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Fell type topologies of quasi-pseudo-metric spaces and the Kuratowski-Painlevé convergence

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Abstract. We study the double Fell topology when this hypertopology is constructed over a quasi-pseudo-metric space. In particular, its relationship with the Wijsman hypertopology is studied. We also propose an extension of the Kuratowski-Painlevé convergence in the bitopological setting.

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1. Introduction and preliminaries

Recently, the study of nonsymmetric structures has received a new drive as a consequence of its applications to Computer Science. This theory began with Smyth (see [20, 21]). He tried to find a convenient category for computation and he proposed the quasi-uniform spaces as the suitable context. Continuing the work of Smyth, other authors have applied the nonsymmetric topology to this area (see [17, 18, 19]). Furthermore, some hypertopologies have been successfully applied to several areas of Computer Science (see [21, 23]). All these facts motivate our interest in the nonsymmetric study of several hypertopologies. In this paper, we continue the work developed by the author in [15].

The Fell topology was introduced by Fell in [8]. In [15] it is introduced a definition for the Fell hypertopology in the nonsymmetric situation. Some satisfactory results about the relationship of some hypertopologies with the Fell topology are obtained in the quasi-uniform setting. We continue this work and obtain extensions of well-known results in the symmetric case about the relationship between the Fell and the Wijsman hypertopologies. We also study a definition for the Kuratowski-Painlevé convergence in the bitopological setting and obtain extensions of interesting results as the Mrowka's Theorem.

Our basic references for quasi-uniform and quasi-pseudo-metric spaces are [9] and [12]. Terms and undefined concepts may be found in such references.

A *quasi-pseudo-metric* on a set X is a nonnegative real valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x,x) = 0$ and (ii) $d(x,y) \leq d(x,z) + d(z,y)$.

If, in addition, d satisfies the condition: (iii) $d(x,y) = 0 \Rightarrow x = y$, then d is said to be a quasi-metric on X.

A quasi-(pseudo-)metric space is a pair (X,d) such that X is a nonempty set and d is a quasi-(pseudo-)metric on X.

If d is a quasi-(pseudo-)metric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x,y) = d(y,x)$ for all $x, y \in X$, is also a quasi-(pseudo-)metric on X, called the *conjugate quasi-pseudo-metric* of d, and the function d^s defined on $X \times X$ by $d^s(x,y) = \max\{d(x,y), d^{-1}(x,y)\}$ for all $x, y \in X$, is a (pseudo-)metric on X.

Each quasi-pseudo-metric d on X generates a topology $\mathcal{T}(d)$ on X which has as a base the family of balls of the form $B_d(x,r) = \{y \in X : d(x,y) < r\}$, where $x \in X$ and $r > 0$. Note that if d is a quasi-metric, then $\mathcal{T}(d)$ is a T_1 topology on X. We denote $\overline{B_d}(x,\varepsilon) = \{y \in X : d(x,y) \leq \varepsilon\}$.

The quasi-pseudo-metric ℓ on $\mathbb{R}$ defined by $\ell(x,y) = \max\{x - y, 0\}$ for all $x, y \in \mathbb{R}$, is called the *lower quasi-pseudo-metric* on $\mathbb{R}$. Its conjugate quasi-pseudo-metric ℓ^{-1} is denoted by u and is called the *upper quasi-pseudo-metric* on $\mathbb{R}$. Note that $\ell^s = \ell \vee u$ is the usual metric on $\mathbb{R}$. A function from a topological space $(X, \mathcal{T})$ to $\mathbb{R}$ is said to be *lower semicontinuous* (resp. *upper semicontinuous*) if it is continuous when we consider the topology generated by the lower (resp. upper) quasi-pseudo-metric on $\mathbb{R}$.

A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ on $X \times X$ which satisfies: (i) $\Delta \subseteq U$ for all $U \in \mathcal{U}$ and (ii) given $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subseteq U$, where $\Delta = \{(x,x) : x \in X\}$ and $V^2 = \{(x,z) \in X \times X :$ exists $y \in X$ such that $(x,y) \in V, (y,z) \in V\}$. The elements of $\mathcal{U}$ are called *entourages*.

The filter $\mathcal{U}^{-1}$, formed by all sets of the form $U^{-1} = \{(x,y) \in X \times X : (y,x) \in U\}$ where $U \in \mathcal{U}$, is a quasi-uniformity on X called the *conjugate quasi-uniformity* of $\mathcal{U}$.

If $\mathcal{U}$ is a quasi-uniformity on X, then the family $\{U^s = U \cap U^{-1} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}^s$ (in fact, it is a uniformity), which is the coarsest uniformity containing $\mathcal{U}$. This uniformity is called the supremum of the quasi-uniformities $\mathcal{U}$ and $\mathcal{U}^{-1}$.

Every quasi-uniformity $\mathcal{U}$ generates a topology $\mathcal{T}(\mathcal{U})$ on X. A neighborhood base for each point $x \in X$ is given by $\{U(x) : U \in \mathcal{U}\}$ where $U(x) = \{y \in X : (x,y) \in U\}$.

Each quasi-pseudo-metric d on X induces a quasi-uniformity $\mathcal{U}_d$ on X which has as a base the family of entourages of the form $\{(x,y) \in X \times X : d(x,y) < 2^{-n}\}$, $n \in \mathbb{N}$. Moreover, $\mathcal{T}(\mathcal{U}_d) = \mathcal{T}(d)$.

In addition of a quasi-uniformity and a quasi-pseudo-metric, we can define on a space another structure which makes precise the concept of nearness. This structure is a relation δ in $\mathcal{P}_0(X)$. We write $A\delta B$ for $(A,B) \in \delta$ and $A\overline{\delta}B$ instead of $(A,B) \notin \delta$.

Definition 1.1. *Let X be a nonempty set. A relation δ in $\mathcal{P}_0(X)$ is a* quasi-proximity *for X if it satisfies the following conditions:*

i) $X\overline{\delta}\varnothing$ *and* $\varnothing\overline{\delta}X$.
ii) $C\delta(A\cup B)$ *if and only if* $C\delta A$ *or* $C\delta B$
$(A\cup B)\delta C$ *if and only if* $A\delta C$ *or* $B\delta C$.
iii) $\{x\}\delta\{x\}$ *for each* $x\in X$.
iv) If $A\overline{\delta}B$, *there exists* $C\in\mathcal{P}_0(X)$ *such that* $A\overline{\delta}C$ *and* $(X\backslash C)\overline{\delta}B$.

The pair (X,δ) is called a quasi-proximity space.

Obviously, if δ is a quasi-proximity on X, then so is the opposite relation δ^{-1}. This quasi-proximity is called the *conjugate quasi-proximity* of δ. A quasi-proximity δ is a *proximity* if $\delta=\delta^{-1}$.

Let A and B be subsets of a quasi-proximity space (X,δ). If $A\delta B$, then A is said to be near B and if $A\overline{\delta}B$, then A is said to be far from B. A set B is said to be a *δ-neighborhood* of a set A if $A\overline{\delta}(X\backslash B)$.

Every quasi-proximity δ on a space X induces in a natural way a topology on X. If $x\in X$, the neighborhoods of x are the δ-neighborhoods of x.

Furthermore, if $(X,\mathcal{U})$ is a quasi-uniform space, then $\mathcal{U}$ induces a quasi-proximity $\delta_{\mathcal{U}}$ such that $A\delta_{\mathcal{U}}B$ if and only if $(A\times B)\cap U\neq\varnothing$ for all $U\in\mathcal{U}$.

A *bitopological space* (see [10, 13]) is a triple (X,P,Q) where X is a set and P and Q are topologies on X. A bitopological space is said to be quasi-pseudo-metrizable (resp. quasi-uniformizable) if there exists a quasi-pseudo-metric d (resp. a quasi-uniformity $\mathcal{U}$) on X such that $\mathcal{T}(d)=P$ and $\mathcal{T}(d^{-1})=Q$ (resp. $\mathcal{T}(\mathcal{U})=P$ and $\mathcal{T}(\mathcal{U}^{-1})=Q$). In this case we say that d (resp. $\mathcal{U}$) is a quasi-pseudo-metric (resp. quasi-uniformity) compatible with the bitopological space (X,P,Q).

Given a topological space $(X,\mathcal{T})$ we denote by $\mathcal{P}_0(X)$ the family of nonempty subsets of X and by $CL_0(X)$ we denote the family of nonempty closed subsets of X. We also shall use $\mathcal{P}(X)=\mathcal{P}_0(X)\cup\varnothing$. If (X,P,Q) is a bitopological space we denote by $CL_0^P(X)$ (resp. $CL_0^Q(X)$, $CL_0^s(X)$) the family of nonempty P-closed (resp. Q-closed, $P\vee Q$-closed) subsets of X.

2. Fell type topologies of quasi-pseudo-metric spaces

In [15] it can be found a discussion about the definition of the Fell hypertopology in the nonsymmetric case. We propose the use of double topological spaces rather than bitopological spaces. We recall some definitions.

Definition 2.1 ([15]). *A double topological space is simply a pair of topological spaces $((X,\tau),(Y,\upsilon))$.*

In the following, we will also use double space.

Definition 2.2 ([15]). *Let (X,P,Q) be a bitopological space. We define the double upper Fell topological space as the double topological space $((CL_0^Q(X),$ $F_P^+)$, $(CL_0^P(X),$ $F_Q^+))$ where F_P^+ is the topology generated by all sets of the form $G^+=\{A\in CL_0^Q(X):A\subseteq G\}$ where G is a P-open set and $X\backslash G$ is*

$P \vee Q$-compact; F_Q^+ is defined in a similar way by writing P instead of Q and Q instead of P. The pair (F_P^+, F_Q^+) is called the double upper Fell topology.

The double lower Fell topological space is defined as the double topological space $((CL_0^Q(X), F_P^-),(CL_0^P(X), F_Q^-))$ where F_P^- is generated by all sets of the form $G^- = \{A \in CL_0^Q(X) : A \cap G \neq \varnothing\}$ where G is P-open; F_Q^- is defined in a similar way by writing P instead of Q and Q instead of P. The pair (F_P^-, F_Q^-) is called the double lower Fell topology.

The double Fell topological space is the double topological space $((CL_0^Q(X), F_P), (CL_0^P(X), F_Q))$ where $F_P = F_P^+ \vee F_P^-$ and $F_Q = F_Q^+ \vee F_Q^-$. The pair (F_P, F_Q) is called the double Fell topology.

In this section we study the relationship between the double Fell topology and the double Wijsman topology. We also motivate the fact of considering $P \vee Q$-compact sets in the definition of the double Fell topology (see Remark 2.10). The results in the symmetric case can be found in [3] and [4].

The following is an extension of the Wijsman hypertopology definition when d is a quasi-pseudo-metric (see [16]).

Definition 2.3. *Let (X, d) be a quasi-pseudo-metric space. Let $P = \mathcal{T}(d)$ and $Q = \mathcal{T}(d^{-1})$. The* double upper Wijsman topological space *is the double topological space $((CL_0^Q(X), \mathcal{T}^+(W_d)), (CL_0^P(X), \mathcal{T}^+(W_{d^{-1}})))$ where $\mathcal{T}^+(W_d)$ is the weakest topology on $CL_0^Q(X)$ such that for each $x \in X$, the functional $d(\cdot, x)$ is lower semicontinuous on $CL_0^Q(X)$. The definition for $\mathcal{T}^+(W_{d^{-1}})$ is symmetric. The pair $(\mathcal{T}^+(W_d), \mathcal{T}^+(W_{d^{-1}}))$ is called the* double upper Wijsman topology*.*

The double lower Wijsman topological space *is the double topological space $((CL_0^Q(X), \mathcal{T}^-(W_d)), (CL_0^P(X), \mathcal{T}^-(W_{d^{-1}})))$ where $\mathcal{T}^-(W_d)$ is the weakest topology on $CL_0^Q(X)$ such that for each $x \in X$, the functional $d(x, \cdot)$ is upper semicontinuous on $CL_0^Q(X)$. The definition for $\mathcal{T}^-(W_{d^{-1}})$ is symmetric. The pair $(\mathcal{T}^-(W_d), \mathcal{T}^-(W_{d^{-1}}))$ is called the* double lower Wijsman topology*.*

The double topological space $((CL_0^Q(X), \mathcal{T}(W_d)), (CL_0^P(X), \mathcal{T}(W_{d^{-1}})))$ where $\mathcal{T}(W_d) = \mathcal{T}^+(W_d) \vee \mathcal{T}^-(W_d)$ and $T(W_{d^{-1}}) = \mathcal{T}^+(W_{d^{-1}}) \vee \mathcal{T}^-(W_{d^{-1}})$ is called the double Wijsman topological space*. The pair $(\mathcal{T}(W_d), \mathcal{T}(W_{d^{-1}}))$ is called the* double Wijsman topology*.*

Proposition 2.4. *Let (X, P, Q) be a quasi-pseudo-metrizable bitopological space. Then $F_P^- = \mathcal{T}^-(W_d)$, $F_Q^- = \mathcal{T}^-(W_{d^{-1}})$ on $\mathcal{P}_0(X)$ and $F_P^+ \subseteq \mathcal{T}^+(W_d)$ on $CL_0^Q(X)$ and $F_Q^+ \subseteq \mathcal{T}^+(W_{d^{-1}})$ on $CL_0^P(X)$ where d is a quasi-pseudo-metric compatible with the bitopological space.*

Proof. It is easy to show that $d(x, \cdot)^{-1}(-\infty, \alpha) = B_d(x, \alpha)^-$ so we obtain that $F_P^- = \mathcal{T}^-(W_d)$ on $\mathcal{P}_0(X)$. In a similar way, it can be proved $F_Q^- = \mathcal{T}^-(W_{d^{-1}})$ on $\mathcal{P}_0(X)$.

Now, we show that $F_P^+ \subseteq \mathcal{T}^+(W_d)$ on $CL_0^Q(X)$. Let G be a P-open set such that $X \backslash G$ is $P \vee Q$-compact, and $A \in G^+$. For all $x \in X \backslash G$, let us consider

$\varepsilon_x = d(A, x)$. Since A is a Q-closed set then $\varepsilon_x > 0$. Choose $0 < \alpha_x < \varepsilon_x$. Thus $\{B_{d^{-1}}(x, \alpha_x) : x \in X \backslash G\}$ is a Q-open cover of $X \backslash G$ so there exists $\{x_1, \ldots, x_n\} \subseteq X \backslash G$ such that

$$X \backslash G \subseteq \bigcup_{i=1}^{n} B_{d^{-1}}(x_i, \alpha_{x_i}).$$

Consider the $\mathcal{T}^+(W_d)$-open set $C = \cap_{i=1}^n d(\cdot, x_i)^{-1}(\alpha_{x_i}, +\infty)$. Clearly, $A \in C$. Let us see that $C \subseteq G^+$. Let $B \in C$ and suppose that $B \cap (X \backslash G) \neq \varnothing$. Given $b \in B \cap (X \backslash G)$ there exists $i \in \{1, \ldots, n\}$ such that $d^{-1}(x_i, b) < \alpha_{x_i}$. A contradiction with $d(B, x_i) > \alpha_{x_i}$. Thus, $B \in C \subseteq G^+$, i.e. $G^+ \in \mathcal{T}^+(W_d)$. Similarly, we prove $F_Q^+ \subseteq \mathcal{T}^+(W_{d^{-1}})$ on $CL_0^P(X)$. □

Remark 2.5. We give an example showing that the above Proposition is not true when we define F_P^+ and $\mathcal{T}^+(W_d)$ on $CL_0^P(X)$ and F_Q^+ and $\mathcal{T}^+(W_{d^{-1}})$ on $CL_0^Q(X)$.

We consider the set $\mathbb{N} \cup \{\infty\}$ with the following quasi-metric:

$$\begin{cases} d(n,m) = 1 & \text{if } n \neq m \\ d(n,n) = 0 & \text{for all } n \in \mathbb{N} \\ d(n,\infty) = \frac{1}{n} & \text{for all } n \in \mathbb{N} \\ d(\infty, n) = 1 & \text{for all } n \in \mathbb{N} \\ d(\infty,\infty) = 0 & \end{cases}.$$

It is evident that $\mathcal{T}(d) = P$ is the discrete topology. Therefore $\mathbb{N}$ is a $\mathcal{T}(d)$-clopen set, and its complement is obviously a $\mathcal{T}(d^s)$-compact set, so we consider the F_P^+-open set $\mathbb{N}^+$. We shall prove that the set $\mathbb{N} \in \mathbb{N}^+$ has not a $\mathcal{T}^+(W_d)$-neighborhood contained in $\mathbb{N}^+$. If $n \in \mathbb{N}$ and $\mathbb{N} \in d(\cdot, n)^{-1}(\alpha, +\infty)$ where $\alpha \in \mathbb{R}$ we deduce that $\alpha < 0$ but $\{\infty\}$ is a $\mathcal{T}(d)$-closed set which belongs to $d(\cdot, n)^{-1}(\alpha, +\infty) = CL_0^P(X)$ so this set is not contained in $\mathbb{N}^+$. On the other hand, if $\mathbb{N} \in d(\cdot, \infty)^{-1}(\alpha, +\infty)$ we obtain the same contradiction.

It is natural to wonder when the Wijsman and Fell hypertopologies agree. The following extension of a concept introduced by Beer in [1] and reformulated in [2], gives us the answer.

Definition 2.6. *Let (X, d) be a quasi-pseudo-metric space. We say that it has* nice closed balls *if the proper closed d-balls and the proper closed d^{-1}-balls are $\mathcal{T}(d^s)$-compact.*

Now, we can extend a result which can be found in [3].

Theorem 2.7. *Let (X, d) be a quasi-pseudo-metric space, $P = \mathcal{T}(d)$ and $Q = \mathcal{T}(d^{-1})$. Then $F_P = \mathcal{T}(W_d)$ on $CL_0^Q(X)$ and $F_Q = \mathcal{T}(W_{d^{-1}})$ on $CL_0^P(X)$ if and only if (X, d) has nice closed balls.*

Proof. Let us suppose that there exists a proper closed d^{-1}-ball $\overline{B_{d^{-1}}}(x, \alpha)$ which is not $P \vee Q$-compact. Therefore, there is $y_0 \in X$ such that $d(y_0, x) > \alpha$

and there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \overline{B_{d^{-1}}}(x,\alpha)$ which does not admit a $P\vee Q$-cluster point in X since $\overline{B_{d^{-1}}}(x,\alpha)$ is a P-closed set. Let $A_n = \overline{\{x_n\}}^Q \cup \overline{\{y_0\}}^Q$ for all $n \in \mathbb{N}$. Let us prove that the sequence $\{A_n\}_{n\in\mathbb{N}}$ F_P-converges to $\overline{\{y_0\}}^Q$. Let $V^+ \cap V_1^- \cap \ldots \cap V_n^-$ be an F_P -open set containing $\overline{\{y_0\}}^Q$. Clearly $A_n \in V_1^- \cap \ldots \cap V_n^-$. Suppose now, to obtain a contradiction, that given $k \in \mathbb{N}$ we can find $n_k \geq k$ such that $A_{n_k} \notin V^+$. Choose $y_{n_k} \in A_{n_k}$ such that $y_{n_k} \notin V$ for all $k \in \mathbb{N}$. Since $A_{n_k} = \overline{\{x_{n_k}\}}^Q \cup \overline{\{y_0\}}^Q$ and $\overline{\{y_0\}}^Q \in V^+$, it is easy to show that $x_{n_k} \notin V$. Hence, since $X\backslash V$ is a $P\vee Q$-compact set, $\{x_{n_k}\}_{k\in\mathbb{N}}$ admits a $P\vee Q$-cluster point z. A contradiction, so $\{A_n\}_{n\in\mathbb{N}}$ is F_P-convergent to $\overline{\{y_0\}}^Q$. Let us show now that $\{d(A_n,x)\}_{n\in\mathbb{N}}$ does not converge to $d(\overline{\{y_0\}}^Q,x)$ in the lower topology of $\mathbb{R}$, i.e. $\{A_n\}_{n\in\mathbb{N}}$ is not $\mathcal{T}(W_d)$-convergent to $\overline{\{y_0\}}^Q$. We have that $d(A_n,x) \leq d(x_n,x) \leq \alpha$. Moreover, $d(\overline{\{y_0\}}^Q,x) > \alpha$ since if $z \in \overline{\{y_0\}}^Q$ then $d(y_0,z) = 0$, so $\alpha < d(y_0,x) \leq d(y_0,z) + d(z,x) = d(z,x)$. Therefore

$$d(\overline{\{y_0\}}^Q,x) - d(A_n,x) \geq d(\overline{\{y_0\}}^Q,x) - \alpha > 0.$$

Consequently, $F_P \neq \mathcal{T}(W_d)$ on $CL_0^Q(X)$. A contradiction.
If there is a proper closed d-ball $\overline{B_d}(x,\alpha)$ which is not $\mathcal{T}(d^s)$-compact, we can prove the statement in a similar way.

Suppose now that (X,d) has nice closed balls. By Proposition 2.4, we only have to show that $\mathcal{T}^+(W_d) \subseteq F_P^+$ on $CL_0^Q(X)$ and $\mathcal{T}^+(W_{d^{-1}}) \subseteq F_Q^+$ on $CL_0^P(X)$. Let $x \in X$ and $\alpha \geq 0$. Let us consider the $\mathcal{T}^+(W_d)$-open set $d(\cdot,x)^{-1}(\alpha,+\infty)$. We first suppose that $\overline{B_{d^{-1}}}(x,\alpha) \neq X$. Fix $\beta > \alpha$ such that $\overline{B_{d^{-1}}}(x,\beta)$ is not equal to X. Then $\overline{B_{d^{-1}}}(x,\beta)$ is $P\vee Q$-compact. If $A \in CL_0^Q(X)$ and $d(A,x) = \alpha$, we can find a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq \overline{B_{d^{-1}}}(x,\beta) \cap A$ such that $\{d(a_n,x)\}_{n\in\mathbb{N}}$ converges to $d(A,x)$. Since $\overline{B_{d^{-1}}}(x,\beta)$ is $P\vee Q$-compact, there is a $P\vee Q$-convergent subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ of $\{a_n\}_{n\in\mathbb{N}}$. If we denote by a its limit we obtain that $a \in A$ and $d(a,x) = \alpha$. Therefore, $d(\cdot,x)^{-1}(\alpha,+\infty) = (X\backslash\overline{B_{d^{-1}}}(x,\alpha))^+$ which is a F_P-open set.

On the other hand, if $\overline{B_{d^{-1}}}(x,\alpha) = X$ then $d(\cdot,x)^{-1}(\alpha,+\infty) = \varnothing \in F_P$.

In a similar way it can be proved $\mathcal{T}^+(W_{d^{-1}}) \subseteq F_Q^+$ on $CL_0^P(X)$. □

Remark 2.8. We observe that by using the above proof, it can be shown: $F_P^+ = \mathcal{T}^+(W_d)$ on $CL_0^Q(X)$ and $F_Q^+ = \mathcal{T}(W_{d^{-1}}^+)$ on $CL_0^P(X)$ if and only if (X,d) has nice closed balls. Therefore, we deduce that the double Fell topology agrees with the double Wijsman topology if and only if the double upper Fell topology agrees with the double upper Wijsman topology.

In the following Remark, we give an example where the above theorem does not work if we change either the definition of the double Fell topology or the definition of nice closed balls. We will use the following definition.

Definition 2.9. *Let (X,P,Q) be a bitopological space. We define the double upper Vietoris topological space as the double topological space $((CL_0^Q(X),V_P^+),$ $(CL_0^P(X),V_Q^+))$ where V_P^+ is the topology generated by all sets of the form*

$G^+ = \{A \in CL_0^Q(X) : A \subseteq G\}$ *where* G *is a* P*-open set;* V_Q^+ *is defined in a similar way by writing* P *instead of* Q *and* Q *instead of* P.

The double lower Vietoris topological space is defined as the double topological space $((CL_0^Q(X), V_P^-), (CL_0^P(X), V_Q^-))$ *where* V_P^- *is generated by all sets of the form* $G^- = \{A \in CL_0^Q(X) : A \cap G \neq \varnothing\}$ *where* G *is* P*-open;* V_Q^- *is defined in a similar way by writing* P *instead of* Q *and* Q *instead of* P.

The double Vietoris topological space is defined as the double topological space $((CL_0^Q(X), V_P),\ (CL_0^P(X), V_Q))$ *where* $V_P = V_P^+ \vee V_P^-$ *and* $V_Q = V_Q^+ \vee V_Q^-$.

Remark 2.10. Now we motivate one fact about the definition of the double Fell topology. We think that, maybe, the natural definition for F_P is to be the topology generated by the sets of the form G^+ and V^- where G and V are P-open sets and $X \backslash G$ is Q-compact. In a similar way, we define the F_Q hypertopology. We give an example where, with this definition, Theorem 2.7 is not true.

Let d be the quasi-metric on $\mathbb{N}$ given by

$$d(n,m) = \begin{cases} \frac{1}{m} & \text{if } n < m \\ 1 & \text{if } n > m \\ 0 & \text{if } n = m \end{cases}.$$

We consider the quasi-metric space $(\mathbb{N}, d)$. Let $P = \mathcal{T}(d)$ and $Q = \mathcal{T}(d^{-1})$. We claim that $F_P = V_P$ on $\mathcal{P}_0(X)$ and $F_Q = V_Q$ on $\mathcal{P}_0(X)$. Since Proposition 2.4 is also true with this definition for the double Fell topology, we can deduce, using that $\mathcal{T}(W_d) \subseteq V_P$ and $\mathcal{T}(W_{d^{-1}}) \subseteq V_Q$ on $\mathcal{P}_0(X)$ (see [16]), that $F_P = \mathcal{T}(W_d)$ on $CL_0^Q(X)$ and $F_Q = \mathcal{T}(W_{d^{-1}})$ on $CL_0^P(X)$ but (X, d) has not nice closed balls. We only have to prove that $V_P^+ \subseteq F_P^+$ and $V_Q^+ \subseteq F_Q^+$.

Let $G \in P$ and we consider the V_P^+-open set G^+. Since G is a $\mathcal{T}(d)$-open set, it easy to prove that $\mathbb{N} \backslash G$ is a finite set, so it is Q-compact. Therefore, $G^+ \in F_P^+$ so $V_P^+ = F_P^+$ on $\mathcal{P}_0(X)$.

On the other hand, let us suppose that $G \in Q$ and we consider the V_Q^+-open set G^+. It is clear that every subset of X is P-compact. Hence, $F_Q^+ = V_Q^+$ on $\mathcal{P}_0(X)$. We observe that this statement is not true if we consider the topology $P \vee Q$, since it is the discrete topology.

We consider the closed ball $\overline{B_d}(n, 1/n) = \{n, n+1, \ldots\}$. It is evident that this set is not $P \vee Q$-compact.

We notice that if we change the definition of a quasi-pseudo-metric space having nice closed balls by saying that a quasi-pseudo-metric space has this property if the proper closed d-balls are $\mathcal{T}(d^{-1})$-compact and the proper closed d^{-1}-balls are $\mathcal{T}(d)$-compact the result is not true either. The preceding example shows that. The above ball is not Q-compact, since it is an infinite set and Q is the discrete topology.

Remark 2.11. We claim that if (X, d) is a quasi-metric space having nice closed balls then $\mathcal{T}(d) = \mathcal{T}(d^{-1})$. Let us show this.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a $\mathcal{T}(d^{-1})$-convergent sequence to x. Then, if $m \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < 1/2m$ for all $n \geq n_0$. In addition, we can find a proper closed d^{-1}-ball with center x (otherwise, since $(X, \mathcal{T}(d^{-1}))$ is a T_1 space, we would have that $X = \{x\}$ and the result is obvious). Let $\overline{B_{d^{-1}}}(x,\alpha)$ be such a ball. Then, $x_n \in \overline{B_{d^{-1}}}(x,\alpha)$ if n is greater or equal than a certain natural number n_1. Hence, $\{x_n\}_{n\in\mathbb{N}}$ admits a $\mathcal{T}(d)$-cluster point y, and, furthermore for each $m \in \mathbb{N}$

$$d(y,x) \leq d(y,x_n) + d(x_n,x) < \frac{1}{m}$$

for a sufficient large n, so $x = y$ and, therefore, $\mathcal{T}(d) \subseteq \mathcal{T}(d^{-1})$.
The other inclusion is similar.

In general, the equality $\mathcal{T}(d) = \mathcal{T}(d^{-1})$ is not true in a quasi-pseudo-metric space (X,d) having nice closed balls. Let $\mathbb{Z}$ be the set of integers. The Khalimsky line consists of $\mathbb{Z}$ with the topology generated by all sets of the form $\{2n-1, 2n, 2n+1\}$, $n \in \mathbb{Z}$. It is introduced in image processing in [11]. Then the quasi-pseudo-metric d defined on $\mathbb{Z}$ by $d(2n, 2n-1) = d(2n, 2n+1) = d(n,n) = 0$ for all $n \in \mathbb{N}$ and $d(x,y) = 1$ otherwise, generates the topology of the Khalimsky line. It is clear that the proper closed d-balls and the proper closed d^{-1}-balls are finite so they are $\mathcal{T}(d^s)$-compact. Furthermore, it is obvious that $\mathcal{T}(d) \neq \mathcal{T}(d^{-1})$.

When we consider a quasi-metric space we obtain the following result.

Corollary 2.12. *Let (X,d) be a quasi-metric space and $\mathcal{T}(d) = P$, $\mathcal{T}(d^{-1}) = Q$. The following statements are equivalent.*

- *i) $F_P = \mathcal{T}(W_d)$ and $F_Q = \mathcal{T}(W_{d^{-1}})$ on $CL_0^s(X)$.*
- *ii) $F_P = \mathcal{T}(W_d)$ on $CL_0^P(X)$ and $F_Q = \mathcal{T}(W_{d^{-1}})$ on $CL_0^Q(X)$.*
- *iii) $F_P = \mathcal{T}(W_d)$ on $CL_0^Q(X)$ and $F_Q = \mathcal{T}(W_{d^{-1}})$ on $CL_0^P(X)$.*
- *iv) (X,d) has nice closed balls.*
- *v) $P = Q$ and (X,d) has nice closed balls.*

Proof. $i) \Rightarrow ii)$ and $i) \Rightarrow iii)$ are obvious. $ii)$ implies $iv)$ can be shown as above, taking into account that (X,P) and (X,Q) are T_1 spaces. By the above Theorem we obtain $iii) \Rightarrow iv)$. $iv) \Rightarrow v)$ is the above Remark. The implication $v) \Rightarrow i)$ is [3, Theorem 5.1.10]. □

Remark 2.13. Let us observe that the above Corollary is not true when we consider a quasi-pseudo-metric space. Let us show that $ii) \Rightarrow iv)$ fails. Consider the quasi-pseudo-metric space $(\mathbb{R}, \ell)$ where ℓ denotes the lower quasi-pseudo-metric. Clearly, we have that $F_P = \mathcal{T}(W_\ell)$ on $CL_0^P(X)$ and $F_Q = \mathcal{T}(W_u)$ on $CL_0^Q(X)$ where $P = \mathcal{T}(\ell)$ and $Q = \mathcal{T}(u)$. However, $(\mathbb{R}, \ell)$ does not have nice closed balls, since the closed ℓ-balls and closed u-balls are not bounded.

3. Other Fell type topologies

As we have already observed, we have various possibilities in order to define the Fell hypertopology in the nonsymmetric situation. This section is devoted to describe the advantages and disadvantages of our definition compared with other ones.

We begin giving the definition that we think is more natural.

Definition 3.1. *Let* (X, P, Q) *be a bitopological space. We define the* double upper fine Fell space *as the double space* $((CL_0^Q(X), FF_P^+), (CL_0^P(X), FF_Q^+))$ *where* FF_P^+ *is the topology generated by all sets of the form* G^+ *where* G *is a* P*-open set and* $X \backslash G$ *is* Q*-compact; the topology* FF_Q^+ *is defined in the corresponding natural way.*

The double fine Fell space *is the double space* $((CL_0^Q(X), FF_P), (CL_0^P(X), FF_Q))$ *where* $FF_P = FF_P^+ \vee F_P^-$ *and* $FF_Q = FF_Q^+ \vee F_Q^-$.

With this definition, not all the results proved in the previous section work (see Remark 2.10).

Another possible definition is suggested by Burdick's investigations ([5, 6, 7]). He looked for a context in which he considered separately the upper and lower Vietoris topologies on a hyperspace and explored the interactions between them.

Definition 3.2. *Let* (X, P, Q) *be a bitopological space. The* double mixed Fell space *is the double space* $((CL_0^Q(X), MF_P), (CL_0^P(X), MF_Q))$ *where* $MF_P = F_P^+ \vee F_Q^-$ *and* $MF_Q = F_Q^+ \vee F_P^-$.

We call this hypertopology mixed, because we interchange the natural lower hypertopologies between the two hyperspaces that we construct. We notice that all results obtained in the previous section are true using this definition whenever we change the definition of the Wijsman lower hypertopology. Let us observe that our main results only use the upper hypertopologies since the double lower Wijsman topology always coincides with the double lower Fell topology. However, we think that is not a natural definition, although it provides a nontrivial topology on the bitopological space $(\mathbb{R}, \mathcal{T}(\ell), \mathcal{T}(u))$.

Furthermore, we can give another definition.

Definition 3.3. *Let* (X, P, Q) *be a bitopological space. The* double mixed fine Fell space *is the double space* $((CL_0^Q(X), MFF_P), (CL_0^P(X), MFF_Q))$ *where* $MFF_P = FF_P^+ \vee F_Q^-$ *and* $MFF_Q = FF_Q^+ \vee F_P^-$.

Unfortunately, the double mixed fine Fell space has the same problems of generalization as the double fine Fell space. However, it is an appropriate Fell type topology to study epiconvergence of lower semicontinuous functions in the double setting, which will be discussed elsewhere.

4. The Kuratowski-Painlevé convergence

In this section, we propose a definition for the Kuratowski-Painlevé convergence in the nonsymmetric case and obtain some results about the relationships of this type of convergence and some hypertopologies.

The Kuratowski-Painlevé convergence was introduced to describe the limit of a net in terms of the members of the net itself. We propose the following definitions.

Definition 4.1. *Let (X, P, Q) be a bitopological space and $\{A_\lambda\}_{\lambda\in\Lambda}$ a net of subsets of X.*

i) A point x_0 belongs to P-LiA_λ (resp. Q-LiA_λ) and we say that x_0 is a P-limit point *(resp.* Q-limit point*) of $\{A_\lambda\}_{\lambda\in\Lambda}$ if each Q-neighborhood (resp. P-neighborhood) of x_0 intersects A_λ for all λ in some residual subset of Λ.*

ii) A point x_0 belongs to P-LsA_λ (resp. Q-LsA_λ) and we say that x_0 is a P-cluster point *(resp.* Q-cluster point*) of $\{A_\lambda\}_{\lambda\in\Lambda}$ if each Q-neighborhood (resp. P-neighborhood) of x_0 intersects A_λ for all λ in some cofinal subset of Λ.*

The proof of the following proposition is straightforward.

Proposition 4.2. *Let (X, P, Q) be a bitopological space. If $\{A_\lambda\}_{\lambda\in\Lambda}$ is a net of subsets of X then P-LiA_λ (resp. Q-LiA_λ) and P-LsA_λ (resp. Q-LsA_λ) are Q-closed sets (resp. P-closed sets).*

Definition 4.3. *Let (X, P, Q) be a bitopological space and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a net of subsets of X and $A \in \mathcal{P}(X)$.*

We say that $\{A_\lambda\}_{\lambda\in\Lambda}$ is P-Kuratowski-Painlevé upper convergent *(resp.* Q-Kuratowski-Painlevé upper convergent*) to A if P-$LsA_\lambda \subseteq A$ (resp. Q-$LsA_\lambda \subseteq A$). We write $A = K_P^+\text{-}\lim A_\lambda$ (resp. $A = K_Q^+\text{-}\lim A_\lambda$).*

We say that $\{A_\lambda\}_{\lambda\in\Lambda}$ is P-Kuratowski-Painlevé lower convergent *(resp.* Q-Kuratowski-Painlevé lower convergent*) to A if $A \subseteq P$-LiA_λ (resp. $A \subseteq Q$-LiA_λ). We write $A = K_P^-\text{-}\lim A_\lambda$ (resp. $A = K_Q^-\text{-}\lim A_\lambda$).*

We say that $\{A_\lambda\}_{\lambda\in\Lambda}$ is P-Kuratowski-Painlevé convergent *(resp.* Q-Kuratowski-Painlevé convergent*) to A if $A = P$-$LiA_\lambda = P$-LsA_λ (resp. $A = Q$-$LiA_\lambda = Q$-LsA_λ). We write $A = K_P\text{-}\lim A_\lambda$ (resp. $A = K_Q\text{-}\lim A_\lambda$).*

With these definitions, we can extend a classical result which gives a relationship between the Kuratowski-Painlevé convergence and the convergence in the Fell topology. We need the following definition.

Definition 4.4 ([15])**.** *A bitopological space (X, P, Q) is said to be locally bicompact if every point has a neighborhood base in P and a neighborhood base in Q whose elements are $P \vee Q$-compact sets.*

Theorem 4.5. *Let (X, P, Q) be a bitopological space, $A \in CL_0^Q(X)$ (resp. $A \in CL_0^P(X)$) and $\{A_\lambda\}_{\lambda\in\Lambda}$ a net in $CL_0^Q(X)$ (resp. $CL_0^P(X)$).*

i) $A = K_P^-\text{-}\lim A_\lambda$ (resp. $A = K_Q^-\text{-}\lim A_\lambda$) if and only if $A = F_Q^-\text{-}\lim A_\lambda$ (resp. $A = F_P^-\text{-}\lim A_\lambda$).

ii) If $A = K_P^+\text{-}\lim A_\lambda$ (resp. $A = K_Q^+\text{-}\lim A_\lambda$) then $A = F_P^+\text{-}\lim A_\lambda$ (resp. $A = F_Q^+\text{-}\lim A_\lambda$).

iii) If (X, P, Q) is a locally bicompact quasi-uniformizable bitopological space and $A = F_P^+ - \lim A_\lambda$ (resp. $A = F_Q^+ - \lim A_\lambda$) then $A = K_P^+ - \lim A_\lambda$ (resp. $A = K_Q^+ - \lim A_\lambda$).

Proof. *i*) This statement is straightforward.

ii) Let us suppose that $A = K_P^+\text{-}\lim A_\lambda$. Then

$$P - Ls(K \cap A_\lambda) \subseteq P - LsA_\lambda \subseteq A$$

for all $P \vee Q$-compact set K. Let K_0 be a $P \vee Q$-compact and P-closed set such that $K \cap A_\lambda \neq \varnothing$ for a cofinal subset of Λ. Let us prove that $P\text{-}LsA_\lambda \neq \varnothing$. Choose $k_\lambda \in K \cap A_\lambda$ for all λ belonging to a cofinal subset Λ_0 of Λ. We obtain that $\{k_\lambda\}_{\lambda\in\Lambda_0}$ admits a $P \vee Q$-cluster point $k \in K$. It is evident that $k \in P\text{-}LsA_\lambda \cap K \subseteq A$ so $A \cap K \neq \varnothing$. Therefore, if $A \cap K = \varnothing$ then $A_\lambda \cap K = \varnothing$ eventually. Hence, $A = F_P^+\text{-}\lim A_\lambda$.

The other statement can be proved in a similar way.

iii) Let us suppose that $P\text{-}LsA_\lambda \not\subseteq A$. Let $x \in (P\text{-}LsA_\lambda)\backslash A$. Since (X, P, Q) is a locally bicompact quasi-uniformizable bitopological space, we can find a P-closed and $P \vee Q$-compact Q-neighborhood V of x such that $V \cap A = \varnothing$ but $A_\lambda \cap V \neq \varnothing$ frequently. Therefore, $A \neq F_P^+\text{-}\lim A_\lambda$ which is a contradiction. Consequently, $P\text{-}LsA_\lambda \subseteq A$.

The same reasoning proves the statement for Q. □

We characterize the Kuratowski-Painlevé convergence in terms of sequences of points.

Proposition 4.6. *Let (X, P, Q) be a quasi-pseudo-metrizable bitopological space and d a quasi-pseudo-metric on X compatible with the bitopological space. A sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ is P-Kuratowski-Painlevé lower convergent to a set A if and only if each point $a \in A$ is the limit of some $\mathcal{T}(d^{-1})$-convergent sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $a_n \in A_n$ for all $n \in \mathbb{N}$.*

Proof. Let us suppose that $\{A_n\}_{n\in\mathbb{N}}$ is F_Q^--convergent to $A \in \mathcal{P}(X)$. Pick $a \in A$ (if $A = \varnothing$ the result is evident). Given $k \in \mathbb{N}$, we have that $A \in B_{d^{-1}}(a, 1/k)^-$ so there exists $n_k \in \mathbb{N}$ such that $A_n \in B_{d^{-1}}(a, 1/k)^-$ for all $n \geq n_k$. We can suppose that $n_1 < n_2 < \ldots < n_k < \ldots$. Therefore, we can find $a_n \in A_n \cap B_{d^{-1}}(a, 1/k)$ for all $n_{k+1} \geq n \geq n_k + 1$. If we consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ where if $n \in \{1, \ldots, n_1\}$ we consider a fixed point $a_n \in A_n$, we have that this sequence is $\mathcal{T}(d^{-1})$-convergent to a.

Conversely, let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence and $A \subseteq X$ satisfying our assumption. If $a \in A \cap G$ where G is a $\mathcal{T}(d^{-1})$-open set, there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ $\mathcal{T}(d^{-1})$-convergent to a verifying that $a_n \in A_n$ for all $n \in \mathbb{N}$. On the other hand, we can find $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $B_{d^{-1}}(a, \varepsilon) \subseteq G$ and $d^{-1}(a, a_n) < \varepsilon$ for all $n \geq n_0$. Therefore, $A_n \in G^-$ for all $n \geq n_0$. □

We can also obtain a characterization of the Kuratowski-Painlevé upper convergence in terms of sequences.

Proposition 4.7. *Let (X, P, Q) be a quasi-pseudo-metrizable bitopological space and d a quasi-pseudo-metric on X compatible with the bitopological space. A sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ is P-Kuratowski-Painlevé upper convergent to a set A if and only if whenever there exist positive integers $n_1 < n_2 < \ldots$ and $a_k \in A_{n_k}$ for all $k \in \mathbb{N}$ such that $\{a_k\}_{k\in\mathbb{N}}$ is $\mathcal{T}(d^{-1})$-convergent to a then $a \in A$.*

Proof. Let us suppose that $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ is P-Kuratowski-Painlevé convergent to A. If $\{a_n\}_{n\in\mathbb{N}}$ is a sequence as in the statement, it is evident that $a \in P\text{-}LsA_n \subseteq A$.

Now, let $a \in P\text{-}LsA_n$ and $\{B_{d^{-1}}(a,1/n) : n \in \mathbb{N}\}$ a countable $\mathcal{T}(d^{-1})$-neighborhood base of a. Choose $n_1 \in \mathbb{N}$ such that $B_{d^{-1}}(a,1) \cap A_{n_1} \neq \varnothing$. Since $\{n \in \mathbb{N} : B_{d^{-1}}(a,1/2) \cap A_n\}$ is infinite, we can find $n_2 > n_1$ verifying $B_{d^{-1}}(a,1/2) \cap A_{n_2} \neq \varnothing$. Following this procedure, we can construct a strictly increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ such that $B_{d^{-1}}(a,1/k) \cap A_{n_k} \neq \varnothing$ for all $k \in \mathbb{N}$. If $a_k \in B_{d^{-1}}(a,1/k) \cap A_{n_k}$ for all $k \in \mathbb{N}$, it is evident that this sequence is $\mathcal{T}(d^{-1})$-convergent to a, so by assumption $a \in A$. □

Now, we can extend an interesting result due to Mrowka (see [14]).

Theorem 4.8 (Mrowka)**.** *Let (X,P,Q) be a bitopological space and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a net in $\mathcal{P}(X)$. Then $\{A_\lambda\}_{\lambda\in\Lambda}$has a P-Kuratowski-Painlevé convergent subnet and a Q-Kuratowski-Painlevé convergent subnet.*

Proof. Let $\mathcal{B}$ be a base for the topology Q. Let us consider the space $\{0,1\}$ with the discrete topology. For each $\lambda \in \Lambda$, we define $f_\lambda : \mathcal{B} \to \{0,1\}$ as follows:

$$f_\lambda(V) = \begin{cases} 1 & \text{if } A_\lambda \cap V \neq \varnothing \\ 0 & \text{if } A_\lambda \cap V = \varnothing \end{cases}.$$

By the Tychonoff's theorem, $\{f_\lambda\}_{\lambda\in\Lambda}$ has a convergent subnet $\{f_{\lambda'}\}_{\lambda'\in\Lambda'}$. For each $V \in \mathcal{B}$, we obtain that $f_{\lambda'}(V) = 1$ eventually if and only if $f_{\lambda'}(V) = 1$ frequently. Therefore, if $A_\lambda \cap V \neq \varnothing$ frequently then $A_\lambda \cap V \neq \varnothing$ eventually, so $\{A_{\lambda'}\}_{\lambda'\in\Lambda'}$ is P-Kuratowski-Painlevé convergent.

The reasoning for P is similar. □

Theorem 4.9. *Let (X,d) be a quasi-pseudo-metric space. Let $P = \mathcal{T}(d)$ and $Q = \mathcal{T}(d^{-1})$. Let us consider $\{A_\lambda\}_{\lambda\in\Lambda}$a net in $CL_0^Q(X)$ and $\{B_\gamma\}_{\gamma\in\Gamma}$ a net in $CL_0^P(X)$.*

i) If $A = \mathcal{T}^+(W_d)\text{-}\lim A_\lambda$ and $B = \mathcal{T}^+(W_{d^{-1}})\text{-}\lim B_\gamma$ then $A = K_P^+\text{-}\lim A_\lambda$ and $B = K_Q^+\text{-}\lim B_\gamma$.

ii) $A = K_P^+\text{-}\lim A_\lambda$ and $B = K_Q^+\text{-}\lim B_\gamma$ implies $A = \mathcal{T}^+(W_d)\text{-}\lim A_\lambda$ and $B = \mathcal{T}^+(W_{d^{-1}})\text{-}\lim B_\gamma$ if and only if (X,d) has nice closed balls.

Proof. i) Let us suppose that $A = \mathcal{T}^+(W_d)\text{-}\lim A_\lambda$ and $B = \mathcal{T}^+(W_{d^{-1}})\text{-}\lim B_\gamma$. Let $a \in P\text{-}LsA_\lambda$ and suppose that $a \notin A$. Thus $d(A,a) > 0$. Therefore, given $0 < \delta < d(A,a)$ since $a \in P\text{-}LsA_\lambda$ we obtain that $d(A_\lambda,a) < \delta$ frequently so

$$d(A,a) - d(A_\lambda,a) > d(A,a) - \delta > 0$$

frequently which contradicts that $d(A,a) = \mathcal{T}(\ell)\text{-}\lim d(A_\lambda,a)$. The same reasoning shows that $B \subseteq Q\text{-}LsB_\gamma$.

ii) This statement is similar to the proof of Theorem 2.7. □

We recall the following definitions (see [15]).

Definition 4.10. *Let (X,P,Q) be a quasi-uniformizable bitopological space and $\mathcal{U}$ a quasi-uniformity compatible with the bitopological space. The* double upper $\mathcal{U}$-proximal topological space *is defined as the double topological space $((CL_0^Q(X),\mathcal{T}^+(\delta_{\mathcal{U}})),(CL_0^P(X),\mathcal{T}^+(\delta_{\mathcal{U}^{-1}})))$ where $\mathcal{T}^+(\delta_{\mathcal{U}})$ is the topology generated by all sets of the form $G^{++}=\{A\in CL_0^Q(X):$ there exists $U\in\mathcal{U}$ such that $U(A)\subseteq G\}$ where G is a P-open set. The topology $\mathcal{T}(\delta_{\mathcal{U}^{-1}})$ is defined in a similar way by writing P instead of Q, Q instead of P and $\mathcal{U}^{-1}$ instead of $\mathcal{U}$. The pair $(\mathcal{T}^+(\delta_{\mathcal{U}}),\mathcal{T}^+(\delta_{\mathcal{U}^{-1}}))$ is called the* double upper $\mathcal{U}$-proximal topology.

The double lower $\mathcal{U}$-proximal topological space *is defined as the double topological space $((CL_0^Q(X),\mathcal{T}^-(\delta_{\mathcal{U}})),(CL_0^P(X),\mathcal{T}^-(\delta_{\mathcal{U}^{-1}})))$ where this double topological space coincides with the double lower Fell topological space. The pair $(\mathcal{T}^-(\delta_{\mathcal{U}}),\mathcal{T}^-(\delta_{\mathcal{U}^{-1}}))$ is called the* double lower $\mathcal{U}$-proximal topology.

The double $\mathcal{U}$-proximal topological space *is defined as the double topological space $((CL_0^Q(X),\mathcal{T}(\delta_{\mathcal{U}})),(CL_0^P(X),\mathcal{T}(\delta_{\mathcal{U}^{-1}})))$ where $\mathcal{T}(\delta_{\mathcal{U}})=\mathcal{T}^+(\delta_{\mathcal{U}})\vee\mathcal{T}^-(\delta_{\mathcal{U}})$ and $\mathcal{T}(\delta_{\mathcal{U}^{-1}})=\mathcal{T}^+(\delta_{\mathcal{U}^{-1}})\vee\mathcal{T}^-(\delta_{\mathcal{U}^{-1}})$. The pair $(\mathcal{T}(\delta_{\mathcal{U}}),\mathcal{T}(\delta_{\mathcal{U}^{-1}}))$ is called the* double $\mathcal{U}$-proximal topology.

Definition 4.11. *Let (X,P,Q) be a quasi-uniformizable bitopological space and $\mathcal{U}$ a quasi-uniformity compatible with (X,P,Q). We say that (X,P,Q) has the* property pairwise star *if*

i) given $A\in CL_0^Q(X)$ and $B\in CL_0^P(X)$ with $A\overline{\delta_{\mathcal{U}}}B$ there exist $\{x_1,\dots,x_n\}\subseteq X$ and $U_1,\dots,U_n\in\mathcal{U}$ such that $A\cap(\cup_{i=1}^n\overline{U_i^{-1}(x_i)}^P)=\varnothing$ and $B\subseteq\cup_{i=1}^n U_i^{-1}(x_i)$.

ii) given $A\in CL_0^P(X)$ and $B\in CL_0^Q(X)$ with $A\overline{\delta_{\mathcal{U}^{-1}}}B$ there exist $\{x_1,\dots,x_n\}\subseteq X$ and $U_1,\dots,U_n\in\mathcal{U}$ such that $A\cap(\cup_{i=1}^n\overline{U_i(x_i)}^Q)=\varnothing$ and $B\subseteq\cup_{i=1}^n U_i(x_i)$.

Definition 4.12. *Let $(X,\mathcal{U})$ be a quasi-uniform space. We say that it has nice closed balls if every proper set of the form $\overline{U(x)}^{\mathcal{T}(\mathcal{U}^{-1})}$ or $\overline{U^{-1}(x)}^{\mathcal{T}(\mathcal{U})}$ is $\mathcal{T}(\mathcal{U}^s)$-compact, where $U\in\mathcal{U}$ and $x\in X$.*

Theorem 4.13. *Let (X,P,Q) be a quasi-uniformizable bitopological space and $\mathcal{U}$ a quasi-uniformity compatible with the bitopological space. Let us consider $\{A_\lambda\}_{\lambda\in\Lambda}$ a net in $CL_0^Q(X)$ and $\{B_\gamma\}_{\gamma\in\Gamma}$ a net in $CL_0^P(X)$.*

i) If $A=\mathcal{T}^+(\delta_{\mathcal{U}})\text{-}\lim A_\lambda$ and $B=\mathcal{T}^+(\delta_{\mathcal{U}^{-1}})\text{-}\lim B_\gamma$ then $A=K_P^+\text{-}\lim A_\lambda$ and $B=K_Q^+\text{-}\lim B_\gamma$.

ii) $A=K_P^+\text{-}\lim A_\lambda$ and $B=K_Q^+\text{-}\lim B_\gamma$ implies $A=\mathcal{T}^+(\delta_{\mathcal{U}})\text{-}\lim A_\lambda$ and $B=\mathcal{T}^+(\delta_{\mathcal{U}^{-1}})\text{-}\lim B_\gamma$ if and only if (X,P,Q) has the property pairwise star and $(X,\mathcal{U})$ has nice closed balls.

Proof. i) Suppose that $A=\mathcal{T}^+(\delta_{\mathcal{U}})\text{-}\lim A_\lambda$ and $B=\mathcal{T}^+(\delta_{\mathcal{U}^{-1}})\text{-}\lim B_\gamma$. If there exists $a\in P\text{-}LsA_\lambda\backslash A$, we can find $U\in\mathcal{U}$ such that $U^{-1}(a)\cap A=\varnothing$. It is easy to prove that $A\in(int_P V(A))^{++}$, where $V\in\mathcal{U}$ and $V^2\subseteq U$. Therefore, $A_\lambda\subseteq(int_P V(A))^{++}$ for all λ in a residual subset of Λ. Furthermore,

$V(A) \cap V^{-1}(a) = \varnothing$. On the other hand, since $a \in P\text{-}LsA_\lambda$ we obtain that $V^{-1}(a) \cap A_\lambda \neq \varnothing$ for a cofinal subset of Λ which is not possible. Therefore, $LsA_\lambda \subseteq A$. $B = \mathcal{T}^+(\delta_{\mathcal{U}^{-1}})\text{-}\lim B_\gamma$ implies $B = K_Q^+\text{-}\lim B_\gamma$ can be proved in a similar way.

ii) Let us suppose that there exist $U \in \mathcal{U}$ and $x \in X$ such that $\overline{U^{-1}(x)}^P$ is a proper set and is not $P \vee Q$-compact. Then, we can find $y_0 \in X \backslash \overline{U^{-1}(x)}^P$ and a net $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq \overline{U^{-1}(x)}^P$ such that it does not admit a $P \vee Q$-cluster point. We can easily deduce that the net $\{A_\lambda\}_{\lambda\in\Lambda}$ is P-Kuratowski-Painlevé upper convergent to $\overline{\{y_0\}}^Q$, where $A_\lambda = \overline{\{x_\lambda\}}^Q \cup \overline{\{y_0\}}^Q$ for all $\lambda \in \Lambda$. On the other hand, if we consider the $\mathcal{T}^+(\delta_{\mathcal{U}})$-open set $(int_P V(\overline{\{y_0\}}^Q))^{++}$ where $V, U_0 \in \mathcal{U}$, $V^2 \subseteq U_0$ and $U_0(y_0) \cap \overline{U^{-1}(x)}^P = \varnothing$, we have that $x_\lambda \notin int_P V(\overline{\{y_0\}}^Q)$ for all $\lambda \in \Lambda$ since if there exists $z \in \overline{\{y_0\}}^Q$ verifying $(z, x_\lambda) \in V$, for some $\lambda \in \Lambda$, we obtain that $(y_0, x_\lambda) \in U_0$ which is not possible. Consequently, $\{A_\lambda\}_{\lambda\in\Lambda}$ is not $\mathcal{T}^+(\delta_{\mathcal{U}})$-convergent to $\overline{\{y_0\}}^Q$. A contradiction. In a similar way, it can be proved that the proper sets of the form $\overline{U(x)}^Q$ are $P \vee Q$-compact. Consequently, $(X, \mathcal{U})$ has nice closed balls.

Therefore, (X, P, Q) is a locally bicompact space. Applying Theorem 4.5, we deduce that the double Fell topology agrees with the double proximal topology which implies (see [15]) that the bitopological space has the property pairwise star.

Conversely, if (X, P, Q) has the property pairwise star and $(X, \mathcal{U})$ has nice closed balls it can be proved (see [15]) that the double upper Fell topology agrees with the double upper proximal topology, and the statement follows directly. □

At last, we establish the relationship of the Kuratowski-Painlevé convergence and the convergence in the Vietoris hypertopology.

Theorem 4.14. *Let (X, P, Q) be a quasi-uniformizable bitopological space. Let us consider $\{A_\lambda\}_{\lambda\in\Lambda}$ a net in $CL_0^Q(X)$ and $\{B_\gamma\}_{\gamma\in\Gamma}$ a net in $CL_0^P(X)$. Then*

- *i) If $A = V_P^+\text{-}\lim A_\lambda$ and $B = V_Q^+\text{-}\lim B_\gamma$ then $A = K_P^+\text{-}\lim A_\lambda$ and $B = K_Q^+\text{-}\lim B_\gamma$.*
- *ii) $A = K_P^+\text{-}\lim A_\lambda$ and $B = K_Q^+\text{-}\lim B_\gamma$ implies $A = V_P^+\text{-}\lim A_\lambda$ and $B = V_Q^+\text{-}\lim B_\gamma$ if and only if $(X, P \vee Q)$ is a compact space.*

Proof. *i*) The proof is similar to the part *i*) of the above theorem.

ii) Let us suppose that $(X, P \vee Q)$ is not a compact space. Therefore, there exists a net $\{x_\lambda\}_{\lambda\in\Lambda}$ that does not admit a $P \vee Q$-cluster point. If we fix $y_0 \in X$ it is easy to prove that $\{\overline{\{x_\lambda\}}^Q \cup \overline{\{y_0\}}^Q\}_{\lambda\in\Lambda}$ is P-Kuratowski-Painlevé upper convergent to $\overline{\{y_0\}}^Q$. We can also prove that the net $\{\overline{\{x_\lambda\}}^P \cup \overline{\{y_0\}}^P\}_{\lambda\in\Lambda}$ is Q-Kuratowski-Painlevé upper convergent to $\overline{\{y_0\}}^P$.
On the other hand, since y_0 is not a $P \vee Q$-cluster point of $\{x_\lambda\}_{\lambda\in\Lambda}$, we can find

$U \in \mathcal{U}$ such that $x_\lambda \notin U^s(y_0)$ for all λ in whatever cofinal subset Λ_0 of Λ. We can choose Λ_0 in such way that we only have to distinguish two possibilities:

i) $x_\lambda \notin U(y_0)$ for all $\lambda \in \Lambda_0$. Therefore, it is evident that if we consider the P-open set $int_P V(\overline{\{y_0\}}^Q)$ where $V \in \mathcal{U}$ and $V^2 \subseteq U$, we have that $x_\lambda \notin V(\overline{\{y_0\}}^Q)$ for all $\lambda \in \Lambda_0$ so $\{\overline{\{x_\lambda\}}^Q \cup \overline{\{y_0\}}^Q\}_{\lambda \in \Lambda}$ is not V_P^+-convergent to $\overline{\{y_0\}}^Q$. A contradiction.

ii) $x_\lambda \notin U^{-1}(y_0)$ for all $\lambda \in \Lambda_0$. Reasoning as above we obtain that $\{\overline{\{x_\lambda\}}^P \cup \overline{\{y_0\}}^P\}_{\lambda \in \Lambda}$ is not V_Q^+-convergent to $\overline{\{y_0\}}^P$. A contradiction.

Conversely, since $(X, P \vee Q)$ is a compact space then $F_P^+ = V_P^+$ on $CL_0^Q(X)$ and $F_Q^+ = V_Q^+$ on $CL_0^P(X)$ (see [15]), so the proof is evident. □

We can also define the Kuratowski-Painlevé convergence in a bitopological sense in a different way.

Definition 4.15. *Let (X, P, Q) be a bitopological space and let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a net of subsets of X and $A \in \mathcal{P}(X)$.*

We say that $\{A_\lambda\}_{\lambda \in \Lambda}$ is P-mixed Kuratowski-Painlevé convergent *(resp.* Q-mixed Kuratowski-Painlevé convergent*) to A if $A = K_Q^-\text{-}\lim A_\lambda$ and $A = K_P^+\text{-}\lim A_\lambda$ (resp. $A = K_P^-\text{-}\lim A_\lambda$ and $A = K_Q^+\text{-}\lim A_\lambda$). We write $A = MK_P\text{-}\lim A_\lambda$ (resp. $A = MK_Q\text{-}\lim A_\lambda$).*

With this definition, we can also wonder under which conditions we can topologize this convergence. For the other definition, the condition was to make the space locally bicompact. We observe that we use this condition only to reconcile the Kuratowski-Painlevé upper convergence with the convergence in the upper Fell topology. So we have that this is an appropriate concept to work with the double Fell topology.

Proposition 4.16. *Let (X, P, Q) be a locally bicompact bitopological space. Then the mixed Kuratowski-Painlevé convergence agrees with the convergence in the double Fell topology.*

Consequently, the concept of Kuratowski-Painlevé convergence is suitable to obtain relationships with the double mixed Fell topology and the topologization of the mixed Kuratowski-Painlevé convergence is the double Fell topology.

It is natural to wonder if we can obtain conditions for the bitopological space (X, P, Q) in order to obtain the coincidence of the Kuratowski-Painlevé convergence and the other Fell topologies defined. We give a positive answer to this question. It is natural to look for other definitions of local compactness in bitopological spaces. The next definition is due to Stoltenberg.

Definition 4.17 ([22]). *Let (X, P, Q) be a bitopological space. We say that P is locally compact with respect to Q if for all $x \in X$ there exists a P-neighborhood G of x such that the Q-closure of G is Q-compact.*

We say that (X, P, Q) is pairwise locally compact *if P is locally compact with respect to Q and Q is locally compact with respect to P.*

This definition is not suitable here. If we want that our techniques work with this definition, we have to define another upper Fell topology for P considering that this topology is generated by the sets of the form G^+ where G is P-open and $X\backslash G$ is P-compact. The upper Fell topology for Q would be defined in a similar way. But this topology does not give good results.

Taking into account this, we propose the following definition.

Definition 4.18. *Let (X,P,Q) be a bitopological space. We say that (X,P,Q) is* bilocally compact *if (X,P) and (X,Q) are locally compact spaces.*

It is clear that if (X,P,Q) is locally bicompact then it is bilocally compact. With this definition we have the following obvious result.

Proposition 4.19. *Let (X,P,Q) be a bitopological space and suppose that $A \in CL_0^Q(X)$ (resp. $A \in CL_0^P(X)$) and that $\{A_\lambda\}_{\lambda\in\Lambda}$ is a net in $CL_0^Q(X)$ (resp. $CL_0^P(X)$). Then*

i) If $A = K_P^+\text{-}\lim A_\lambda$ (resp. $A = K_Q^+\text{-}\lim A_\lambda$) then $A = FF_P^+\text{-}\lim A_\lambda$ (resp. $A = FF_Q^+\text{-}\lim A_\lambda$).

ii) If (X,P,Q) is a bilocally compact pairwise Hausdorff bitopological space and $A = FF_P^+\text{-}\lim A_\lambda$ (resp. $A = FF_Q^+\text{-}\lim A_\lambda$) then $A = K_P^+\text{-}\lim A_\lambda$ (resp. $A = K_Q^+\text{-}\lim A_\lambda$).

Proof. The proof is similar to Theorem 4.5. □

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