Structure preserving integrators for solving linear quadratic optimal control problems with applications to describe the flight of a quadrotor

P. Bader, S. Blanes, E. Ponsoda

Instituto de Matemática Multidisciplinar, Building 8G, second floor, Universitat Politècnica de València. 46022 Valencia, Spain.

Abstract

In this paper we present linear quadratic methods for optimal nonlinear control problems. These techniques lead to a matrix Riccati differential equation that should be solved numerically. The solution is a symmetric positive definite time-dependent matrix which controls the stability of the equation for the state. This property is not preserved, in general, by the numerical integrators and we propose second order exponential integrators methods which unconditionally preserve this property and analyse higher order exponential methods. This method can be applied to the integration of nonlinear problems if they are previously appropriately linearized. The algorithm obtained is applied for the control of a quadrotor which is an unmanned flying vehicle. Both the trajectory following as the correction of the angles have been achieved.

Keywords: Optimal control, linear quadratic methods, matrix Riccati differential equation, second order exponential integrators
2000 MSC: 49J15, 49N10, 34A26

1. Introduction

Nonlinear control problems have attracted during the last years the interest of researchers in different fields like control of airplanes, helicopters, rockets, satellites, etc. Linear quadratic optimal control problems have been extensively studied, but most realistic problems are inherently nonlinear. In addition, nonlinear control theory can improve the performance of the controller and enable tracking of aggressive trajectories [13].

To solve nonlinear optimal control problems requires the numerical integration of systems of coupled non-autonomous and nonlinear equations with

Email addresses: `phiba@imm.upv.es` (P. Bader), `serblaza@imm.upv.es` (S. Blanes), `eponsoda@imm.upv.es` (E. Ponsoda)

boundary conditions, where the equations for the state vector and for the control vector are coupled. Simple, fast, accurate and reliable numerical algorithms are of great interest for real time integrations. It is usual to solve the nonlinear problems by linearization, which then have to be solved using linear quadratic (LQ) methods. In general, they usually require the integration of matrix Riccati differential equations (RDEs) iteratively many times. These RDEs usually present an algebraic structure which make the solutions to be symmetric positive definite matrices which play an important role for the qualitative and quantitative solutions of both the control as well as the state vector (e.g. to stabilize the equation for the state vector).

Geometric numerical integrators are numerical algorithms which preserve most of the qualitative properties of the exact solution. However, some qualitative properties like preservation of positivity (which is a relevant property in this problem) is not unconditionally preserved by most methods, included geometric integrators. We show that some low order exponential integrators unconditionally preserve this property, and higher order methods preserve it under mild constraints on the time step. We refer to these methods as structure preserving integrators, and they allow to use relatively large time steps while showing a high performance for stiff problems or problems which strongly vary along the evolution.

It is usual to solve the nonlinear problems by linearization, and this can be done in different ways. We consider three techniques to linearize the equations and the linear equations are then numerically solved using some exponential integrators which preserve the relevant properties of the solution. Since the nonlinear problems are solved by linearization, we first consider in detail the linear problem.

The paper is organized as follows. In section 2 we consider in detail the linear case. We study the algebraic structure of the equations and the qualitative properties of the solutions. We next consider some exponential integrators and we analyze the preservation of the qualitative properties of the solution by the proposed methods. In section 3 we consider the nonlinear case which, after linearization, can be treated as a particular case of the non-autonomous linear one. Finally, in section 4, the numerical algorithm is applied to a particular example (the control of the flight of a quadrotor corresponding to an unmanned microhelicopter) in order to test the accuracy of the exponential methods. Numerical results and conclusions are included.

2. Linear quadratic (LQ) methods in optimal control problems

Let us consider the general LQ optimal control problem

$$\min_{u\in L^2}\int_0^{t_f} \left(X^T(t)Q(t)X(t)+u^T(t)R(t)u(t)\right)\,dt \tag{1a}$$

$$\text{subject to}\quad \dot{X}(t)=A(t)X(t)+B(t)u(t),\quad X(0)=X_0, \tag{1b}$$

where $\dot{X}(t)$ is the time-derivative of the state vector $X(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ is the control, $R(t) \in \mathbb{R}^{m \times m}$ is symmetric and positive definite, $Q(t) \in \mathbb{R}^{n \times n}$ is symmetric positive semi-definite, $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$, and M^T denotes the transpose of a matrix M.

Problems of the type (1) are frequent in many areas like game theory, quantum mechanics, economy, environment problems, etc., see [6, 18], or in engineering models [2, ch. 5].

The optimal control problem (1) is solved, assuming some controllability conditions, by the linear feedback controller [21]

$$u(t) = -K(t)X(t), \tag{2}$$

with the gain matrix

$$K(t) \,=\, R^{-1}(t)B^T(t)P(t)\,,$$

and $P(t)$ verifying the matrix RDE

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) + P(t)B(t)R^{-1}(t)B^T(t)P(t) - Q(t), \tag{3}$$

with the final condition $P(t_f) = 0$. The backward time integration of this equation with $Q(t)$ and $R(t)$ symmetric positive definite matrices has as solution $P(t)$ a symmetric and positive definite matrix [1] (if $Q(t)$ is positive semidefinite it requires the problem to be stabilizable and detectable). To compute the optimal control, $u(t)$, we solve for $P(t)$ and plugging the control law into (1b) yields a linear equation for the state vector

$$\dot{X}(t) = \big(A(t) - B(t)R^{-1}(t)B^T(t)P(t)\big)\, X(t), \quad X(0) = X_0$$

to be integrated forward in time with which the control is readily computed. Notice that $S(t) = B(t)R^{-1}(t)B^T(t)$ is a semi-definite positive symmetric matrix (positive definite if $\ rank\, B = n$) and $P(t)$ is a positive definite matrix. Thus, its product is also a semi-positive matrix, and this is very important for the stability of the solution for the state vector and ultimately for the control. A numerical integrator which do not preserve the positivity of $P(t)$ can lead into unstable methods when solving the state vector.

In this paper, exponential integrators, which belong to the class of Lie group methods (see [7, 19] and references therein), are proposed in order to solve the RDE (3). They are geometric integrators because they preserve some of the qualitative properties of the exact solution and also frequently provide accurate results.

2.1. Structure preserving integrators

We are interested in the search of numerical integrators which preserve both the symmetry as well as the positivity of $P(t)$. While symmetry is a property preserved by most of the methods, the preservation of positivity is a more challenging task.

For our analysis, we consider convenient to review some results from the numerical integration of differential equations. Given the ordinary differential equation (ODE)

$$\dot{x} = f(x,t), \qquad x(t_0) = x_0 \in \mathbb{R}^n \tag{4}$$

the exact solution at time $t = t_0 + h$ can formally be written as a map from the initial conditions to the final solutions, $x(t_0 + h) = \Phi_h(x_0)$ and, for sufficiently small h, one can also consider it as the exact solution at time $t = t_0 + h$ of an autonomous ODE

$$\dot{x} = f_h(x), \qquad x(t_0) = x_0$$

where f_h is the vector field associated to the Lie operator $\dfrac{1}{h}\log(\Phi_h)$.

In a similar way, a numerical integrator for solving the equation (4) which is used with a time step, h, can be seen as the exact solution at time $t = t_0 + h$ of a perturbed problem (backward error analysis)

$$\dot{x} = \tilde{f}_h(x), \qquad x(t_0) = x_0.$$

If $\tilde{f}_h - f_h = \mathcal{O}(h^{p+1})$ we say that the method is of order p. The qualitative properties of the exact solution, Φ_h, is related to the algebraic structure of the vector field f_h. If the the vector field, $\tilde{f}_h$, associated numerical integrator, shares the same algebraic structure, the numerical integrator will preserve these qualitative properties.

Given the RDE

$$\dot{P} = PA(t) + A^T(t)P - PS(t)P + Q(t), \qquad P(t_0) = 0$$

with $Q(t), S(t)$ symmetric and positive definite matrices then $P(t)$, for $t > t_0$, is also a symmetric and positive definite matrix. This equation is equivalent to (3) since in that case the equation is integrated backward in time.

A numerical integrator which can be considered as the exact solution of a perturbed matrix RDE

$$\dot{P} = P\tilde{A}_h + \tilde{A}_h^T P - P\tilde{S}_h P + \tilde{Q}_h, \qquad P(t_0) = 0$$

with $\tilde{Q}_h, \tilde{S}_h$ symmetric and positive definite matrices preserves the symmetry and positivity of the exact solution. The same result applies if the numerical integrator is given by a composition of maps such that each one, separately, can be seen as the exact solution of a matrix RDE with the same structure.

We will refer to these methods as positive preserving integrators. If this property is preserved for all $h > 0$, we say it is unconditionally positive preserving and, if it exists $h^* > 0$ such that this property is preserved for $0 < h < h^*$ we will refer to it as conditionally positive preserving.

In general, standard methods do not preserve positivity. We show, however, that some second order exponential integrators preserve this positivity unconditionally and higher order ones are conditionally preserving the property for a relatively large range of values of h^* which depends on the smoothness in the time dependency of the matrices $A(t), S(t), Q(t)$.

At this stage, it is convenient to write the RDE (3) as a linear differential equation

$$\frac{d}{dt}\begin{bmatrix} V(t) \\ W(t) \end{bmatrix} = \begin{bmatrix} -A(t)^T & -Q(t) \\ -S(t) & A(t) \end{bmatrix} \begin{bmatrix} V(t) \\ W(t) \end{bmatrix}, \quad \begin{bmatrix} V(t_f) \\ W(t_f) \end{bmatrix} = \begin{bmatrix} P_f \\ I \end{bmatrix}, \tag{5}$$

where $P_f = 0$, $S(t) = B(t)R^{-1}(t)B^T(t)$, and the solution $P(t)$ of problem (3) to be integrated backward in time is given by

$$P(t) = V(t)W(t)^{-1}, \quad P(t), V(t), W(t) \in \mathbb{R}^{n\times n},$$

in the region where $W(t)$ is invertible (see, for instance, [7] or [20], and references therein). If R and Q are positive definite matrices, this problem has always solution.

It is then clear that if a numerical integrator for the equation (5) can be seen as the exact solution of an autonomous perturbed linear equation

$$\frac{d}{dt}\begin{bmatrix} V(t) \\ W(t) \end{bmatrix} = \begin{bmatrix} -\tilde{A}_h^T & -\tilde{Q}_h \\ -\tilde{S}_h & \tilde{A}_h \end{bmatrix} \begin{bmatrix} V(t) \\ W(t) \end{bmatrix}, \quad \begin{bmatrix} V(t_f) \\ W(t_f) \end{bmatrix} = \begin{bmatrix} P_f \\ I \end{bmatrix},$$

where $\tilde{Q}_h$ and $\tilde{S}_h$ are symmetric and positive definite matrices, then the numerical solution is symmetric and positive definite.

In general, high order standard methods like Runge-Kutta methods do not preserve positivity. Explicit methods applied to the linear problem do not preserve positivity unconditionally, but to show this result for implicit methods requires a more detailed analysis, and it is stated in the following theorem.

Theorem 2.1. *The second order implicit midpoint and trapezoidal Runge-Kutta methods do not preserve the positivity unconditionally for the solution of the RDE (5).*

Proof 2.2. *It suffices to prove it for the scalar non-autonomous problem*

$$\dot{p} = -q - 2a(t)\,p + s\,p^2, \qquad p(t_f) = 0$$

with $q, s > 0$ and $a : [0, t_f] \to \mathbb{R}$.

Firstly, we study the implicit midpoint method for which one iteration backwards in time is given by

$$p_{n+1} = -p_n + \frac{-2 + 2ah + 2\sqrt{(-1 + a(t_f - \frac{hn}{2})h)^2 + h(2p_n + hq)s}}{hs}.$$

The first iteration starting from the initial value $p_0 = 0$ yields $p_1 > 0$. A simple way to produce a negative value $p_2 < 0$ is given by letting $a(t) = 0$ for $t \in [t_f - h, t_f]$. Then, $p_2 < 0$ is equivalent to $a(t_f - 3h/2) =: a < 0$ and $h > -2/a$. Given a time-step h, it is thus easy to construct a continuous

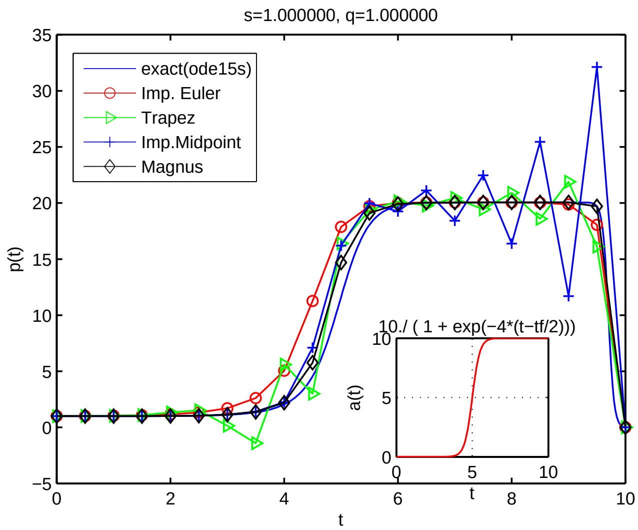


Figure 1: $h = 1/2$

(smooth) function $a(t)$ such that positivity is not preserved. For the trapezoidal rule, letting $a(t_f - 2h) = a(t_f - 3h) = a < 0$, the condition for $p_3 < 0$ reads

$$a < \frac{(4 + h^2 q)\left(-h^2 q + 2\left(5 + \sqrt{1 + 2h^2 q}\right)\right)}{4h\,(-4 + h^2 q)} \quad \wedge \quad h < \frac{2}{\sqrt{q}}.$$

a stronger (sufficient) criterion is ***Comment:****(which we can even make stronger to simplify more (and always remove the more accurate previous restriction))*

$$a < -\frac{3}{h} + hq\left(-\frac{1}{4} + \frac{6}{-4 + h^2 q}\right) \quad \wedge \quad h < \frac{2}{\sqrt{q}}.$$

We remark, that, given $a < 0$, the method produces negative values p_3 for a range of time-steps h, i.e., for larger time-steps h, it is less prone to negativity.

Comment:*Write about results for trapezoidal rule, and why we dont use it later on, also write about possibly complex results in intermediate steps*

If we are interested on high order numerical integrators, a different class of methods has to be explored. We consider a particular class of exponential integrators referred as Magnus integrators (see [4] and references therein).

2.2. Magnus integrators

Magnus integrators can be considered as a special class of exponential integrators as well as Lie group integrators. When they are used to numerically solve, e.g. the eq. (5), one can interpret the numerical solution as the exact solution of a slightly perturbed linear system with a similar structure (i.e. replacing in each time step the time-dependent matrices $Q(t), R(t), \ldots$ by perturbed constant ones $\tilde{Q}_h, \tilde{R}_h, \ldots$, obtained from appropriate time averaging) and then in some cases it is possible to guarantee that the numerical solution for $P(t)$ is, for example, a symmetric and positive definite matrix.

Given the general linear equation

$$y' = M(t)\, y\,, \qquad\qquad y(t_0) = y_0\,; \tag{6}$$

with $y \in \mathbb{R}^p$, and if we denote by $\Phi(t,t_0) \in \mathbb{R}^{p\times p}$ the fundamental solution, such that $y(t) = \Phi(t,t_0)y(t_0)$, the Magnus expansion gives us the formal solution (under certain convergence conditions, see [4] and references therein) as

$$\Phi(t,t_0) = \exp\left(\Omega(t,t_0)\right)$$

where $\Omega(t,t_0) = \sum_{n=1}^{\infty}\Omega_n(t,t_0)$ and each $\Omega_n(t,t_0)$ is an element of the Lie algebra generated by $M(t)$ given by n-dimensional integrals involving $n-1$ nested commutators of $M(t)$ at different instants. The first two terms are given by

$$\Omega_1(t,t_0) = \int_{t_0}^{t} M(s)\, ds, \qquad \Omega_2(t,t_0) = \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} [M(t_1), M(t_2)]\, dt_2$$

where $[A,B] = AB - BA$.

In the region of convergence of the Magnus expansion, the exact solution at time $t = t_0 + h$ is equivalent to the exact solution of the autonomous linear equation

$$y' = \frac{1}{h}\Omega(t_0+h,t_0)\, y\,, \qquad\qquad y(t_0) = y_0\,.$$

It is well known that the set of matrices

$$\begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \tag{7}$$

with $A,B,C \in \mathbb{R}^{n\times n}$ and $B = B^T,\ C = C^T$ form the algebra of symplectic matrices. This algebraic property is preserved by the commutators and then any truncated Magnus expansion preserves symplecticity for this problem. However, the additional property about the positivity (or negativity) on the skewdiagonal matrices B, C is not always guaranteed when the series is truncated. We analyse the low order methods and show that it is possible to build second order Magnus integrators which unconditionally preserve this positivity.

The first term in the expansion, applied to (5), which do not contain commutators is given by

$$\Omega_1(t,t_0) = \begin{bmatrix} -\displaystyle\int_{t_0}^{t} A(s)^T\, ds & -\displaystyle\int_{t_0}^{t} Q(s)\, ds \\ -\displaystyle\int_{t_0}^{t} S(s)\, ds & \displaystyle\int_{t_0}^{t} A(s)\, ds \end{bmatrix}. \tag{8}$$

Then, if we truncate the series to the first term, and approximate the integrals for a time interval $t \in [t_0, t_0 + h]$ using a quadrature rule of second or higher order, we obtain a second order method.

It is well known that, if $Q(t)$ is a symmetric positive definite matrix for $t \in [t_0, t_0 + h]$ then $\hat{Q}_h = \displaystyle\int_{t_0}^{t_0+h} Q(s)\, ds$ is also a symmetric positive definite matrix. Suppose now that the integral is approximated using a quadrature rule

$$\tilde{Q}_h \equiv h \sum_{i=1}^{k} b_i Q(t_0 + c_i h) \simeq \int_{t_0}^{t_0+h} Q(s)\, ds$$

with $c_i \in [0,1]$, $i = 1, \ldots, k$. If $\sum_i b_i > 0$, we have:

a) If $b_i > 0$, $i = 1, \ldots, k$, then $\tilde{Q}_h$ is a symmetric positive definite matrix.

b) If $\exists\, b_j < 0$, for some value of j and $\|Q(t_m) - Q(t_n)\| < C|t_m - t_n|$, $\forall t_m, t_n \in [t_0, t_0 + h]$, then $\exists\ h^* > 0$ such that $\tilde{Q}_h$ is a symmetric positive definite matrix for $0 < h < h^*$, and h^* depends on C.

The same results also apply to $\tilde{S}_h$.

To have a second order method which preserves positivity, it suffices to consider the first term in the Magnus expansion (8) and to approximate the integrals by a second or higher order rule with the constraint that all $b_i > 0$. The most natural choices are the midpoint rule

$$\Psi_h^{[2]} = \exp\left(hM(t + h/2)\right) = \Phi(t + h, t) + \mathcal{O}(h^3),$$

or the trapezoidal rule

$$\Psi_h^{[2]} = \exp\left(\frac{h}{2}\left[M(t + h) + M(t)\right]\right) = \Phi(t + h, t) + \mathcal{O}(h^3). \qquad (9)$$

However, from the computational point of view, in order to save evaluations on the numerical algorithm, we found more efficient the trapezoidal rule. If we consider the RDE (3) that corresponds to (6) with the data (5) and consider an equidistant time grid $t_n = t_0 + nh$, $0 \le n \le N$, with constant time step $h = (t_f - t_0)/N$ and taking into account that this equation has to be solved backward in time, we obtain

$$\begin{bmatrix} V_n \\ W_n \end{bmatrix} = \exp\left(-\frac{h}{2}\left[M(t_n) + M(t_{n+1})\right]\right) \begin{bmatrix} V_{n+1} \\ W_{n+1} \end{bmatrix} \;\Rightarrow\; \tilde{P}_n = V_n\, W_n^{-1},$$

By construction, $\tilde{P}_n$ is a symmetric positive definite matrix. In addition, it is also a time symmetric second order approximation to $P(t_n)$. In this way, the matrix functions $A(t_n)$, $B(t_n)$, $Q(t_n)$, $R(t_n)$ are computed at the same mesh points as the approximations $\tilde{P}_h$ of $P(t)$ are computed and, as we will see, they can be reused for the forward time integration of the state vector.

Let us consider the state vector to be integrated forward in time which takes the form

$$\dot{X} = (A(t) - S(t)\tilde{P}_h)X$$

where we denote by $\tilde{P}_h$ the numerical approximations to $P(t)$ computed on the mesh and $\tilde{P}_{h,n} \simeq P(t_n)$. Notice that at the instant $t = t_f - h$ we have that $\tilde{P}_{h,N-1} = P(t_f - h) + \mathcal{O}(h^3)$ (local error) but at $t = t_0$, after N steps, we have $\tilde{P}_{h,0} = P(t_0) + \mathcal{O}(h^2)$ (global error). This accuracy suffices to get a second order approximation for the numerical approximation to the state vector.

If we use the same Magnus expansion for the numerical integration of the state vector with the trapezoidal rule we have the algorithm

$$X_{n+1} = \exp\left(\frac{h}{2}\left[D_{n+1} + D_n\right]\right) X_n\,, \qquad D_m = A_m - S_m\tilde{P}_{h,m}, \quad m = n, n+1$$

where $A_m = A(t_m)$, $S_m = S(t_m)$.

Finally, the controls which allow us to reach the final state in a nearly optimal way are

$$u_n \;=\; -R^{-1}(t_n)B^T(t_n)P_n\, X_n\,.$$

Higher order Magnus integrators. Let us now consider high order Magnus integrators. They usually require to compute matrix commutators. If we truncate up to the second term in the Magnus expansion, $\Psi_h \equiv \exp(\Omega_1 + \Omega_2)$, it agrees with exact solution up to order four, i.e. $\Psi_h = \Phi(t+h,t) + \mathcal{O}(h^5)$. We have that $\Omega_1 + \Omega_2$ belong to the algebra of symplectic matrices, as given in (7), where the skewdiagonal matrices take an involved form. It is possible to show that it conditionally preserves positivity, but it is not unconditionally preserved as it happens with $\exp(\Omega_1)$.

For simplicity in the analysis, we consider commutator-free Magnus integrators (see [4, 5] and references therein). If we denote

$$M^{(0)} = \int_{t_n}^{t_n+h} M(s)\, ds, \qquad M^{(1)} = \frac{1}{h}\int_{t_n}^{t_n+h} \big(s - (t_n + \frac{h}{2})\big) M(s)\, ds$$

we have that the following commutator-free composition gives an approximation to fourth-order

$$\Psi_{CF}^{[4]} = \exp\left(\frac{1}{2}M^{(0)} + 2M^{(1)}\right)\, \exp\left(\frac{1}{2}M^{(0)} - 2M^{(1)}\right) = \Phi(t_0+h, t_0) + \mathcal{O}(h^5).$$

If we approximate the integrals using the fourth-order Gaussian quadrature rule we have

$$\Psi_G^{[4]} = \exp\left(h(\beta M_1 + \alpha M_2)\right) \exp\left(h(\alpha M_1 + \beta M_2)\right),$$

where $M_i \equiv M(t_n + c_i h)$, $i = 1, 2$, $c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}$, $c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}$, $\alpha = \frac{1}{4} - \frac{\sqrt{3}}{6} = -0.038\ldots < 0$, $\beta = \frac{1}{4} + \frac{\sqrt{3}}{6}$. This composition will not preserve positivity unconditionally when applied to solve the RDE because $\alpha < 0$. However, since

$\alpha + \beta = \frac{1}{2}$ the positivity will be preserved for relatively large values of the time step.

If we approximate the integral using the Simpson rule we have

$$\Psi_G^{[4]} = \exp\left(\frac{h}{12}(-M_1 + 4M_2 + 3M_3)\right) \exp\left(\frac{h}{12}(3M_1 + 4M_2 - M_3)\right),$$

where $M_1 \equiv M(t_n), M_2 \equiv M(t_n + h/2), M_3 \equiv M(t_n + h)$. As previously, one of the coefficients is again negative and it does not preserve positivity unconditionally if we apply the method to solve the RDE.

If we first integrate backward in time the matrix RDE with one of the fourth-order commutator-free methods and next we want to use the same method for the state vector, we need to use a time step twice larger for the forward integration (prefearably with the Simpson rule). The main goal of this paper is to present a simple, fast, accurate and reliable numerical scheme for nonlinear problems. As we will see, nonlinear problems are solved after linearization, and the linear equations are solved iteratively. The solution at each iteration is plugged into the following iteration, and this requires to use a fixed mesh for all methods. For this reason, we consider that the most convenient algorithm is the second order Magnus integrators with the trapezoidal rule, being this the methods used in the numerical examples.

3. The nonlinear control problem

Many problems in engineering are described by optimal control problem of the form

$$\min_{u \in L^2} \int_0^{t_f} \left(X^T(t)Q(t, X(t))X(t) + u^T(t)R(t, X(t))u(t)\right) dt \qquad (10a)$$

$$\text{subject to} \quad \dot{X}(t) = f_A(t, X(t)) + f_B(t, X(t), u(t)), \quad X(0) = X_0. \qquad (10b)$$

This nonlinear optimal control problem is considerably more involved than the linear case. It is then usual to solve the nonlinear problem by linearization, and this can be done in different ways. In the following we present three of them and compare their performance when the linear equations are solved using exponential integrators.

Quasilinearization. For $f_A(t, 0) = 0$ and $f_B(t, X, u) \neq 0$ for all t, X in the appropriate domains, the state equation (10b) can be written in a non-unique way as

$$\dot{X}(t) = A(t, X)X(t) + B(t, X, u)u(t), \quad X(0) = X_0. \qquad (11)$$

The formulation (11) is the basic ingredient for the State Dependent Riccati Equation (SDRE) control technique [15, 16]. Then, formal similarity to the linear problem (1) motivates the imitation of the optimal LQ controller by defining

$$u(t) = -R^{-1}(t)B^T(t, X(t))P(t, X)X(t) \qquad (12a)$$

where $P(t, X)$ solves the now state-dependent algebraic Riccati equation

$$0 = -PA(t, X) - A(t, X)^T P + PB(t, X)R(t, X)^{-1}B(t, X)^T P - Q(t, X). \quad (12b)$$

One has to choose the unique positive definite solution of the algebraic Riccati equation and, combining (12a) with (10b), the closed-loop nonlinear dynamics are given by

$$\dot{X} = \left(A(t, X) - B(t, X)R(t, X)^{-1}B(t, X)^T P(t, X)\right) X, \quad X(0) = X_0. \quad (12c)$$

The usual approach is to start from $X(0) = X_0$, and then to advance step by step in time by first computing P from (12b) at each step and then applying the Forward Euler method on (12c). The application of higher order methods, such as Runge-Kutta schemes, requires to solve implicit systems with (12b) and can thus be costly. In addition, if one is interested in aggressive trajectories, the algebraic equation (12b) can considerably differ from the solution of the corresponding Riccati differential equation, which affects to the solution of the state vector, X, and ultimately the choice of the control in (12a).

Waveform relaxation. Alternatively, we can linearize (12c), by iterating

$$\frac{d}{dt}X^{n+1} = \left(A(t, X^n) - B(t, X^n)R(t, X^n)^{-1}B(t, X^n)^T P(t, X^n)\right) X^{n+1}, \quad (13)$$

We start with a guess solution $X^0(t)$, and iteratively obtain a sequence of solutions, $X^1(t)$, $X^2(t), \ldots, X^n(t)$. The iteration is stops once consecutive solutions differ by less than a given tolerance. Here, $P(t, X^n(t))$ at each iteration is obtained from

$$\dot{P} = -PA^n(t) - A^n(t)^T P + PB^n(t)R^n(t)^{-1}B^n(t)^T P - Q^n(t), \quad P(t_f) = 0, \quad (14)$$

with $A^n(t) \equiv A(t, X^n(t))$, $B^n(t) \equiv B(t, X^n(t))$, etc.

This procedure is similar to what is known as waveform relaxation [24], however, the backward integration for P limits the parallelizability in this application. This approach corresponds to freezing the nonlinear parts in (11) at the previous state and then applying the optimal control law (2). It is worth noting that this technique can handle inhomogeneities by slightly adapting the control law, at the cost of solving an inhomogeneous linear system, see below. The algorithm is illustrated in Table 1.

Taylor-type linearization. Similarly to [22], we can Taylor-expand the vector field in (10b) around an approximate solution $X^n(t)$ and use optimal LQ controls for the approximated equation. The iteration step reads then

$$\dot{X}^{n+1}(t) = \bar{A}^n(t)X^{n+1}(t) + \bar{B}^n(t)u^{n+1}(t) + \bar{C}^n(t), \quad (15)$$

where

$$\begin{aligned}
\bar{A}^n(t) &= D_X f_A\left(t, X^n(t)\right) + D_X f_B\left(t, X^n(t), u^n(t)\right) \\
\bar{B}^n(t) &= D_U f_B(t, X^n, U^n) \\
\bar{C}^n(t) &= f_A(t, X^n) + f_B(t, X^n, u^n) - \left(\bar{A}^n(t) \cdot X^n + \bar{B}^n(t) \cdot u^n\right),
\end{aligned}$$

A1: waveform relaxation	**A2: linearization**
$n := 0$; *guess* : $X^0(t), u^0(t)$ **do** $n := n+1$ compute : $A^{n-1}(t), B^{n-1}(t)$ solve $(t_f \to 0)$: eq. (14) *for* P^{n-1} solve $(0 \to t_f)$: eq. (13) *for* X^n **while** $\lvert X^n - X^{n-1}\rvert > tolerance$ *Check for feasibility of* X^n	$n := 0$; *guess* : $X^0(t), u^0(t)$ **do** $n := n+1$ compute : $\bar{A}^{n-1}(t), \bar{B}^{n-1}(t), \bar{C}^{n-1}(t)$ solve $(t_f \to 0)$: eq. (3) *for* P^{n-1} solve $(t_f \to 0)$: eq. (16) *for* V^n solve $(0 \to t_f)$: eq. (15) *for* X^n **while** $\lvert X^n - X^{n-1}\rvert > tolerance$ *Check for feasibility of* X^n

Table 1: Algorithm (A1) for the waveform relaxation and algorithm (A2) for the Taylor-type linearization.

and D_X denotes the derivative with respect to X, etc. One starts with an initial guess, $X^0(t)$ and the iteration stops once consecutive iterations differ by less than a given tolerance.

The inhomogeneity $\bar{C}^n$ can be treated as a disturbance input and compensated by the controller [10]. The optimal control then becomes

$$u^{n+1}(t) = -R^n(t)^{-1}\bar{B}^n(t)^T\left(P^n(t)X^{n+1}(t) + V^n(t)\right)$$

where $P^n(t)$ satisfies (3) with replacements $A \to \bar{A}^n$ and $B \to \bar{B}^n$, etc. and $V^n(t)$ is given by

$$\dot{V} = \left(P\bar{B}R^{-1}\bar{B}^T - \bar{A}^T\right)V - P\bar{C}, \quad V(t_f) = 0 \tag{16}$$

at each iteration. The linearization procedure is summarized in Table 1.

NOTE: We can solve non homogeneous equations with Magnus integrators as follows. Given the non-homogeneous equation

$$y' = M(t)\,y + C(t), \qquad y(t_0) = y_0\,;$$

it can be formulated as a homogeneous one in the following way [7],

$$\frac{d}{dt}\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} M(t) & C(t) \\ 0_n^T & 0 \end{bmatrix}\begin{bmatrix} y \\ 1 \end{bmatrix}, \quad [y(0), 1]^T = [y_0, 1]^T,$$

where $0_n^T = [0, \ldots, 0] \in \mathbb{R}^n$.

4. Modeling the control of a quadrotor UAV

The optimal control of Unmanned Air Vehicles (UAV) has attracted a great attention in recent years [11, 14]. Helicopters are classified as Vertical Take Off Landing (VTOL) aircraft and are among the most complex flying objects because their flight dynamics is nonlinear and their variables are strongly coupled.

In this section, we address the optimal control of a quadrotor, i.e., a vehicle with four propellers, whose rotational speeds are independent, placed around a main body [3, 9, 12, 14, 17]. Linear techniques to control the system have been frequently used. However, to improve the performance, the nonlinear nature of the quadrotor has to be taken into account.

The controllers are designed based on a simplified description of the system behavior (linearized models). While this is satisfactory at hover and low velocities, it does not predict correctly the system behavior during fast maneuvers (most controllers are specifically designed for low velocities). In order to reach the desired final position as fast as possible, real time calculations are necessary and hence more efficient and elaborated algorithms have to be designed.

LQ optimal controllers are widely used, in particular for the control of small aircrafts [3, 23], where they have shown to produce better results than other standard methods, like proportional integral derivative methods (PID) [9]. The techniques presented here, however, are valid for the general optimal LQ control problem (1).

For the illustration of our methods, we consider a VTOL quadrotor, based on the model presented in [14, 23] (and references therein). Figure 2 describes the configuration of the system, where ϕ, θ and ψ denote the rolling, pitching and yawing angles, respectively.

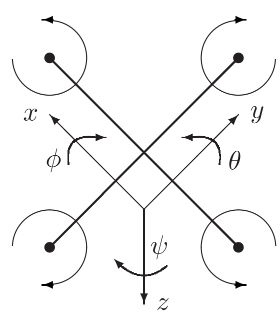


Figure 2: Quadrotor schematic

In general, one assumes some standard general conditions: symmetric and rigid structure of the flying robot, the center of mass is in the center of the planar quadrotor and the propellers are rigid. However, more realistic problems have time-varying parameters [25], require a time-dependent state reference [17] or involve nonlinear equations [15, 23].

There are several ways to perform these computational tasks, among the simplest is the so called Pearson method that proposes $\dot{P}(t) = 0$ and thus simplifies (3) to an algebraic Riccati equation whose symmetric and nonnegative solution is chosen. For time-dependent problems, however, better results are obtained with the Sage-Eisenberg method [9], i.e., to compute $P(t)$ by integrating (3) backwards in time. Note that this requires to store the values $P(t)$ on an appropriate time mesh.

To describe the helicopter flight, it is more realistic to consider the nonlinear model (11). Nonlinear control theory can improve the performance of the controller and enable the tracking of aggressive trajectories [13]. We remark that inhomogeneities $f_A(t,0) = b(t)$, e.g. from gravitational forces, can be treated as disturbances, by adding new state variables or by taking advantage of non-vanishing states, e.g. the altitude of the UAV when hover is searched [15]. Usually, B is assumed to be independent of u.

An analysis of the dynamics of the quadrotor shows that the control of the attitude can be separated from the translation of the UAV [23] and we focus our attention on the stabilization of the attitude, neglecting the gyroscopic effect. The state vector is given by

$$X(t) = \left(\phi(t), \dot{\phi}(t), \theta(t), \dot{\theta}(t), \psi(t), \dot{\psi}(t)\right)^T \in \mathbb{R}^6,$$

and the input vector $u \in \mathbb{R}^3$ is formed by linear combinations of the thrust of each propeller.

The system designer can choose the weight matrices to tune the behavior of the control according to the requirements, $R(t)$ is used to suppress certain movements and $Q(t)$ limits the use of the control inputs. Usually, these matrices are chosen constant, nonnegative definite, and often even diagonal, see [3, p. 67], [14, 17]. For the numerical experiments we consider the problem (10) with the following values taken from [8, 23]

$$\begin{array}{lll} a_{1,2} = a_{3,4} = a_{5,6} = 1, & a_{2,4} = \lambda\alpha_1 I_1 \dot{\psi}, & a_{2,6} = \lambda(1-\alpha_1) I_1 \dot{\theta} \\ a_{4,2} = \lambda\alpha_2 I_2 \dot{\psi}, & a_{4,6} = \lambda(1-\alpha_2) I_2 \dot{\phi}, & a_{6,2} = \lambda\alpha_3 I_3 \dot{\theta}, \\ a_{6,4} = \lambda(1-\alpha_3) I_3 \dot{\phi}, & b_{2,1} = L/I_x, \quad b_{4,2} = L/I_y, & b_{6,3} = 1/I_z \end{array} \tag{17}$$

where α_i reflects the non-uniqueness in the SDRE formulation, λ denotes the inflow ratio, L is the length of the arms connecting the propellers with the center, $I_1 = (I_y - I_z)/I_x$, $I_2 = (I_z - I_x)/I_y$, $I_3 = (I_x - I_y)/I_z$. Here, $m_{i,j}$ denotes the element located at i-th row and j-th column of the matrix M. Other entries of $A(t) \in \mathbb{R}^{6\times 6}$ and $B(t) \in \mathbb{R}^{6\times 3}$ not indicated in (17) are null elements.

The numerical values are extracted from [8] and are given in the SI units

$$I_x = 0.0075, \quad I_y = 0.0075, \quad I_z = 0.0130, \quad L = 0.23, \quad \lambda = 1, \quad \alpha_i = 1.$$

The weight matrices are fixed at

$$Q = 0.01 \cdot \mathrm{diag}\{\,1, 0.1, 1, 0.1, 1, 0.1\} \in \mathbb{R}^{6\times 6}, \quad R = \mathrm{diag}\{1, 0.1, 1\} \in \mathbb{R}^{3\times 3}.$$

We set the time frame to $t_f = 10$ seconds, with a stepsize of $h = 0.125s$ and initial state

$$X_0 = (70\,^\circ,\, 10,\, 70\,^\circ,\, 20,\, -130\,^\circ, -1)^T,$$

that corresponds to a disadvantageous orientation and high rotational velocities that is sought to be stabilized at $0 \in \mathbb{R}^6$.

We have implemented a variety of methods to test against the Magnus integrators presented in section 2.2. As initial condition, we have taken $X^0(t) = (1-t/t_f)X_0$ and the iteration was stopped when $||X^n - X^{n-1}||_2 < 10^{-3}$. Some experimental results is given in Table 2, where we can see that the Magnus based method (9), approximates the optimal control best. However, we have to remark that the SDRE method is for the given parameters about a factor ten faster, due to necessary iterations for the other schemes. In more difficult settings, e.g. in the case of trajectory following and obstacle avoidance, stronger time dependencies of the parameters are expected, and thus, an even greater advantage of the exponential method towards achieving optimality in the control.

	Type	$X(t)$	$P(t)$	$V(t)$	Cost	It.
`S1`)	SDRE	Euler	are	N/A	0.1114	
`S2`)		Impl. Euler (IE)	are	N/A	0.1021	
		Optimal	$\Rightarrow$		0.0977	
`W1`)	WAVE	Euler	Euler	N/A	0.1071	3
`W2`)		IE	IE	N/A	0.1036	3
`W3`)		Magnus (9)	Magnus	N/A	0.0926	3
		Optimal	$\Rightarrow$	N/A	0.0888	
`T1`)	TAYLOR	Euler	Euler	Euler	N/A	Inf
`T2`)		IE	IE	IE	0.0789	12
`T3`)		Magnus	Magnus	Magnus	0.0707	12
		Optimal	$\Rightarrow$		0.0707	

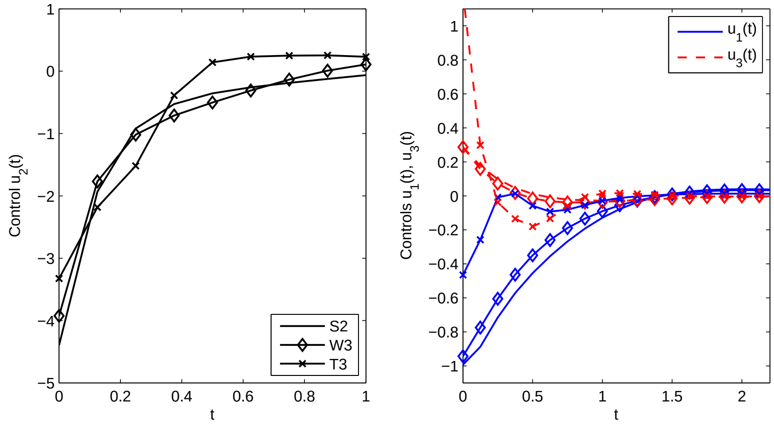
Table 2: Comparison of numerical methods, Type indicates the linearization procedure given by section 3, and It. denotes the number of iterations necessary until convergence. The co

Figure 3 shows the controls obtained for the schemes `S2`, `W3`, `T3` and Figure 4 shows the dynamics of the quadrotor subject to the obtained controls. We can appreciate how the Magnus methods maximize the use of the controls to reach an overall minimum of the cost functional.

From the numerical experiments we conclude that Lie group methods like Magnus integrators are very useful tools for solving optimal control problems of UAV. The results shown for a quadrotor easily extend to other helicopters. In addition, for more involved trajectories the structure of the equations will play a more important role and Lie group methods can provide efficient numerical algorithms.

Acknowledgements

This work has been partially supported by Ministerio de Ciencia e Innovación (Spain) under the coordinated project MTM2010-18246-C03 (co-financed by the ERDF of the European Union) and MTM2009-08587, and the Universitat Politècnica de València throughout the project 2087. P. Bader also acknowledges the support through the FPU fellowship AP2009-1892.



,

Figure 3: Evolution of the control vector. The left column shows the control that has been less penalized u_2. All curves are given for all methods `S2` (line), `W3` (diamond) and `T3` (cross).

[1] H. Abou-Kandil, G. Freiling, V. Ionescu & G. Jank, (2003) *Matrix Riccati equations in control and systems theory*. Virkäuser Verlag, Basel.

[2] B. D. O. Anderson & J. B. Moore, (2007) *Optimal control. Linear quadratic methods*. Dover Publications. New York.

[3] C. Balas, (2007) Modelling and linear control of a quadrotor. *MSc Thesis*, School of Engineering, Cranfield University. England.

[4] S. Blanes, F. Casas, J. A. Oteo & J. Ros, (2009) The Magnus expansion and some of its applications. *Physics Reports*, **470**, pp. 151–238.

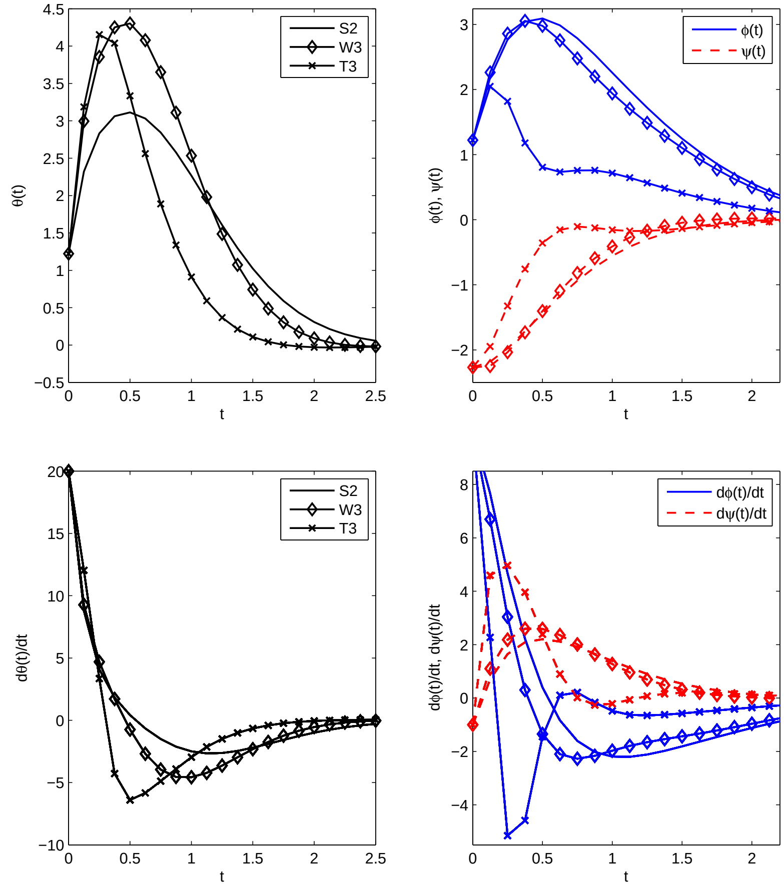
[5] S. Blanes & P. C. Moan, (2006) Fourth- and sixth-order commutator-free Magnus integrators for linear and non-linear dynamical systems. *Appl. Num. Math.*, **56**, pp. 1519-1537.

[6] S. Blanes & E. Ponsoda (2012) Magnus integrators for solving linear-quadratic differential games. *J. Comp. Appl. Math.*, **236**, pp. 3394–3408.

[7] S. Blanes & E. Ponsoda (2012) Time-averaging and exponential integrators for non-homogeneous linear IVPs and BVPs. *Appl. Num. Math*, **62**, pp. 875–894.

[8] S. Bouabdallah, (2006) Design and control of quadrotors with application to autonomous flying. *Ph.D. dissertation*, EPFL.

[9] S. Bouabdallah, A. Noth & R. Siegwart, (2004) PID vs LQ control techniques applied to an indoor micro quadrotor. *Proc. of the IEEE/RSJ Int. Conf. on Intelligent Robots and Systems (IROS 2004)*, **3**, pp. 2451–2456.



,

Figure 4: Evolution of the orientation of the quadrotor (top row) and angular velocities (bottom). The left column shows the the coordinates $\theta(t)$ and $\dot{\theta}(t)$. All curves are given for all methods `S2` (line), `W3` (diamond) and `T3` (cross).

[10] A. Bryson Jr. & Y. C. Ho, (1975) *Applied Optimal Control,* Halsted.

[11] A. Budiyono & S. S. Wibowo, (2007) Optimal tracking controller design for a small scale helicopter. *J. Bionic Eng.*, **4**, pp. 271–280.

[12] P. Castillo, A. Dzul & R. Lozano, (2004) Real-time stabilization and tracking of four rotor mini-rotorcraft. *IEEE Trans. Control Syst Tech.*, **12**, pp. 510–516.

[13] P. Castillo, R. Lozano & A. Dzul, (2005) RStabilization of a mini rotorcraft with four rotors, Experimental implementation of linear and nonlinear control laws. *IEEE Control Systems Magazine*, pp. 45–45, Dec. 2005.

[14] P. Castillo, R. Lozano & A. E. Dzul, (2005) *Modelling and control of mini-flying machines.* Advances in Industrial Control Series. Springer. London. England.

[15] T. Çimen, (2008) State-dependent Riccati equation (SDRE) control: A survey. *Proc. of the 17th IFAC World Congress(IFAC'08) Seoul, Korea*, pp. 3761–3775.

[16] J. Cloutier, (1997) State-Dependent Riccati Equation Techniques: An Overview. *Proc. of the American Control Conference*, Albuquerque, New Mexico, **2**, pp.932–936.

[17] I. D. Cowling, J. F. Whidborne & A. K. Cooke, (2006) Optimal trajectory planning and LQR control for a quadrotor UAV. *Proc. UKACC Int. Conf. Control 2006 (ICC 2006)*, Glasgow, UK.

[18] J. Engwerda (2005) *LQ dynamic optimization and differential games.* John Wiley and sons.

[19] A. Iserles, H. Z. Munthe-Kaas, S. P. Nørsett & A. Zanna, (2000) Lie group methods. *Acta Numerica*, **9**, 215–365.

[20] L. Jódar & E. Ponsoda, (1995) Non-autonomous Riccati-type matrix differential equations: existence interval, construction of continuous numerical solutions and error bounds. *IMA J. Num. Anal.*, **15**, 61-74.

[21] D. Kirk, (2004) *Optimal control theory, an Introduction.* Dover Publ., Mineola, New York.

[22] E. Ponsoda, S. Blanes & P. Bader, (2011) New efficient numerical methods to describe the heat transfer in a solid medium. *Math. Comput. Mod.*. **54**, pp. 1858-1862.

[23] H. Voos, (2006) Nonlinear state-dependent Riccati equation control of a quadrotor UAV. *Proc. Int. Conf. Control Appl.,* Munich, Germany, pp. 2547–2552.

[24] J. White, F. Odeh, A.S. Vincentelli & A.Ruehli, (1985) Waveform relaxation: theory and practice. *Trans. Soc. Comput. Simulation*, **2**, pp. 95–133.

[25] R. Zhang, Q. Quan & K.-Y. Cai, (2011) Attitude control of a quadrotor aircraft subject to a class of time-varying disturbance. *IET Control Theory Appl.*, **5**, pp. 1140–1146.