [5], [6] and [9]. Furthemore, when $C(A)$ is nonnegative it is doubly stochastic. Applications of the combined matrix can be found in [7] and [9]. In [2], we have studied when the combined matrix of some classes of matrices are nonnegative. More precisely, we have studied the nonnegativity of the combined matrices of totally positive (nonnegative) matrices and totally negative (nonpositive) matrices.

In this work, we study the nonnegativity of combined matrices of nonsingular sign regular matrices. Sign regular matrices are defined by its signature vector and have different applications, some of them can be seen in [1, 8, 10]. The paper is organized as follows. In section 2, we give our notation and some lemmas that help to prove the main results. In section 3, we give the results on the nonnegativity of the combined matrices of sign regular matrices. Finally, in section 4 we write our conclusions.

2. Notation and previous results. We consider $n \times n$ nonsingular real matrices. Given an $n \times n$ matrix A, we denote by N the set of indexes $\{1, ..., n\}$. Given two indices $i, j \in N$, we denote by A_{ij} the (i, j) minor, i.e., the determinant of the submatrix obtained from A by deleting row i and column j.

The nonnegativity of a matrix is considered entry-wise. That is, a real matrix $A = [a_{ij}]$ is nonnegative (positive) if $a_{ij} \geq 0$ $(a_{ij} > 0)$ for all $i, j \in N$, and we will denote it by $A \geq 0$ $(A > 0)$.

Definition 2.1. *An $n \times n$ real matrix $A = [a_{ij}]$ is said to have a checkerboard pattern if* $\mathrm{sign}(a_{ij}) = (-1)^{i+j}$ *or $a_{ij} = 0$ for all $i, j \in N$.*

A signature is a vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ whose entries take values from the set {+1,-1}.

Definition 2.2. *An $n \times n$ matrix A is called sign regular of order k with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ if for each $j = 1, \ldots, k$, the sign of all its minors of order j coincides with ε_j. When $k = n$, a sign regular matrix of order k is simply called sign regular matrix (see [10]).*

A sign regular matrix A has associated a signature $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_k = +1$ if all its minors of order k are positive or equal to zero and $\varepsilon_k = -1$ if they are negative

*Research supported by Spanish DGI grant number MTM2010-18674.

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or equal to zero. It can not be the case that all minors of order k are zero, since A is a nonsingular matrix. Then $\varepsilon_k = \pm 1$.

We denote by S the matrix

$$S = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^{n-1} \end{bmatrix}. \tag{2.1}$$

If A is a nonsingular sign regular matrix A, the signature of the matrix $SA^{-1}S$ is obtained from the signature of A as the following result shows.

THEOREM 2.3 (Theorem 3.3 of [1]). *Let A be a nonsingular sign regular matrix with signature ε. Then the matrix $SA^{-1}S$ is also sign regular and the entries of its signature satisfy*

$$\varepsilon_i(SA^{-1}S) = \varepsilon_n \varepsilon_{n-i}$$

with convention $\varepsilon_j = 1$ when $j = 0$.

Recall that the Hadamard (or entry–wise) product of two $n \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the matrix $A \circ B = [a_{ij} b_{ij}]$.

DEFINITION 2.4. *The combined matrix of a nonsingular real matrix A is defined as $C(A) = A \circ (A^{-1})^T$.*

Then, if $A = [a_{ij}]$ and $A^{-1} = \left[\frac{1}{\det(A)}(-1)^{i+j} A_{ji}\right]$, the combined matrix is

$$C(A) = \left[\frac{1}{\det(A)}(-1)^{i+j} a_{ij} A_{ij}\right] = [c_{ij}].$$

From this definition we can see that the entries $\varepsilon_1, \varepsilon_{n-1}, \varepsilon_n$ of the signature of A, play an important role in determining whether $C(A)$ is nonnegative or not. $C(A)$ has some curious properties (see [9]). Among them, the row and column sums are equal to one. As a consequence, if $C(A)$ is nonnegative, $C(A)$ is doubly stochastic. This class of matrices has interesting properties and applications (see [3]).

LEMMA 2.5. *If A is an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ and $C(A)$ is nonnegative, then $C(A)$ or $-C(A)$ has a checkerboard pattern. More precisely, we have:*

1. *If $\varepsilon_1 \varepsilon_{n-1} \varepsilon_n = 1$, then $C(A)$ has the following zero pattern*

$$\begin{bmatrix} * & 0 & * & \cdots & * & 0 \\ 0 & * & 0 & \cdots & 0 & * \\ * & 0 & * & \cdots & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & * & \cdots & * & 0 \\ 0 & * & 0 & \cdots & 0 & * \end{bmatrix} \quad \textit{or} \quad \begin{bmatrix} * & 0 & * & \cdots & 0 & * \\ 0 & * & 0 & \cdots & * & 0 \\ * & 0 & * & \cdots & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & 0 & \cdots & * & 0 \\ * & 0 & * & \cdots & 0 & * \end{bmatrix},$$

if $n = 2k$ and if $n = 2k - 1$, respectively.

2. If $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = -1$, then $C(A)$ has the following zero pattern

$$\begin{bmatrix} 0 & * & 0 & \cdots & 0 & * \\ * & 0 & * & \cdots & * & 0 \\ 0 & * & 0 & \cdots & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & 0 & \cdots & 0 & * \\ * & 0 & * & \cdots & * & 0 \end{bmatrix} \quad or \quad \begin{bmatrix} 0 & * & 0 & \cdots & * & 0 \\ * & 0 & * & \cdots & 0 & * \\ 0 & * & 0 & \cdots & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & * & \cdots & 0 & * \\ 0 & * & 0 & \cdots & * & 0 \end{bmatrix},$$

if $n = 2k$ and if $n = 2k - 1$, respectively.

*The entries marked by * are nonnegative.*

Proof. The checkerboard pattern of $C(A)$ and $-C(A)$ follows from the c_{ij} entries and Definition 2.1. Furthermore, each case follows from the definition of $C(A)$ and the facts that A is sign regular and $C(A) \geq 0$. □

Throughout the paper we consider that the index $k \in \mathbb{N}$. It is used to define $n = 2k$ or $n = 2k - 1$. Note that the diagonal and the antidiagonal (formed by the entries whose indices i, j satisfy $j = n - i + 1$) are important in the study of the combined matrices of sign regular matrices. We denote by J the antidiagonal matrix whose antidiagonal entries are 1.

LEMMA 2.6. *(See Lemma 7 of [8]) Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$.*

1. If $\varepsilon_2 = 1$, then
  - *(a) $a_{11} \neq 0, a_{22} \neq 0, \ldots, a_{nn} \neq 0$,*
  - *(b) $a_{ij} = 0$ and $i > j \Rightarrow a_{tl} = 0$, for all $t \geq i$ and $l \leq j$,*
  - *(c) $a_{ij} = 0$ and $i < j \Rightarrow a_{tl} = 0$, for all $t \leq i$ and $l \geq j$.*

2. If $\varepsilon_2 = -1$, then
  - *(a) $a_{1n} \neq 0, a_{2,n-1} \neq 0, \ldots, a_{n1} \neq 0$,*
  - *(b) $a_{ij} = 0$ and $j > n - i + 1 \Rightarrow a_{tl} = 0$, for all $t \geq i$ and $l \geq j$,*
  - *(c) $a_{ij} = 0$ and $j < n - i + 1 \Rightarrow a_{tl} = 0$, for all $t \leq i$ and $l \leq j$.*

Note that matrices $SA^{-1}S$ and $SA^{-T}S$ have the same signature, i.e., $\varepsilon(SA^{-1}S) = \varepsilon(SA^{-T}S)$.

LEMMA 2.7. *Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ and S given in* (2.1).

1. If $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$ and $C(A) \geq 0$, then $C(A) = A \circ (SA^{-T}S)$.

2. If $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = -1$ and $C(A) \geq 0$, then $C(A) = -(A \circ (SA^{-T}S))$.

Proof. 1. Let denote by c_{ij} the entries of $C(A)$ and by m_{ij} the entries of $A \circ (SA^{-T}S)$. Then

$$m_{ij} = \begin{cases} c_{ij} & \text{if} \quad i + j = 2k, \\ -c_{ij} & \text{if} \quad i + j = 2k - 1, \end{cases}$$

for some $k \in \mathbb{N}$. Since $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$ and $C(A) \geq 0$, from Lemma 2.5, $c_{ij} = 0$ if $i + j = 2k - 1$, thus, $c_{ij} = m_{ij}$ for all i, j.

2. Reasoning in a similar way, $c_{ij} = 0$ if $i + j = 2k$, thus $-c_{ij} = m_{ij}$ for all i, j. □

3. Nonnegativity of combined matrices. In this section we give the main results of the paper. More precisely, we study when the combined matrix of sign regular matrices is nonnegative. First, let us give some lemmas that help to write the proof of all cases. These cases of sign regular matrices are related to some elements of the signature associated to the matrix.

LEMMA 3.1. *Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. Let the combined matrix $C(A) \geq 0$. If $\varepsilon_2(\varepsilon_{n-2}\varepsilon_n) = -1$, then $c_{ij} = 0$ for each of the following subsets of indices i, j :*

1. *$j > i$ and $j > n - i + 1$.*
2. *$j > i$ and $j < n - i + 1$.*
3. *$j < i$ and $j > n - i + 1$.*
4. *$j < i$ and $j < n - i + 1$.*

Proof. Consider the case $\varepsilon_2 = 1$ and $\varepsilon_{n-2}\varepsilon_n = -1$. Note that $\varepsilon_2(SA^{-T}S) = -1$ by Theorem 2.3. Let us give the proof of part 1, that is, we are going to prove that $c_{ij} = 0$ for all i, j such that $j > i$ and $j > n - i + 1$. We mark by $\bullet$ these entries in the combined matrix

$$C(A) = \begin{bmatrix} * & * & \cdots & * & * & * & * \\ * & * & \cdots & * & * & * & \bullet \\ * & * & \ddots & * & \vdots & \bullet & \bullet \\ \vdots & \vdots & \cdots & \ddots & \bullet & \vdots & \vdots \\ * & * & \cdots & * & \ddots & \bullet & \bullet \\ * & * & \cdots & * & * & * & \bullet \\ * & * & \cdots & * & * & * & * \end{bmatrix}.$$

To reach a contradiction suppose that there exists an entry $c_{ij} \neq 0$ with $j > i$ and $j > n - i + 1$. Then, by definition of $C(A)$, $a_{ij} \neq 0$ and $A_{ij} \neq 0$. Since $C(A) \geq 0$, $c_{i,j-1} = 0$ by Lemma 2.5. Then $a_{i,j-1} = 0$ or $A_{i,j-1} = 0$. However $a_{i,j-1}$ can not be zero, since A is sign regular with $\varepsilon_2 = 1$. Otherwise, $a_{i,j-1} = 0$ implies $a_{ij} = 0$ by Lemma 2.6 part 1.c, which contradicts $a_{ij} \neq 0$. Further, in this case, $A_{i,j-1}$ can not be zero. If so, since $SA^{-T}S$ is sign regular with $\varepsilon_2(SA^{-T}S) = -1$, applying Lemma 2.6 part 2.b, $A_{i,j-1} = 0$ implies $A_{ij} = 0$, which contradicts $A_{ij} \neq 0$.

Then, $c_{ij} = 0$ for all i, j such that $j > i$ and $j > n - i + 1$.

Parts 2, 3 and 4 follow with the same arguments.

The reverse case, that is, when $\varepsilon_2 = -1$ and $\varepsilon_{n-2}\varepsilon_n = 1$, has a similar proof. $\square$

LEMMA 3.2. *Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ and $C(A) \geq 0$. If $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$, $\varepsilon_2(\varepsilon_{n-2}\varepsilon_n) = -1$ and $n = 2k-1$, then $C(A)$ is either diagonal or antidiagonal.*

Proof. Note that at least one entry in each row of $C(A)$ is positive, since $C(A)$ is doubly stochastic. Let us consider the case $\varepsilon_2 = 1$, $\varepsilon_{n-2}\varepsilon_n = -1$. Note that $\varepsilon_2(SA^{-T}S) = -1$ by Theorem 2.3, and $C(A) = A \circ (SA^{-T}S)$ by Lemma 2.7. Further, the only nonzero entry in the first row of $C(A)$ must be $c_{11} \neq 0$ or $c_{1n} \neq 0$ by Lemma 3.1.

First, consider $c_{1n} \neq 0$. We are going to prove that $C(A) = J$. In this case $c_{1n} \neq 0$ implies $a_{1n} \neq 0$. Since $\varepsilon_2 = 1$ and $a_{1n} \neq 0$, applying conveniently Lemma 2.6 to A, it follows that $a_{ij} \neq 0$, for all $i \leq j$. On the other hand, by Lemma 2.5, we have $c_{i,n-i} = 0$, for all $i = 1, 2, \ldots, n-1$. Then it follows that $A_{i,n-i} = 0$, for all $i < \frac{n+1}{2}$, since $a_{ij} \neq 0$, for all $i \leq j$. Even more, as $\varepsilon_2(SA^{-T}S) = -1$ and $A_{i,n-i} = 0$, applying conveniently Lemma 2.6 to $SA^{-T}S$, it follows that, $A_{ii} = 0$ and therefore $c_{ii} = 0$, for all i such that $1 \leq i < \frac{n+1}{2}$.

Furthermore, considering that $c_{1n} \neq 0$, by Lemma 2.5, $c_{i,n-i+2} = 0$, for all $i > 1$. Then $A_{i,n-i+2} = 0$, for all i such that $1 < i \leq \frac{n+1}{2}$, since $a_{ij} \neq 0$ for all $i < j$. Considering that $\varepsilon_{n-2}\varepsilon_n = -1$ and $A_{i,n-i+2} = 0$, applying conveniently Lemma 2.6 to $SA^{-T}S$, it follows that $A_{ii} = 0$ and therefore $c_{ii} = 0$, for all i such that $\frac{n+1}{2} < i \leq n$.

A:	A.1:	
$\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$	$\varepsilon_2 = 1$, $\varepsilon_{n-2}\varepsilon_n = 1$	$C(A) \geq 0 \Leftrightarrow C(A) = I$
	A.2:	$n = 2k$: $C(A)$ is not nonnegative
	$\varepsilon_2 = -1$, $\varepsilon_{n-2}\varepsilon_n = -1$	$n = 2k-1$: $C(A) \geq 0 \Leftrightarrow C(A) = J$
	A.3:	$n = 2k$: $C(A) \geq 0 \Leftrightarrow C(A)$=I
	$\varepsilon_2 = 1$, $\varepsilon_{n-2}\varepsilon_n = -1$	$n = 2k-1$,
		$a_{1n} \neq 0$: $C(A) \geq 0 \Leftrightarrow C(A) = J$
		$a_{1n} = 0$: $C(A) \geq 0 \Leftrightarrow C(A) = I$
	A.4:	$n = 2k$: $C(A) \geq 0 \Leftrightarrow C(A)$=I
	$\varepsilon_2 = -1$, $\varepsilon_{n-2}\varepsilon_n = 1$	$n = 2k-1$,
		$a_{11} \neq 0$: $C(A) \geq 0 \Leftrightarrow C(A) = I$
		$a_{11} = 0$: $C(A) \geq 0 \Leftrightarrow C(A) = J$
B:	B.1:	
$\varepsilon_1\varepsilon_{n-1}\varepsilon_n = -1$	$\varepsilon_2 = 1$, $\varepsilon_{n-2}\varepsilon_n = 1$	$C(A)$ is not nonnegative
	B.2:	$n = 2k$: $C(A) \geq 0 \Leftrightarrow C(A) = J$
	$\varepsilon_2 = -1$, $\varepsilon_{n-2}\varepsilon_n = -1$	$n = 2k-1$: $C(A)$ is not nonnegative
	B.3:	$n = 2k$: $C(A) \geq 0 \Leftrightarrow C(A) = J$
	$\varepsilon_2\varepsilon_{n-2}\varepsilon_n = -1$	$n = 2k-1$: $C(A)$ is not nonnegative

TABLE 3.1

Nonnegative combined matrices of sign regular matrices for $n \geq 4$. If $n = 3$, cases A.1, A.2 and B.3 are only possible.

In conclusion, $c_{ii} = 0$ for all $i \neq \frac{n+1}{2}$ and then, by Lemma 3.1, $C(A) = J$.

Second, consider now $c_{11} \neq 0$. We are going to prove that $C(A) = I$. In this case $c_{11} \neq 0$ implies $A_{11} \neq 0$. Since $\varepsilon_2(SA^{-T}S) = -1$ and $A_{11} \neq 0$, applying conveniently Lemma 2.6 to $SA^{-T}S$, it follows that $A_{ij} \neq 0$, for all $i \leq n-j+1$. By Lemma 2.5, $c_{i,i+1} = 0$, for all $i = 1, 2, \ldots, n-1$, then it follows that $a_{i,i+1} = 0$, for all $i < \frac{n+1}{2}$, since $A_{ij} \neq 0$, for all $i \leq n-j+1$. Even more, as $\varepsilon_2 = 1$ and $a_{i,i+1} = 0$, applying conveniently Lemma 2.6 to A, it follows that $a_{i,n-i+1} = 0$, and therefore $c_{i,n-i+1} = 0$, for all i such that $1 \leq i < \frac{n+1}{2}$.

Further, considering that $c_{11} \neq 0$, we have $c_{i,i-1} = 0$, for all $i > 1$, by Lemma 2.5. Then $a_{i,i-1} = 0$, for all i such that $1 < i < n-j+1$, since $A_{ij} \neq 0$, for all $i \leq n-j+1$. Considering that $\varepsilon_2 = 1$ and $a_{i,i-1} = 0$, applying conveniently Lemma 2.6 to A, it follows that $a_{i,n-i+1} = 0$ and therefore $c_{i,n-i+1} = 0$, for all i such that $\frac{n+1}{2} < i \leq n$.

In conclusion $c_{i,n-i+1} = 0$, for all $i \neq \frac{n+1}{2}$, and then $C(A) = I$ by Lemma 3.1.

The case $\varepsilon_2 = -1$, $\varepsilon_{n-2}\varepsilon_n = 1$ can be proved with the same arguments. $\square$

To analyze the nonnegativity of the combined matrix of an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ we must consider the cases of Table 3.1.

THEOREM 3.3. ***(Case** A.1) Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ such that $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$. If $\varepsilon_2 = 1$ and $\varepsilon_{n-2}\varepsilon_n = 1$, then $C(A) \geq 0$ if and only if $C(A) = I$.*

Proof. ($\Leftarrow$) It is obvious.

($\Rightarrow$) Note that, by Theorem 2.3, $SA^{-T}S$ is sign regular, with $\varepsilon_2(SA^{-T}S) = 1$, and $C(A) = A \circ (SA^{-T}S)$ by Lemma 2.7. Since $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$, by Lemma 2.5, $c_{i,i+1} = 0$, then $a_{i,i+1} = 0$ or $A_{i,i+1} = 0$ for all $i = 1, 2, \ldots, n-1$. Also, since $c_{j+1,j} = 0$, then $a_{j+1,j} = 0$ or $A_{j+1,j} = 0$ for all $j = 2, 3, \ldots, n$. Applying conveniently Lemma 2.6 to

A and to $SA^{-T}S$ it follows that the only nonzero entries in $C(A)$ are in the diagonal. On the other hand, since $C(A)$ is nonnegative and doubly stochastic it follows that $C(A) = I$. ◻

Note that, case $A.1$ contains the totally nonnegative matrices (TNN). We have proven that the only nonnegative combined matrix of a TNN matrix is the identity matrix (see Theorem 2.1 of [2]).

THEOREM 3.4. *(**Case** $A.2$) Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ such that $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$, $\varepsilon_2 = -1$ and $\varepsilon_{n-2}\varepsilon_n = -1$.*

1. *If $n = 2k$, $k \in \mathbb{N}$, then $C(A)$ is not nonnegative.*
2. *If $n = 2k-1$, $k \in \mathbb{N}$, then $C(A) \geq 0 \Leftrightarrow C(A) = J$.*

Proof. 1. To reach a contradiction, let us assume that $C(A) \geq 0$. Since $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$ and $n = 2k$, it follows from Lemma 2.5 that $c_{i,n-i+1} = 0$, for all $i \in N$.

On the other hand, since A is sign regular with $\varepsilon_{n-2}\varepsilon_n = -1$, it follows from Theorem 2.3 that $\varepsilon_2(SA^{-T}S) = -1$. By Lemma 2.7, $C(A) = A \circ (SA^{-T}S)$. Then, applying conveniently Lemma 2.6 to A and to $SA^{-T}S$, it follows that $a_{i,n-i+1} \neq 0$ and $A_{i,n-i+1} \neq 0$ and so, $c_{i,n-i+1} \neq 0$ for all $i \in N$. This contradicts the fact that the $c_{i,n-i+1} = 0$. Therefore, $C(A)$ is not nonnegative.

2. $(\Leftarrow)$ It is obvious.

$(\Rightarrow)$ Let us assume that $C(A) \geq 0$. Note that $C(A) = A \circ (SA^{-T}S)$, by Lemma 2.7. Since A is sign regular with $\varepsilon_{n-2}\varepsilon_n = -1$, then $\varepsilon_2(SA^{-T}S) = -1$ by Theorem 2.3. Applying conveniently Lemma 2.6 to A and to $SA^{-T}S$, it follows that $a_{i,n-i+1} \neq 0$ and $A_{i,n-i+1} \neq 0$, then $c_{i,n-i+1} \neq 0$ for all $i \in N$.

Since $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$ and $n = 2k-1$, by Lemma 2.5, $c_{i,n-i} = 0$, then $a_{i,n-i} = 0$ or $A_{i,n-i} = 0$ for all $i = 1, 2, \ldots, n-1$. In a similar way $c_{i,n-i+2} = 0$, then $a_{i,n-i+2} = 0$ or $A_{i,n-i+2} = 0$ for all $i = 2, 3, \ldots, n$. Applying again Lemma 2.6, it follows $c_{ij} = 0$ for all i, j such that $j \neq n-i+1$.

Given that $C(A) \geq 0$ is doubly stochastic, it follows that $C(A) = J$. ◻

THEOREM 3.5. *(**Case** $A.3$) Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ such that $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$, $\varepsilon_2 = 1$ and $\varepsilon_{n-2}\varepsilon_n = -1$.*

1. *If $n = 2k$, $k \in \mathbb{N}$, then $C(A) \geq 0 \Leftrightarrow C(A)$=I.*
2. *If $n = 2k-1$, $k \in \mathbb{N}$ and $a_{1n} \neq 0$, then $C(A) \geq 0 \Leftrightarrow C(A) = J$.*
3. *If $n = 2k-1$, $k \in \mathbb{N}$ and $a_{1n} = 0$, then $C(A) \geq 0 \Leftrightarrow C(A) = I$.*

Proof. 1. $(\Leftarrow)$ It is straightforward.

$(\Rightarrow)$ Let us assume that $C(A) \geq 0$. Since $\varepsilon_2(\varepsilon_{n-2}\varepsilon_n) = -1$, the nonzero entries of $C(A)$ can be only in the diagonal or antidiagonal by Lemma 3.1. Since $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$ and $n = 2k$, by Lemma 2.5, $c_{i,n-i+1} = 0$, for all $i \in N$. Therefore, $C(A) = I$ since $C(A) \geq 0$ is doubly stochastic.

2. $(\Leftarrow)$ It is obvious.

$(\Rightarrow)$ Let us assume that $C(A) \geq 0$. Since $\varepsilon_2(\varepsilon_{n-2}\varepsilon_n) = -1$ and $n = 2k-1$, we deduce that $C(A)$ is diagonal or antidiagonal by Lemma 3.2. We have $\varepsilon_2(SA^{-T}S) = -1$ by Theorem 2.3. Further, $A_{1n} \neq 0$ by Lemma 2.6 applied to $SA^{-T}S$. As $a_{1n} \neq 0$, then $c_{1n} \neq 0$. We conclude that $C(A)$ is antidiagonal. Since $C(A) \geq 0$ is doubly stochastic, $C(A) = J$.

3. $(\Leftarrow)$ It is obvious.

$(\Rightarrow)$ Let us assume that $C(A) \geq 0$. Since $\varepsilon_2(\varepsilon_{n-2}\varepsilon_n) = -1$ and $n = 2k-1$, we deduce that $C(A)$ is diagonal or antidiagonal by Lemma 3.2. Since $a_{1n} = 0$ implies $c_{1n} = 0$, we have that $C(A)$ is diagonal. $C(A) \geq 0$ doubly stochastic implies $C(A) = I$. ◻

Theorem 3.6. *(**Case** A.4) Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ such that $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$, $\varepsilon_2 = -1$ and $\varepsilon_{n-2}\varepsilon_n = 1$.*

1. *If $n = 2k$, $k \in \mathbb{N}$, then $C(A) \geq 0 \Leftrightarrow C(A)=I$.*
2. *If $n = 2k-1$, $k \in \mathbb{N}$ and $a_{11} \neq 0$, then $C(A) \geq 0 \Leftrightarrow C(A) = I$.*
3. *If $n = 2k-1$, $k \in \mathbb{N}$ and $a_{11} = 0$, then $C(A) \geq 0 \Leftrightarrow C(A) = J$.*

Proof. 1. Similar to the proof of Theorem 3.5, case 1.

2. ($\Leftarrow$) It is obvious.

($\Rightarrow$) Let us assume that $C(A) \geq 0$. Since $\varepsilon_2(\varepsilon_{n-2}\varepsilon_n) = -1$ and $n = 2k-1$, we deduce that $C(A)$ is diagonal or antidiagonal by Lemma 3.2. We have $\varepsilon_2(SA^{-T}S) = 1$ by Theorem 2.3. This implies $A_{11} \neq 0$ by Lemma 2.6. Since $a_{11} \neq 0$, then $c_{11} \neq 0$. We conclude that $C(A)$ is diagonal. Since $C(A) \geq 0$ is doubly stochastic, then $C(A) = I$.

3. ($\Leftarrow$) It is obvious.

($\Rightarrow$) Let us assume that $C(A) \geq 0$. Since $\varepsilon_2(\varepsilon_{n-2}\varepsilon_n) = -1$ and $n = 2k-1$, we deduce that $C(A)$ is diagonal or antidiagonal by Lemma 3.2. We have $a_{11} = 0$ and this implies $c_{11} = 0$. Then $C(A)$ is antidiagonal. Since $C(A) \geq 0$ is doubly stochastic, we conclude that $C(A) = J$. □

Theorem 3.7. *(**Case** B.1). Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ such that $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = -1$. If $\varepsilon_2 = 1$ and $\varepsilon_{n-2}\varepsilon_n = 1$, then $C(A)$ is not nonnegative.*

Proof. Let us work by contradiction. Assume that $C(A) \geq 0$. Since $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = -1$, we have $c_{ii} = 0$, for all $i \in N$ by Lemma 2.5. Since A is sign regular with $\varepsilon_{n-2}\varepsilon_n = 1$, we have $\varepsilon_2(SA^{-T}S) = 1$ by Theorem 2.3. Furthermore, $C(A) = -A \circ (SA^{-T}S))$ by Lemma 2.7. Applying Lemma 2.6 to A and to $SA^{-T}S$, we obtain $c_{ii} \neq 0$ for all $i \in N$. This contradicts the fact $c_{ii} = 0$. Hence $C(A)$ is not nonnegative. □

Theorem 3.8. *(**Case** B.2). Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ such that $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = -1$, $\varepsilon_2 = -1$ and $\varepsilon_{n-2}\varepsilon_n = -1$.*

1. *If $n = 2k$, $k \in \mathbb{N}$, then $C(A) \geq 0 \Leftrightarrow C(A) = J$.*
2. *If $n = 2k-1$, $k \in \mathbb{N}$, then $C(A)$ is not nonnegative.*

Proof. 1. ($\Leftarrow$) It is straightforward.

($\Rightarrow$) Let us assume that $C(A) \geq 0$. Since A is sign regular with $\varepsilon_{n-2}\varepsilon_n = -1$, then $\varepsilon_2(SA^{-T}S) = -1$ by Theorem 2.3. We have $C(A) = -A \circ (SA^{-T}S)$ by Lemma 2.7.

Since $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = -1$ and $n = 2k$, we have $c_{i,n-i} = 0$ for $i = 1, 2, \dots, n-1$ and $c_{i,n-i+2} = 0$ for $i = 2, 3, \dots, n$ by Lemma 2.5. Applying conveniently Lemma 2.6 to A and to $SA^{-T}S$ it follows that $a_{ij} = 0$ or $A_{ij} = 0$, and consequently $c_{ij} = 0$, for all $i, j \in N$ such that $j \neq n-i+1$. Given that $C(A) \geq 0$ is doubly stochastic, it follows that $C(A) = J$.

2. To reach a contradiction, let us assume that $C(A) \geq 0$. Since $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = -1$ and $n = 2k-1$, we have $c_{ii} = c_{i,n-i+1} = 0$ for all $i \in N$ by Lemma 2.5. Further, since A is sign regular with $\varepsilon_{n-2}\varepsilon_n = -1$, then $\varepsilon_2(SA^{-T}S) = -1$ by Theorem 2.3. In addition, $C(A) = -A \circ (SA^{-T}S)$ by Lemma 2.7. Since $\varepsilon_2 = -1$ and $\varepsilon_2(SA^{-T}S) = -1$, we have $a_{i,n-i+1} \neq 0$ and $A_{i,n-i+1} \neq 0$ for all $i \in N$ by Lemma 2.6. Then, $c_{i,n-i+1} \neq 0$ for all $i \in N$. This is a contradiction. Hence $C(A)$ is not nonnegative. □

Theorem 3.9. *(**Case** B.3). Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ such that $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = -1$, and $\varepsilon_2(\varepsilon_{n-2}\varepsilon_n) = -1$.*

1. *If $n = 2k$, $k \in \mathbb{N}$, then $C(A) \geq 0 \Leftrightarrow C(A) = J$*
2. *If $n = 2k-1$, $k \in \mathbb{N}$, then $C(A)$ is not nonnegative.*

Proof. 1. ($\Leftarrow$) It is obvious.

($\Rightarrow$) Let us assume that $C(A) \geq 0$. Since $\varepsilon_2(\varepsilon_{n-2}\varepsilon_n) = -1$, we have $c_{ij} = 0$ for all $i, j \in N$ whenever $j \neq i$ and $j \neq n - i + 1$ by Lemma 3.1. Since $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = -1$ and $n = 2k$, we have $c_{ii} = 0$ for all $i \in N$ by Lemma 2.5. Consequently $c_{ij} = 0$ for all $i, j \in N$ with $j \neq n - i + 1$. Since $C(A) \geq 0$ is doubly stochastic, $C(A) = J$.

2. To reach a contradiction, let us assume that $C(A) \geq 0$. Since $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = -1$ and $n = 2k - 1$, we have $c_{ii} = 0$ and $c_{i,n-i+1} = 0$ for all $i \in N$ by Lemma 2.5. However, since $\varepsilon_2(\varepsilon_{n-2}\varepsilon_n) = -1$, we conclude that $C(A) = 0$ by Lemma 3.1. This is a contradiction. Therefore, $C(A)$ is not nonnegative. □

Note that the case $B.3$ contains the totally nonpositive matrices (TNP). In [2] (Theorem 2.7), we have proven for 2×2 TNP matrices that $C(A) \geq 0$ is equivalent to $C(A) = J$. In addition, Theorem 2.6 of the same paper states that $C(A)$ is not nonnegative for any TNP matrix of order greater than 2.

4. Conclusions. We have studied the nonnegativity of the combined matrix of sign regular matrices. We have proven that, when the combined matrix of a sign regular matrix is nonnegative, it is either the identity matrix I or the antidiagonal matrix J. The main tool we have used are the values of some coordinates of the signature vector ε of the sign regular matrix. In this way, the coordinates $\varepsilon_1(A)$, $\varepsilon_{n-1}(A)$ and $\varepsilon_n(A)$ play a crucial role in determining when $C(A)$ is nonnegative. Further, the entries $\varepsilon_2(A)$ and $\varepsilon_{n-2}(A)$ are important in this study since the second entry of the signature of the matrix $SA^{-T}S$ is $\varepsilon_2(SA^{-T}S) = \varepsilon_2(A)\varepsilon_{n-2}(A)$. We have displayed in Table 3.1, for any sign regular matrix, a summary of all possibilities of being $C(A) \geq 0$. Furthermore, we have fitted the cases TNN and TNP with the corresponding general case.

Acknowledgment. We thank the referee for his report that improves this paper.

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