IJCSS International Journal of Complex Systems in Science

Int. J. Complex Systems in Science
vol. **6(1)** (2016), pp. 23–28

Stabilization of positive linear continuous-time systems by using a Brauer's theorem

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Abstract. In this paper we study the stability property of positive linear continuous-time systems. This property is useful to study the asymptotic behavior of a dynamical system and specifically, in positive systems. Stabilization of linear systems using feedbacks has been deeply studied during the last decades. Motivated by some results, in this paper we find conditions on the system such that the eigenvalues of the closed loop system are in the open left half plane of the complex plane $\mathbb{C}$. We do this by applying a Brauer's theorem.

Keywords: Continuous-time system, positive system, stability, feedback, Brauer's theorem
MSC 2000: 93C05,93D15

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Received: July 27th, 2016
Published: November 30th, 2016

1. Introduction

Positive systems are often found in the modeling of biology, hydrology, engineering and industrial processes whose variables represent quantities that don't make sense unless they are nonnegative; for example, time in stochastic game algorithms, money and goods in Leontief model, data packets flowing in a network, quantity of bacteria in a epidemiological model, etc. (see for example [2, 12] and the references therein). The practical importance of these systems is widely visible, as the nonnegative property occurs quite frequently in numerous applications and in nature [9].

Stability is one of the most important topics discussed in control systems and it is not different for the case of positive systems. The use of state-feedbacks for the stabilization of linear systems has been considered during the last decades. For example, a feedback stabilization for linear time-invariant control systems with saturating quantized measurements is given in [4, 6].

It is known that the results obtained for linear systems cannot be applied to positive linear systems. Note that the theory of positive systems is more complicated by the nonnegativity restrictions. Nevertheless there are several papers on positive systems. For example, the evolution of the disease in an epidemiological model is treated in [1], the study of reachability and controllability properties is carried out in [7, 10] and a survey of the topics on positive systems is given in [11].

In this paper we consider a linear continuous-time system. It is known that a linear system is said to be positive if for every nonnegative initial state and for every nonnegative control sequence its trajectory is nonnegative [9]. This system is asymptotically stable if and only if the real parts of all eigenvalues of the state matrix A are negative. A linear continuous-time invariant system (LTI) is exponentially stable if and only if the system has eigenvalues with strictly negative real parts [13].

In [5] we use an application of a Brauer's theorem to stabilize a SISO positive linear discrete-time system. In this paper, we give a condition to stabilize a positive continuous-time system using a state-feedback. That is, we find conditions on the system such that the eigenvalues of the closed loop system are in the open left half plane of the complex plane $\mathbb{C}$.

From now on the following notation will be used. The set of $n \times m$ real matrices with nonnegative entries will be denoted by $\mathbb{R}_+^{n\times m}$, therefore $A = [a_{ij}] \in \mathbb{R}_+^{n\times m}$ if all its entries are nonnegative, i.e. $a_{ij} \geq 0$ for $i = 1, \ldots, n$; $j = 1, \ldots, m$. A real square matrix $M = [m_{ij}] \in R^{n\times n}$ is called Metzler matrix if all its off-diagonal entries are nonnegative, i.e. $m_{ij} \geq 0$, with $i \neq j$. The set of $n \times n$ Metzler matrices will be denoted by $\mathcal{M}_n$. Finally, the identity $n \times n$ matrix will by denoted by I_n.

The paper is organized as follows: in Section 2 spectral properties of Metzler matrices are considered and the problem is presented. In Section 3 some conditions to stabilize a positive linear continuous-time system are obtained. Finally, the conclusions and remarks are in Section 4.

2. Positive linear continuous-time system

It is well known that Metzler matrices are connected with positive continuous-time dynamical systems. In addition, the spectrum of the state matrices plays an important role in the behaviour of the positive dynamical systems.

We recall the definition of a positive linear continuous-time system (see, for instance, [11]). Consider the system described by the following equation

$$\dot{x} = Ax + bu. \tag{1}$$

The system (1) is positive if and only if for any initial condition $x_0 \in \mathbb{R}^n_+$ and every control vector $u \in \mathbb{R}^m_+$, we have the state vector $x \in \mathbb{R}^n_+$. The system (1) is positive if and only if the state matrix $A = [a_{ij}] \in \mathcal{M}_n$ and the control matrix $b = [b_i] \in \mathbb{R}^{n\times m}_+$.

Metzler matrices have some interesting properties and in this paper we show some useful properties for our study (for more information see [14]).

Let $\rho(A) = \max\{|\lambda|;\ \lambda \in \sigma(A)\}$ be the spectral radius of the matrix A and let $\mu(A) = \max\{\mathrm{Re}\lambda;\ \lambda \in \sigma(A)\}$ be the growth constant of A. Now, we give the following result.

Proposition 1 *[15] Consider $A \in \mathcal{M}_n$. Then*

- (i) *$\mu(A)$ is an eigenvalue of A and there exists a nonnegative eigenvector $x \geq 0, \quad x \neq 0$, such that $Ax = \mu(A)x$.*
- (ii) *If $\lambda \neq \mu(A)$ is any other eigenvalue of A then $\mathrm{Re}\lambda < \mu(A)$.*
- (iii) *Given $\alpha \in \mathbb{R}$, there exists a nonzero vector $x \geq 0$ such that $Ax \geq \alpha x$ if and only if $\mu(A) \geq \alpha$.*
- (iv) *For some $t \geq 0$, $(tI_n - A)^{-1}$ exists and it is nonnegative if and only if $t > \mu(A)$.*

A system (1) is stable if the state matrix A is a Hurwitz matrix. Recall that a matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz or stable if all the eigenvalues of A are in the open left half of the complex plane $\mathbb{C}$.

From now on we consider that the system (1) is not stable and we give some conditions in order to construct a state-feedback $F \in \mathbb{R}^{n\times 1}$, for the control law

$$u = -F^T x,$$

such that the closed-loop system

$$\dot{x} = (A - bF^T)x$$

be positive and asymptotically stable. That is, the closed-loop matrix $A - bF^T$ would be Metzler and Hurwitz.

3. Stabilization problem

In this section we present our main results to solve the stabilization problem presented in Section 2. To that end, we consider the Brauer's Theorem that gives the relationship among the eigenvalues of an arbitrary matrix A and the updated matrix $\bar{A}$ by a rank-one additive perturbation.

Theorem 1 *[16] Let A be an $n \times n$ arbitrary matrix with the eigenvalues $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Let x_k be an eigenvector of A associated with the eigenvalue λ_k, and let q be any n-dimensional vector. Then, the matrix $\bar{A} = A + x_k q^T$ has the eigenvalues $\{\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + x_k^T q, \lambda_{k+1}, \ldots, \lambda_n\}$.*

This theorem was used in [3] to stabilize control systems, including the case when the system is noncontrollable.

Proposition 2 *[3] Consider the pair (A, b) that represents a single-input single-output linear time invariant control system. Let $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and let x_k be an eigenvector of A^T associated with λ_k. If $b^T x_k \neq 0$, then there exists a vector F such that $\sigma(A + bF^T) = \{\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + x_k^T b, \lambda_{k+1}, \ldots, \lambda_n\}$.*

Using these results we solve the stability problem for the system (1). Without loss of generality, we consider that λ_1 is an eigenvector of the state matrix A such that $\mathrm{Re}\lambda_1 \geq 0$.

Proposition 3 *Consider the positive linear continuous-time system given by (1). Let $x_1 \in \mathbb{R}^{n\times 1}$ be an eigenvector of A associated with the eigenvalue λ_1. If there exists $\alpha \in \mathbb{R}$, with $F = \alpha x_1$, for the control law*

$$u = -F^T x,$$

the closed-loop system

$$\dot{x} = (A - bF^T)x$$

satisfies the following conditions:

a) $\widehat{b} = b^T x_1 \neq 0$,

b) for each row of A with zero elements, the corresponding element of the vector b is equal to zero,

c) and

$$\max\left\{\frac{-a_{ij}}{b_i x_{1_j}^T}\right\}_{\substack{i \neq j \\ i \neq 0}} \leq \alpha < \frac{-\rho}{\widehat{b}},$$

$i = 1, \ldots, n,\ j = 1, \ldots, n,\ b_i x_{1_j}^T \neq 0.$

then the closed-loop system is positive and asymptotically stable.

Note that the first assumption of the above Proposition is needed only to assure the change of an specific eigenvalue. Otherwise no eigenvalue changes. When this condition holds for all eigenvectors of A^T, then it is said that the pair (A, b) is completely controllable and, in this case, the solution of the system (1) is unique [8].

4. Conclusions

In this paper we apply a statistics result proved by A. Brauer to solve the stabilization problem of a positive linear continuous-time system. For that, we use a state-feedback and give some sufficient conditions for the matrices that characterize the control system. Note that the result does not only include the asymptotically stabilization but also the positivity conditions.

Acknowledgements

The authors would like to thank the referees and the editor for their comments and useful suggestions for improvement of the manuscript. This research was partially supported by Spanish Grant MTM2013-43678-P.

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