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Additional Information

Computing American Option Price under Regime Switching with Rationality Parameter

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Abstract

American put option pricing under regime switching is modelled by a system of coupled partial differential equations. The proposed model combines better the reality of the market by incorporating the regime switching jointly with the emotional behaviour of traders using the rationality parameter approach recently introduced by Tågholt Gad and Lund Petersen to cope with possible irrational exercise policy. The classical rational exercise is recovered as a limit case of the rational parameter. The resulting nonlinear system of PDEs is solved by a weighted finite difference method, also known as θ -method. In order to avoid the need of an iterative method for the nonlinear system, the term with rationality parameter and the coupling term are treated explicitly. Next, the resulting linear system is solved by Thomas algorithm. Stability conditions for the numerical scheme are studied by using the von Neumann approach. Numerical examples illustrate the efficiency and accuracy of the proposed method.

Keywords: American regime-switching option pricing, rational exercise, partial differential system, weighted finite difference scheme, numerical analysis, computing

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1. Introduction

In financial derivatives pricing problems, when the stochastic process for the underlying asset is too simple, as when assuming constant parameters [3], the model does not replicate the market price. This drawback has been overcome
5 in the literature by introducing stochastic volatility, jump-diffusion and regime switching models for the underlying price evolution.

Since the paper of Buffington and Elliot [5] the switching model has attracted much attention, mainly due to its ability to model non-constant real scenarios when market switches from time to time among different regimes. It
10 is well known that regime switching models are computationally inexpensive when compared to stochastic volatility jump-diffusion models and provide versatile applications in other fields, like electricity markets [2], valuation of stock loans [34], forestry valuation [6], natural gas [7] and insurance [17].

In this paper we consider a continuous time Markov chain α_t taking values
15 among different regimes, where I is the total number of regimes considered in the market. Thus, each regime is labelled by an integer i with $1 \leq i \leq I$. Hence, the regime space of α_t is $\Omega = \{1, 2, \dots, I\}$. Let $Q = (q_{i,j})_{I \times I}$ be the given generator matrix of α_t . Following [32], the entries $q_{i,j}$ satisfy:

$$q_{i,j} \geq 0, \text{ if } i \neq j; \quad q_{i,i} = -\sum_{j \neq i} q_{i,j}, \quad 1 \leq i \leq I. \quad (1)$$

Under the risk-neutral measure, see Elliot et al. [14] for details, the stochastic
20 process for the underlying asset S_t satisfies the following stochastic differential equation:

$$\frac{dS_t}{S_t} = r_{\alpha_t} dt + \sigma_{\alpha_t} d\tilde{B}_t, \quad t \leq 0, \quad (2)$$

where σ_{α_t} is the volatility of the asset S_t , r_{α_t} is the risk-free interest rate, both depending on the Markov chain α_t , and \tilde{B}_t is a standard Brownian motion defined on some given risk-neutral probability space, independent on the Markov
25 chain α_t .

Here we consider the American put option on the asset $S_t = S$ with strike price E and maturity $T < \infty$. For $1 \leq i \leq I$, let $V_i(S, \tau)$ denote the option price functions (i.e.

$$(V_i)_\tau = V_i(S_\tau, \tau)$$

is the option price process in regime i), where $\tau = T - t$ denotes the time to maturity and the regime $\alpha_t = i$. Then, $V_i(S, \tau)$, $1 \leq i \leq I$, satisfy the free boundary value problem for $0 < \tau \leq T$, see [22]:

$$\begin{cases} \frac{\partial V_i}{\partial \tau} = \frac{\sigma_i^2}{2} S^2 \frac{\partial^2 V_i}{\partial S^2} + r_i S \frac{\partial V_i}{\partial S} - r_i V_i + \sum_{l \neq i} q_{i,l} (V_l - V_i), & S > S_i^*(\tau), \\ V_i(S, \tau) = E - S, & 0 \leq S \leq S_i^*(\tau), \end{cases} \quad (3)$$

where $S_i^*(\tau)$ denotes the optimal stopping boundary of the option under regime i . Initial conditions are

$$V_i(S, 0) = \max(E - S, 0), \quad S_i^*(0) = E, \quad i = 1, \dots, I. \quad (4)$$

Boundary conditions for $i = 1, \dots, I$ are as follows

$$\lim_{S \rightarrow \infty} V_i(S, \tau) = 0, \quad (5)$$

$$V_i(S_i^*(\tau), \tau) = E - S_i^*(\tau), \quad (6)$$

$$\frac{\partial V_i}{\partial S}(S_i^*(\tau), \tau) = -1. \quad (7)$$

30 Several different numerical methods for solving problem (3) have been proposed. Lattice methods [19, 25] are popular for practitioners because they are easy to implement. The penalty method [23, 22, 33] adds a penalty term into each equation of the coupled system. After considering American options pricing under regime switching model as a Hamilton Jacobi Bellman problem, in
35 [1] iterated optimal stopping [24] and local policy iteration [27] methods are compared.

Recently, in [13] the front-fixing method (see [9]) has been employed for valuation of American option under regime switching model, by incorporating

free boundary into the PDE as a new unknown variable. In such paper efficient
40 explicit finite difference methods are shown.

Unlike the direct approach of a European option pricing problem where
the price is given by the solution of a partial differential equation (PDE), it is
well known that the price of an American option is described by the solution of
partial differential inequality (see [31]). Once the inequality has been discretized,
45 a linear complementarity problem (LCP) arises with the additional algebraic
complexity. Although the LCPs are satisfactorily addressed (see [15] and the
references therein), the possibility of computing an American option pricing
problem by solving a PDE problems could be interesting not only from the
computational point of view, but also from the reliability of the computed price.

50 On the other hand, very recently the rationality parameter approach pro-
posed by Tågholt Gad and Lund Pedersen in [30] allows to incorporate the
possibility of an irrational exercise policy in the American option. In this set-
ting, the computation of the price can be obtained by solving a PDE problem
with an additional nonlinear term in the corresponding European option pricing
55 formulation. Moreover, when the rationality parameter in the previous model
tends to infinity, the formulation tends to the one of the classical American
option pricing problem with rational exercise. Therefore, by addressing the
solution of the model for large enough values of the rationality parameter pro-
vides an approximation of the classical American option price. In the absence
60 of regime switching, the numerical solution of American option pricing models
with irrational exercise has been recently addressed [10].

The main aim of this paper is to propose a new model that simultaneously
incorporates the advantages of the regime switching jointly with those of the
rationality parameter approach. Additionally, we propose and develop the nu-
65 merical analysis of a suitable family of weighted finite differences schemes to
solve numerically this nonlinear model.

The plan of the work is as follows. In Section 2, the new model that takes
into account the irrational behaviour under regime switching is described and a
suitable change of variables and unknown transforms the original PDE problem

70 into an equivalent one with constant coefficients in the differential part. Section
 3 deals with the construction of a one parameter family of finite difference
 methods, also known as weighted schemes [28]. Next, relevant numerical analysis
 issues as positivity, stability and consistency are studied in Section 4. Numerical
 simulations are included in Section 5, paying special attention to the limit case of
 75 classical American options with rational exercise, also the order of convergence
 and the comparison with other methods is presented. Section 6 contains the
 concluding remarks.

2. Regime switching model with rationality parameter

For the sake of simplicity, we summarize the rationality parameter approach
 80 in the incorporation of the irrational behaviour of the option trader into the
 mathematical formulation of the model throughout the intensity function (see
 Theorem 2 in [30]). Then, Feynman-Kac theorem gives the option price as the
 solution of a nonlinear PDE problem. The rationality parameter λ involved
 in the intensity functions plays the role of a unifying modelling tool in such
 85 way that for $\lambda = 0$ provides the European style and for large values of λ one
 approximates to the classical American option price with rational exercise.

For an intensity function $f : [-E, E] \rightarrow [0, \infty)$, in the regime switching
 setting we assume that the relation between the profitability and the stochastic
 exercise intensity is $f((E - S)^+ - V_i(S, \tau))$ for each regime. After incorporating
 90 this term to the system of PDEs satisfied by the call option price in the regime
 switching model, we obtain

$$\begin{aligned} \frac{\partial V_i}{\partial \tau} = & \frac{\sigma_i^2}{2} S^2 \frac{\partial^2 V_i}{\partial S^2} + r_i S \frac{\partial V_i}{\partial S} - r_i V_i + f((E - S)^+ - V_i) ((E - S)^+ - V_i) \\ & + \sum_{l \neq i} q_{i,l} (V_l - V_i), \quad S > 0, 0 < \tau \leq T, \end{aligned} \quad (8)$$

for $i = 1, \dots, I$, jointly with the initial and boundary conditions:

$$V_i(S, 0) = \max(E - S, 0), \quad (9)$$

$$\lim_{S \rightarrow \infty} V_i(S, \tau) = 0, \quad (10)$$

$$\begin{aligned} \frac{\partial V_i}{\partial \tau}(0, \tau) &= -r_i V_i(0, \tau) + f(E - V_i(0, \tau))(E - V_i(0, \tau)) \\ &\quad + \sum_{l \neq i} q_{i,l}(V_l(0, \tau) - V_i(0, \tau)), \quad i = 1, \dots, I. \end{aligned} \quad (11)$$

Note that since the spatial domain is $S > 0$ in rationality parameter model, an additional boundary condition at the point $S = 0$ has to be included to treat the problem numerically. We assume that the PDE (8) holds at $S = 0$, so that equation (11) is established.

In the previous framework, the family of intensity functions f depends on some parameter λ , that has to satisfy the condition of rationality parameter. In this paper we consider the parameter dependent intensity functions proposed in [30]

$$f_1(x) = \begin{cases} \lambda, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases} \quad (12)$$

as well as the two following intensity functions introduced in [10], that are the smooth analogue of stepwise function (12):

$$f_2(x) = \frac{2\lambda}{1 + e^{-\lambda^2 x}}, \quad (13)$$

$$f_3(x) = \lambda \left(1 + \frac{2}{\pi} \arctan \lambda^2 x \right). \quad (14)$$

Since a closed form solution is not available for the nonlinear system of equations (8)-(11), the solution has to be computed numerically.

In order to construct an effective finite difference scheme with constant coefficients in the differential part, let us firstly introduce the following normalized

transformation

$$x = \ln \frac{S}{E}, \quad u_i = \frac{V_i(S, \tau)}{E}, \quad i = 1, \dots, I. \quad (15)$$

Then, problem (8)-(11) takes the following equivalent form:

$$\begin{aligned} \frac{\partial u_i}{\partial \tau} &= \frac{\sigma_i^2}{2} \frac{\partial^2 u}{\partial x^2} + \left(r_i - \frac{\sigma_i^2}{2} \right) \frac{\partial u_i}{\partial x} - r_i u_i + \sum_{l \neq i} q_{i,l} (u_l - u_i) \\ &+ f(E(1 - e^x)^+ - E u_i) ((1 - e^x)^+ - u_i), \quad i = 1, \dots, I, \end{aligned} \quad (16)$$

with the new initial and boundary conditions

$$u_i(x, 0) = (1 - e^x)^+, \quad (17)$$

$$\lim_{x \rightarrow \infty} u_i(x, \tau) = 0, \quad (18)$$

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{\partial u_i}{\partial \tau}(x, \tau) &= \lim_{x \rightarrow -\infty} -r_i u_i(x, \tau) \\ &+ f(E(1 - u_i(x, \tau))) (1 - u_i(0, \tau)) \\ &+ \sum_{l \neq i} q_{i,l} (u_l(x, \tau) - u_i(x, \tau)). \end{aligned} \quad (19)$$

In next sections we propose and analyze the numerical solution of the problem (16)-(19) by using the one parameter family of a weighted finite differences scheme, also called θ -method.

3. Weighted finite difference scheme (θ -method) for PDE problem

The bounded computational domain is chosen as $[x_{min}, x_{max}] \times [0, T]$, where $x_{min} = -3$, $x_{max} = 3$, that is sufficiently large to translate limit conditions (18) and (19) into boundary conditions at $x = x_{min}$ and $x = x_{max}$, respectively. A uniform grid of $M + 1$ spatial nodes and $N + 1$ temporal nodes is introduced with step sizes $h = \frac{x_{max} - x_{min}}{M}$ and $k = \frac{T}{N}$, respectively. The nodes are denoted as follows

$$x_j = x_{min} + jh, \quad j = 0, \dots, M, \quad \tau^n = nk, \quad n = 0, \dots, N. \quad (20)$$

For each regime $i = 1, \dots, I$, the finite differences approximation of the solution at the node (x_j, τ^n) is denoted by $u_{i,j}^n$. Then, the weighted finite difference scheme with parameter $\theta \in [0, 1]$, by using central differences in space and forward difference in time takes the following form

$$\begin{aligned}
& -\theta a_i u_{i,j-1}^{n+1} + b_i u_{i,j}^{n+1} - \theta c_i u_{i,j+1}^{n+1} = (1 - \theta) a_i u_{i,j-1}^n + \tilde{b}_i u_{i,j}^n + (1 - \theta) c_i u_{i,j+1}^n \\
& + k f_{i,j}^n (u_{i,j}^0 - u_{i,j}^n) + k \sum_{l \neq i} q_{i,l} (u_{l,j}^n - u_{i,j}^n), \quad j = 1, \dots, M - 1, \quad n = 0, \dots, N - 1,
\end{aligned} \tag{21}$$

where the involved coefficients are

$$\begin{aligned}
a_i &= \frac{\sigma_i^2 k}{2 h^2} - \left(r_i - \frac{\sigma_i^2}{2} \right) \frac{k}{2h}, \\
b_i &= 1 + \theta \left(\sigma_i^2 \frac{k}{h^2} + r_i k \right), \\
\tilde{b}_i &= 1 - (1 - \theta) \left(\sigma_i^2 \frac{k}{h^2} + r_i k \right), \\
c_i &= \frac{\sigma_i^2 k}{2 h^2} + \left(r_i - \frac{\sigma_i^2}{2} \right) \frac{k}{2h},
\end{aligned} \tag{22}$$

and the rationality function term is denoted by $f_{i,j}^n = f(E(u_{i,j}^0 - u_{i,j}^n))$. Note that the θ -method is used for the differential part while the other terms are treated explicitly for the computational convenience [23]. Note that case $\theta = 0$, $\theta = 1/2$ and $\theta = 1$ corresponds to the so called fully explicit, Crank-Nicolson and fully implicit schemes, respectively.

The initial condition is discretized as follows

$$u_{i,j}^0 = (1 - e^{x_j})^+, \quad j = 0, \dots, M, \quad i = 1, \dots, I. \tag{23}$$

For each regime $i = 1, \dots, I$ and each time level $n = 0, \dots, N - 1$, the discrete form of the boundary condition at the point $x_0 = x_{min}$ is obtained by using a forward in time explicit finite difference scheme, thus leading to

$$u_{i,0}^{n+1} = (1 - r_i k) u_{i,0}^n + k f_{i,0}^n (1 - u_{i,0}^n) + k \sum_{l \neq i} q_{i,l} (u_{l,0}^n - u_{i,0}^n). \tag{24}$$

Also for each regime $i = 1, \dots, I$ and each time level $n = 0, \dots, N - 1$, at the boundary point $x_M = x_{max}$ the boundary condition $u_{i,M}^{n+1} = 0$ is imposed.

Since the nonlinear terms in (21) are taken at the previous time level, the
 130 system of equations (21) is linear with tridiagonal matrix, so that it can be solved by Thomas algorithm.

4. Qualitative properties of the scheme

In this section, some qualitative properties of the proposed numerical method (positivity, stability and consistency) are studied. First, we start with the approximation at $x = x_{min} = x_0$. Since the value at this left boundary is described
 135 by a differential equation, one has to guarantee that solution is stable and oscillations do not occur at this point. The following Lemma provides conditions for boundedness of the numerical solution at the point x_{min} .

Lemma 4.1. *With the previous notation, if*

$$k < \min_{1 \leq i \leq I} \frac{1}{r_i + C\lambda - q_{i,i}}, \quad (25)$$

then we have

$$0 \leq u_{i,0}^n \leq 1, \quad i = 1, \dots, I, \quad n = 0, \dots, N, \quad (26)$$

where the constant C appearing in 25 is defined as

$$C = \lim_{\lambda \rightarrow \infty} \frac{f^\lambda(x)}{\lambda}, \quad (27)$$

so that $C = 1$ for f_1 and $C = 2$ for f_2 and f_3 .

140 *Proof.* Let us consider boundary condition (24). Note that $u_{i,0}^0 \in [0, 1]$. Next, assume that $u_{i,0}^n \in [0, 1]$ for each regime $i = 1, \dots, I$ and fixed n . Then

$$\begin{aligned} u_{i,0}^{n+1} &\leq (1 - kr_i)u_{i,0}^n + kf_{i,0}^n(1 - u_{i,0}^n) - kq_{i,i}(1 - u_{i,0}^n) \\ &\leq (1 - kr_i - kf_{i,0}^n + kq_{i,i})u_{i,0}^n + k(f_{i,0}^n - q_{i,i}) \leq 1, \end{aligned} \quad (28)$$

provided that

$$1 - k(r_i + f_{i,0}^n - q_{i,i}) \geq 0. \quad (29)$$

In that case, since $u_{i,0}^n \leq 1$, (28) holds true if

$$1 - kr_i - kf_{i,0}^n + kq_{i,i} \leq 1 - k(f_{i,0}^n - q_{i,i}), \quad (30)$$

that is obvious for any k , such that (29) holds. Therefore, for (26) it is necessary to choose k satisfying

$$k \leq \frac{1}{r_i + f_{i,0}^n - q_{i,i}}. \quad (31)$$

Since (31) has to be fulfilled for any fixed n , $f_{i,0}^n$ can be bounded by the limit values: λ for f_1 and 2λ for f_2 and f_3 . Therefore, condition (25) is proved. Note, that the non-negativity of $u_{i,0}^n$ follows from (24). once rewritten in the following form

$$u_{i,0}^{n+1} = (1 - k(r_i - q_{i,i}))u_{i,0}^n + kf_{i,0}^n(1 - u_{i,0}^n) + k \sum_{l \neq i} q_{i,l}u_{l,0}^n, \quad (32)$$

since each term in (32) is non-negative.

□

All the forthcoming results are valid under condition (25). In the following subsection the positivity of the proposed method is studied.

4.1. Positivity

For the sake of clarity, let us recall some results in matrix analysis.

A matrix $B = (b_{ij}) \in \mathbb{R}^{m \times n}$ is a non-negative matrix, if $b_{ij} \geq 0$ for $1 \leq i \leq m$, $1 \leq j \leq n$.

For any square matrix $B \in \mathbb{R}^{n \times n}$ the maximum of the moduli of its eigenvalues is called spectral radius $\rho(B)$. The $n \times n$ identity matrix is denoted by I .

A $n \times n$ matrix B is called an M-matrix if it can be expressed in the form $B = sI - \tilde{B}$, where $\tilde{B} = (\tilde{b}_{ij})$ with $\tilde{b}_{ij} \geq 0$, and $s \geq \rho(\tilde{B})$. A matrix B is a non-singular M-matrix if and only if it is inverse-positive, that is $B^{-1} \geq 0$ (see statement F_{15} in [26]). Matrix B having all positive diagonal elements is a

M-matrix if there exists a positive diagonal matrix D , such that BD is strictly diagonally dominant (see statement N_{39} in [26]).

Next result guarantees the positivity of the numerical solution $\{u_{i,j}^n\}$ under certain conditions on the step sizes.

Theorem 4.1. *With the previous notation, the finite difference scheme (21) preserves the non-negativity of the numerical solution under the following conditions*

$$h < \min \{h_1, h_2\}, \quad k < \min_{i=1, \dots, I} \frac{h^2}{\sigma_i^2 + (r_i + q_{i,i} - 2\lambda) h^2}, \quad (33)$$

where

$$h_1 = \min_{i=1, \dots, I} \frac{\sigma_i^2}{\left| r_i - \frac{\sigma_i^2}{2} \right|}, \quad h_2 = \min_{i=1, \dots, I} \frac{\sigma_i}{\sqrt{|r_i + q_{i,i} - 2\lambda|}}. \quad (34)$$

In the case that the denominators of h_1 and h_2 vanish, only the second inequality of (33) is needed for all $h > 0$.

Proof. First, let us consider $\theta = 0$. In this case, the scheme (21) can be rewritten in following form

$$u_{i,j}^{n+1} = a_i u_{i,j-1}^n + (\tilde{b}_i + q_{i,i} k - f_{i,j}^n k) u_{i,j}^n + c_i u_{i,j+1}^n + k f_{i,j}^n u_{i,j}^0 + k \sum_{l \neq i} q_{i,l} u_{l,j}^n, \quad (35)$$

where a_i , \tilde{b}_i and c_i are defined by (22). Under conditions (33) the coefficients a_i , $(\tilde{b}_i + q_{i,i} k - f_{i,j}^n k)$ and c_i in (35) are positive. Moreover, note that the value of the intensity function $f_{i,j}^n$ is non-negative by the definition and the last term is a linear combination of non-negative elements at the previous time level. Thus, providing positive solution $\{u_{i,j}^n\}$ at the time level n , non-negativity of $\{u_{i,j}^{n+1}\}$ is established.

For the remaining values of θ , let us consider the vector form of scheme (21)

$$A_i u_i^{n+1} = \beta_i^n, \quad (36)$$

where $u_i^{n+1} = [u_{i,1}^{n+1} \ u_{i,2}^{n+1} \ \dots \ u_{i,M-1}^{n+1}]^T$, A_i is the tridiagonal constant matrix

$$A_i = \begin{pmatrix} b_i & -\theta c_i & 0 & 0 & \dots & 0 \\ -\theta a_i & b_i & -\theta c_i & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & -\theta a_i & b_i & -\theta c_i \\ 0 & 0 & \dots & 0 & -\theta a_i & b_i \end{pmatrix}, \quad (37)$$

and β_i^n is a vector of $M - 1$ components $\beta_{i,j}^n$, such that

$$\beta_{i,j}^n = (1-\theta)a_i u_{i,j-1}^n + \tilde{b}_i u_{i,j}^n + (1-\theta)c_i u_{i,j+1}^n + k f_{i,j}^n (u_{i,j}^0 - u_{i,j}^n) + k \sum_{l \neq i} q_{i,l} (u_{l,j}^n - u_{i,j}^n) \quad (38)$$

for $j = 2, \dots, M - 1$ and

$$\begin{aligned} \beta_{i,1}^n &= (1-\theta)a_i u_{i,0}^n + \tilde{b}_i u_{i,1}^n + (1-\theta)c_i u_{i,2}^n + k f_{i,1}^n (u_{i,1}^0 - u_{i,1}^n) \\ &\quad + k \sum_{l \neq i} q_{i,l} (u_{l,1}^n - u_{i,1}^n) + \theta a_i u_{i,0}^{n+1}. \end{aligned} \quad (39)$$

Note that from (22), if conditions (33) hold, then the coefficients the coefficients a_i and c_i are non-negative and also we have

$$0 \leq \theta(a_i + c_i) < b_i. \quad (40)$$

180 Consequently A_i is a strictly diagonally dominant matrix, and then a non-singular M-matrix. Therefore, the inverse matrix A_i^{-1} does not contain negative elements [26]. As it has been shown for $\theta = 0$, if conditions (33) are fulfilled then β_i^n is a non-negative vector. Therefore, the solution

$$u_i^{n+1} = A_i^{-1} \beta_i^n. \quad (41)$$

is non-negative for all $\theta \in [0, 1]$.

185

□

Under condition (25), in next subsection we prove that constraints (33) are sufficient to obtain the stability of the explicit scheme, while the stability of scheme (21) for $\theta \geq \frac{1}{2}$ does not require any extra conditions.

4.2. Stability

190 Following the stability criteria given in [11], p. 94 and in [8], p. 5, let us introduce the following definition of stability for the proposed problem.

Definition 4.1. *The numerical scheme (21) is said to be $\|\cdot\|_\infty$ -stable in the domain $[x_{min}, x_{max}] \times [0, T]$, if for every partition with $k = \Delta\tau$, $h = \Delta x$, $Nk = T$ and $Mh = x_{max} - x_{min}$ and for every regime $i = 1, \dots, I$,*

$$\|u_i^{n+1}\|_\infty \leq C \|u_i^n\|_\infty, \quad (42)$$

where C is independent on h, k .

Stability analysis is provided following von Neumann method. This approach is usually applied to schemes for linear equations. However, such method has
195 been used also for the variable coefficients case by freezing at each level (see [12], [16], [29], p. 59).

Theorem 4.2. *With the previous notation, explicit finite difference scheme (21) with $\theta = 0$ is conditionally stable with stability conditions (33).*

Proof. In order to avoid notational misunderstanding among the imaginary unit
200 with the regime index i used in previous section, only inside this proof we denote the regime index by R .

An initial error vector for every regime g_R^0 , $R = 1, \dots, I$, is expressed as a finite complex Fourier series, so that at x_j the solution $u_{i,j}^n$ can be rewritten as follows

$$u_{R,j}^n = g_R^n e^{ij\phi}, \quad j = 1, \dots, M-1, \quad R = 1, \dots, I, \quad (43)$$

where $i = (-1)^{1/2}$ is the imaginary unit and ϕ is a phase angle. Then, the scheme is stable if for every regime $R = 1, \dots, I$ the amplification factor $G_R = \frac{g_R^{n+1}}{g_R^n}$ satisfies the relation

$$|G_R| \leq 1 + Kk = 1 + O(k), \quad (44)$$

205 where the positive number K is independent on h, k and ϕ (see [28], p. 68, [29], p. 50).

For the sake of simplicity of the notation, the index of the regime R is skipped in the unknowns, the coefficients and the parameters, understanding that the calculations are done for each regime. The numerical scheme (35) is rewritten
 210 in the following way:

$$g^{n+1}e^{ij\phi} = ag^n e^{i(j-1)\phi} + (\tilde{b}_i + q_{i,ik})g^n e^{ij\phi} + cg^n e^{i(j+1)\phi} + kf(E(g^0 - g^n)e^{ij\phi})((g^0 - g^n)e^{ij\phi}) + k \sum_{l \neq R} q_{R,l} g_l^n e^{ij\phi}.$$

Next, dividing both parts by $g^n e^{ij\phi}$ and denoting

$$A_1 = ae^{-i\phi} + \tilde{b}_i + q_{i,ik} + ce^{i\phi}, \quad (45)$$

$$A_2 = f(E(g^0 - g^n)e^{ij\phi}) \left(\frac{g^0}{g^n} - 1 \right), \quad (46)$$

$$A_3 = \sum_{l \neq R} q_{R,l} \frac{g_l^n}{g^n}, \quad (47)$$

then expression (45) takes the following form:

$$G = A_1 + k(A_2 + A_3). \quad (48)$$

Next, if $|A_1| \leq 1$, $|A_2 + A_3| \leq K$ then expression (48) satisfies (44). Thus,

$$|A_1|^2 = \left(1 - 2 \frac{\sigma^2 k \sin^2 \frac{\phi}{2}}{h^2} - (r - q)k \right)^2 + \frac{k^2}{h^2} \left(r - \frac{\sigma^2}{2} \right)^2 \sin^2 \phi. \quad (49)$$

Moreover, when positivity conditions (33) are fulfilled then

$$\begin{cases} k \left(\frac{\sigma^2}{h^2} + (r - q) \right) \leq 1, \\ k \left(r - \frac{\sigma^2}{2} + \sigma^2(r - q) \right) \leq \sigma^2, \end{cases} \quad (50)$$

so that $|A_1| \leq 1$.

Now, the coupling term A_3 can be bounded as follows

$$|A_3| = \sum_{l \neq R} q_{R,l} \left| \frac{g_l^n}{g^n} \right| \leq \max_{l \neq R} \left| \frac{g_l^n}{g^n} \right| |q_{R,R}| = \left| \frac{g_{l_0(n)}^n}{g^n} \right| |q_{R,R}| = C(n), \quad (51)$$

where $C(n)$ is independent on ϕ , h and k and depends only on the frozen index

215 n .

Since intensity functions (12)-(14) are bounded, one can conclude that A_2 is also bounded by some constant independent on h , k and ϕ .

□

Analogous approach is used to study stability of the scheme (21) with $\theta > 0$.

220 **Theorem 4.3.** *With the previous notation the scheme (21) is stable for $\theta \geq \frac{1}{2}$.*

Proof. The procedure of von Neumann method is retaken (see [28], p. 68, [29], p.50) and the solution is presented in the form (43). Then, after dividing both sides of the identity by $g^n e^{ij\phi}$, the numerical scheme (21) takes the following form:

$$\begin{aligned} & \frac{g^{n+1}}{g^n} \left[1 + k\theta \left(\frac{\sigma^2}{h^2} + r - \frac{\sigma^2}{2h^2} (e^{-i\phi} + e^{i\phi}) + \frac{r - \frac{\sigma^2}{2}}{2h} (e^{-i\phi} - e^{i\phi}) \right) \right] \\ & = 1 + k(1 - \theta) \left[-\frac{\sigma^2}{h^2} - r + \frac{\sigma^2}{2h^2} (e^{-i\phi} + e^{i\phi}) - \frac{r - \frac{\sigma^2}{2}}{2h} (e^{-i\phi} - e^{i\phi}) \right] \quad (52) \\ & + k \left[f_{i,j}^n \left(\frac{g^0}{g^n} - 1 \right) + \sum_{l \neq R} q_{R,l} \left(\frac{g_l^n}{g^n} - 1 \right) \right]. \end{aligned}$$

225 Let us denote

$$A_1 = 1 + k\theta \left(\frac{\sigma^2}{h^2} + r - \frac{\sigma^2}{2h^2} (e^{-i\phi} + e^{i\phi}) + \frac{r - \frac{\sigma^2}{2}}{2h} (e^{-i\phi} - e^{i\phi}) \right), \quad (53)$$

$$A_2 = 1 - k(1 - \theta) \left[\frac{\sigma^2}{h^2} + r - \frac{\sigma^2}{2h^2} (e^{-i\phi} + e^{i\phi}) + \frac{r - \frac{\sigma^2}{2}}{2h} (e^{-i\phi} - e^{i\phi}) \right] \quad (54)$$

Note that

$$\begin{aligned} |A_1|^2 &= 1 + \theta^2 \left(2\sigma^2 \frac{k}{h^2} \sin^2 \frac{\phi}{2} + rk \right)^2 + 2\theta \left(2\sigma^2 \frac{k}{h^2} \sin^2 \frac{\phi}{2} + rk \right) \\ &+ \left(\theta \frac{k}{h} \left(r - \frac{\sigma^2}{2} \right) \sin \phi \right)^2 > 1. \end{aligned} \quad (55)$$

Next, taking into account that the rationality term is bounded and (51), expression (52) is bounded as follows

$$|A_1||G| \leq |A_2| + C(n)k. \quad (56)$$

Since last term in (52) is order $O(k)$ and $|A_1| > 1$, due to the condition (44), for stability it is sufficient to guarantee that

$$\frac{|A_2|}{|A_1|} \leq 1. \quad (57)$$

Next, note that inequality (57) is equivalent to

$$(1 - 2\theta) \left[4\sigma^2 \frac{k^2}{h^2} \sin^2 \frac{\phi}{2} \left(\frac{\sigma^2}{h^2} \sin^2 \frac{\phi}{2} + r \right) + r^2 k^2 + \left(r - \frac{\sigma^2}{2} \right)^2 \frac{k^2}{h^2} \sin^2 \phi \right] - 4\sigma^2 \frac{k}{h^2} \sin^2 \frac{\phi}{2} - 2rk < 0, \quad (58)$$

that holds true for any $\theta \geq \frac{1}{2}$. Therefore, for $\theta \geq \frac{1}{2}$ the scheme is stable in accordance with the known property of the weighted scheme for linear equations (see [28], p. 29). \square

Remark. For $0 \leq \theta < \frac{1}{2}$ conditions (33) are sufficient for stability.

4.3. Consistency

In this subsection we study consistency of the finite difference scheme (21) with PDE (16). Following the definition given in [28], consistency of a numerical scheme with respect to a partial differential equation means that the exact solution of the PDE approximates well the exact theoretical solution of the finite difference scheme as the temporal and spatial discretization steps size tend to zero.

Let us rewrite the finite difference scheme (21) with parameter $\theta \in [0, 1]$, for every fixed regime $i = 1, \dots, I$, $1 \leq j \leq M - 1$, $0 \leq n \leq N - 1$, as follows

$$\begin{aligned} F(u_{i,j}^{n+\theta}) &= \frac{u_{i,j}^{n+1} - u_{i,j}^n}{k} - \theta \frac{\sigma_i^2}{2h^2} (u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j+1}^{n+1}) \\ &\quad - (1 - \theta) \frac{\sigma_i^2}{2h^2} (u_{i,j-1}^n - 2u_{i,j}^n + u_{i,j+1}^n) - \left(r_i - \frac{\sigma_i^2}{2} \right) \frac{\theta}{2h} (u_{i,j+1}^{n+1} - u_{i,j-1}^{n+1}) \\ &\quad - \left(r_i - \frac{\sigma_i^2}{2} \right) \frac{1 - \theta}{2h} (u_{i,j+1}^{n+1} - u_{i,j-1}^{n+1}) + r_i \theta u_{i,j}^{n+1} + r_i (1 - \theta) u_{i,j}^n \\ &\quad - f_{i,j}^n (u_{i,j}^0 - u_{i,j}^n) - \sum_{l \neq i} q_{i,l} (u_{l,j}^n - u_{l,j}^n) = 0. \end{aligned} \quad (59)$$

Furthermore, let us rewrite the PDE system (16) as follows

$$L(u_i) = \frac{\partial u_i}{\partial \tau} - \frac{\sigma_i^2}{2} \frac{\partial^2 u_i}{\partial x^2} - \left(r_i - \frac{\sigma_i^2}{2} \right) \frac{\partial u_i}{\partial x} + (r_i - q_{i,i}) u_i - f((1 - e^x)^+ - u_i) - \sum_{l \neq i} q_{i,l} u_l = 0, \quad i = 1, \dots, I. \quad (60)$$

Next, by denoting the value of the exact solution of the PDE at the mesh point (x_j, τ^n) by $\tilde{u}_{i,j}^n = u_i(x_j, \tau^n)$, the local truncation error $T_{i,j}^n$ is

$$T_{i,j}^n(\tilde{u}_i) = F(\tilde{u}_{i,j}^n) - L(\tilde{u}_{i,j}^n). \quad (61)$$

Note that if $T_{i,j}^n$ tends to zero as $h \rightarrow 0$ and $k \rightarrow 0$, then the consistency of the scheme is guaranteed, Assuming that $u_i(x, \tau)$, $i = 1, \dots, I$, is continuously differentiable four times with respect to x and three times with respect to τ [28] and using the Taylor's expansion around the point $(x_j, \tau^{n+\theta})$, one gets

$$u_i(x_j, \tau^n) = u_i(x_j, \tau^{n+\theta}) - k\theta \frac{\partial u_i}{\partial \tau}(x_j, \tau^{n+\theta}) + \frac{k^2\theta^2}{2} \frac{\partial^2 u_i}{\partial \tau^2}(x_j, \tau^{n+\theta}) + O(k^3), \quad (62)$$

$$u_i(x_j, \tau^{n+1}) = u_i(x_j, \tau^{n+\theta}) + k(1-\theta) \frac{\partial u_i}{\partial \tau}(x_j, \tau^{n+\theta}) + \frac{k^2(1-\theta)^2}{2} \frac{\partial^2 u_i}{\partial \tau^2}(x_j, \tau^{n+\theta}) + O(k^3). \quad (63)$$

Thus, we have

$$\frac{\partial u_i}{\partial \tau}(x_j, \tau^{n+\theta}) = \frac{\tilde{u}_{i,j}^{n+1} - \tilde{u}_{i,j}^n}{k} + (1-2\theta)k \frac{\partial^2 u_i}{\partial \tau^2}(x_j, \tau^{n+\theta}) + O(k^2). \quad (64)$$

Note that the choice $\theta = \frac{1}{2}$ removes the term $O(k)$, so that only in this case we get a second order in time approximation.

Analogously, one obtains that

$$\frac{\partial u_i}{\partial x}(x_j, \tau^{n+\theta}) = \theta \frac{\tilde{u}_{j+1}^{n+1} - \tilde{u}_{j-1}^{n+1}}{2h} + (1-\theta) \frac{\tilde{u}_{j+1}^n - \tilde{u}_{j-1}^n}{2h} + O(h^2), \quad (65)$$

$$\begin{aligned} \frac{\partial^2 u_i}{\partial x^2}(x_j, \tau^{n+\theta}) &= \theta \frac{\tilde{u}_{j+1}^{n+1} - 2\tilde{u}_j^{n+1} + \tilde{u}_{j-1}^{n+1}}{h^2} \\ &+ (1-\theta) \frac{\tilde{u}_{j+1}^n - 2\tilde{u}_j^{n+1} + \tilde{u}_{j-1}^n}{h^2} + O(h^2). \end{aligned} \quad (66)$$

250 Next, by replacing (64), (65) and (66) into (60), the consistency of the scheme (59) for the PDE system (60) is established and the truncation error (61) takes the following form

$$T_j^n(\tilde{u}_i) = (1 - 2\theta)k \frac{\partial^2 u_i}{\partial \tau^2}(x_j, \tau^{n+\theta}) + O(k^2) + O(h^2) \quad \forall i = 1, \dots, I. \quad (67)$$

From the previous equation it follows the order of convergence in k and h of the methods for $\theta = 0, 1/2, 1$. These orders of convergence will be illustrated in 255 the forthcoming section of numerical results.

5. Numerical examples

In this section the numerical solution of the more classical American option pricing problem under regime switching model and rational exercise is found by using a large enough value of the rationality parameter. Also, the dependence 260 on value of rationality parameter is shown, as well as the convergence rates of the proposed θ -method for various θ and the stability conditions. In Example 1 we show that for sufficiently large values of rationality parameter the solution of the problem (8) tends to American option under regime switching and rational exercise, as it was shown in [30] for American put option without regime 265 switching.

5.1. Example 1.

Let us consider a two regimes model with the following parameters (see Example 1 in [22]):

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.05 \end{pmatrix}, \quad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.3 \end{pmatrix}, \quad Q = \begin{pmatrix} -6 & 6 \\ 9 & -9 \end{pmatrix}, \quad T = 1, \quad E = 9. \quad (68)$$

In Table 5.1 the results of numerical solution by the proposed explicit scheme at the point $S = E$ with various intensity functions are collected for rationality parameter λ from 1 to 1000. The results show that the solution tends to

λ	f_1		f_2		f_3	
	Regime 1	Regime 2	Regime 1	Regime 2	Regime 1	Regime 2
1	1.9060	1.8229	1.6472	1.5407	1.6592	1.5532
10	1.9545	1.8656	1.9596	1.8705	1.9150	1.8240
100	1.9700	1.8805	1.9710	1.8815	1.9661	1.8765
1000	1.9718	1.8819	1.9719	1.8819	1.9714	1.8819
10000	1.9720	1.8820	1.9721	1.8820	1.9720	1.8820
FF	1.9713	1.8817	1.9713	1.8817	1.9713	1.8817
Tree	1.9722	1.8819	1.9722	1.8819	1.9722	1.8819

Table 1: Convergence of the solution for various intensity functions f_1, f_2, f_3 to American option price and comparison with front-fixing (FF) and Tree methods. The tests are done with explicit scheme ($\theta = 0$), $h = 10^{-2}$ and time step $k = 10^{-4}$.

270 American option price as $\lambda \rightarrow \infty$. We compare the results with other known techniques, such as front-fixing method, proposed for regime switching model in [13] and the tree method proposed in [25]. The obtained results for the fully implicit scheme ($\theta = 1$) and Crank-Nicolson method ($\theta = \frac{1}{2}$) have not been shown in Table 5.1, because they are very close to the those obtained with the

275 explicit scheme. As one can see in this Table, the difference between the results of applying various intensity functions vanishes for the large enough rationality parameter. The intensity function family f_1 that corresponds to the penalty method, as well as its smooth analogue f_3 are convenient for the American option pricing problem due to their stability properties shown in [10].

280 In Figure 1 the option price at $\tau = T$ is presented for the a two regimes model when using the proposed explicit ($\theta = 0$) method. The intensity function is taken in the form (13) with rationality parameter $\lambda = 10^3$. In this example, the step sizes $h = 10^{-2}$, $k = 10^{-4}$ have been chosen. In Figure 2 we present the

solution of the problem with the set of parameters (68) and the matrix Q given

285 by

$$Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (69)$$

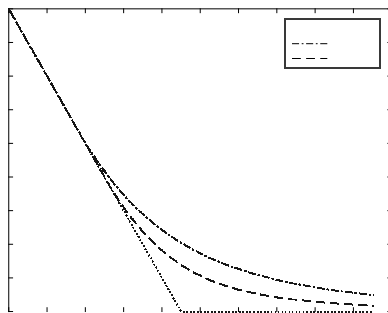
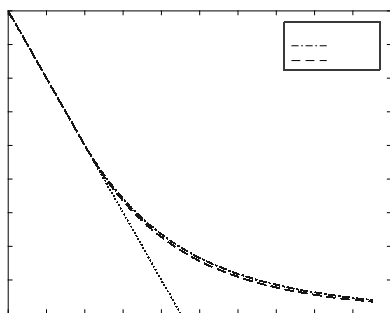


Figure 1: Numerical solution of the problem with parameters (68) by the proposed explicit FDM.

Figure 2: Numerical solution of the problem with parameters (68) with matrix (69) by the proposed explicit FDM.

In Figures 3 and 4 the numerical solutions with two different values of rationality parameter $\lambda = 1$ and $\lambda = 10^3$ are presented.

In Example 2 the numerical convergence rates of the proposed method for various families of intensity functions and rationality parameters are presented. Thus, numerical results for the fully implicit ($\theta = 1$), Crank-Nicolson ($\theta = 1/2$) and explicit ($\theta = 0$) schemes in the differential part of the PDE are shown. 290

5.2. Example 2: Convergence rate

Let us consider the problem (16) with parameters (68). Since the exact solution of the problem is not analytically available, the following formula can be used for the estimation of the convergence rate in space, γ_h :

$$\gamma_h = \log_2 \frac{\|U_{h/2} - U_h\|}{\|U_{h/4} - U_{h/2}\|}. \quad (70)$$

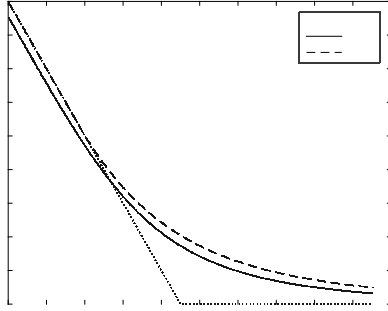


Figure 3: Numerical solution of the problem with parameters (68) by the proposed explicit FDM with various rationality parameters λ (Regime 1).

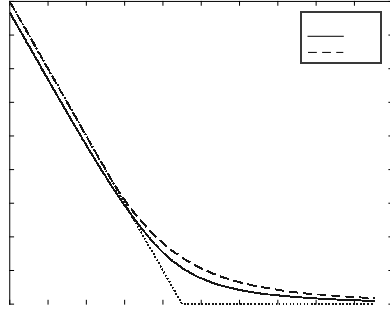


Figure 4: Numerical solution of the problem with parameters (68) by the proposed explicit FDM with various rationality parameters λ (Regime 2).

	Regime 1			Regime 2		
θ	0	0.5	1	0	0.5	1
f_1	2.0084	2.0003	2.0007	2.0143	2.0004	2.0015
f_2	2.0083	2.0003	2.0005	2.0142	2.0007	2.0013
f_3	2.0079	2.0002	2.0001	2.0156	2.0005	2.0004

Table 2: Convergence rate in space of the proposed θ -scheme for $\lambda = 10^3$.

For this purpose, we have obtained a series of numerical results with fixed time step $k = 2 \cdot 10^{-5}$ and the spatial steps $h = 2 \cdot 10^{-2}$, $h/2 = 10^{-2}$ and $h/4 = 5 \cdot 10^{-3}$. The convergence rate γ_h has been calculated by formula (70) for the proposed scheme with $\theta = 0, 0.5, 1$. The results are collected in Table 2 showing the expected orders for the approximation with $\lambda = 10^3$ and various intensity function families.

Analogous formula can be used in order to estimate the convergence rate in time, γ_k , for a fixed space step h :

$$\gamma_k = \log_2 \frac{\|U_{k/2} - U_k\|}{\|U_{k/4} - U_{k/2}\|}. \quad (71)$$

	Regime 1			Regime 2		
θ	0	0.5	1	0	0.5	1
f_1	1.0013	1.7795	1.0007	1.0013	1.8889	1.0010
f_2	1.0009	1.7802	1.0007	1.0009	1.9017	1.0007
f_3	1.0010	1.8543	1.0001	1.0010	1.8943	1.0000

Table 3: Convergence rate in time of the proposed θ -scheme for $\lambda = 10^3$.

In this case, the spatial step has been fixed to $h = 5 \cdot 10^{-3}$, while the chosen
300 time steps are $k = 2 \cdot 10^{-5}$, $k/2 = 10^{-5}$ and $k/4 = 5 \cdot 10^{-6}$. The convergence
rates γ_k of the proposed method for various intensity function families (12)-(13)
are presented in Table 3. The numerical convergence rates are in agreement
with the theoretical study of consistency developed in Section 3.3.

In the previous section the stability conditions for the proposed weighted
305 scheme have been found. The forthcoming Example 3 shows that the stability
condition is crucial.

5.3. Example 3: Stability

Let us consider the explicit finite difference scheme (21) for the problem with
parameters (68). Figures 5 - 6 show the numerical solution for regime 1 and
310 2, respectively. More precisely, by taking a fixed value $h = 10^{-2}$, the dashed
lines show that the numerical solution is stable for $k = 10^{-4}$, when the step size
conditions (33) are satisfied. However, when $k = 1.56 \cdot 10^{-4}$, conditions (33)
are not fulfilled and stability is not guaranteed, so that spurious oscillations can
occur, thus leading to inaccurate numerical approximations that are shown in
315 the solid line.

6. Conclusion

The main result of this paper is to combine simultaneously two recent mod-
els of American option pricing. On one hand, the regime switching approach