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Common fixed point theorems for a countable family of fuzzy mappings

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Abstract. In this paper we prove fixed point theorems for countable families of fuzzy mappings satisfying contractive-type conditions and a rational inequality in left K-sequentially complete quasi-pseudo-metric spaces. These results generalize the corresponding ones obtained by other authors.

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1. Introduction

Heilpern [4] introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of Nadler's [7] fixed point theorem for multivalued mappings. Bose and Sahani [1] extended Heilpern's fixed point theorem to a pair of fuzzy contraction mappings. Park and Jeong [8] proved the existence of common fixed points for pairs of fuzzy mappings satisfying contractive-type conditions and rational inequality in complete metric spaces. In [2] the authors extended the theorems of [8] to left K-sequentially complete quasi-pseudo-metric spaces and in [3] they obtained fixed point theorems for fuzzy mappings in Smyth-sequentially complete quasi-metric spaces. This study was motivated by the efficiency of quasi-pseudo-metric spaces as tools to formulate and solve problems in theoretical computer science. In this paper we generalize the theorems of [2] and present a partial generalization for theorem 3.1 of [1] to countable families of fuzzy mappings in left K-sequentially complete quasi-pseudo-metric spaces.

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2. Preliminaries

Recall that (X, d) is a quasi-pseudo-metric space, and d is called a quasi-pseudo-metric if d is a non-negative real valued function on $X \times X$, which satisfies $d(x, x) = 0$ and $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in X$. If d is a quasi-pseudo-metric on X, then the function $d^{-1} : X \times X \to \mathbb{R}$, defined by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$, is also a quasi-pseudo-metric on X. Only if confusion is possible, we write d-closed or d^{-1}-closed, for example, to distinguish the topological concept in (X, d) or (X, d^{-1}).

We will make use of the following notion, which has been studied under different names by various authors (see e.g. [5], [9]).

Definition 2.1. *A sequence (x_n) in a quasi-pseudo-metric space (X, d) is called left K-Cauchy if for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $d(x_r, x_s) < \varepsilon$ for all $r, s \in \mathbb{N}$ with $k \leq r \leq s$. (X, d) is said to be left K-sequentially complete if each left K-Cauchy sequence in X converges (with respect to the topology $\mathcal{T}(d)$).*

A fuzzy set in X is an element of I^X where $I = [0, 1]$. The r-level set of A, denoted by A_r, is defined by $A_r = \{x \in X : A(x) \geq r\}$ if $r \in (0, 1]$, and $A_0 = cl\,\{x \in X : A(x) > 0\}$. For $x \in X$ we denote by $\{x\}$ the characteristic function of the ordinary subset $\{x\}$ of X. If $A,\ B \in I^X$, as usual in fuzzy theory, we denote $A \subset B$ when $A(x) \leq B(x)$, for each $x \in X$.

Let (X, d) be a quasi-pseudo-metric space. We consider the families of [2]

$$W'(X) = \{A \in I^X : A_1 \text{ is nonempty and } d\text{-closed}\}$$

$$W^*(X) = \{A \in W'(X) : A_1 \text{ is } d^{-1}\text{-countably compact}\}$$

and the following concepts for $A,\ B \in W'(X)$:

- $p(A, B) = \inf\{d(x, y) : x \in A_1, y \in B_1\} = d(A_1, B_1)$,
- $\delta(A, B) = \sup\{d(x, y) : x \in A_0, y \in B_0\}$ and
- $D(A, B) = \sup\{H(A_r, B_r) : r \in I\}$,

where $H(A_r, B_r)$ is the Hausdorff distance deduced from the quasi-pseudo-metric d.

We will use the following lemmas for a quasi-pseudo-metric space (X, d).

Lemma 2.2. *Let $x \in X$ and $A \in W'(X)$. Then $\{x\} \subset A$ if and only if $p(x, A) = 0$.*

Lemma 2.3. *$p(x, A) \leq d(x, y) + p(y, A)$, for any $x, y \in X,\ A \in W'(X)$.*

Lemma 2.4. *If $\{x_0\} \subset A$ then $p(x_0, B) \leq D(A, B)$ for each $A, B \in W'(X)$.*

Lemma 2.5. *Suppose $K \neq \varnothing$ is countably compact in the quasi-pseudo-metric space (X, d^{-1}). If $z \in X$, then there exists $k_0 \in K$ such that $d(z, K) = d(z, k_0)$.*

3. Fixed point theorems

First we generalize the theorems of [2] to countable families of fuzzy mappings. From now on (X, d) will be a quasi-pseudo-metric space.

Definition 3.1. *F is said to be a fuzzy mapping if F is a mapping from the set X into $W'(X)$. We say that $z \in X$ is a **fixed point** of F if $z \in F(z)_1$, i.e., $\{z\} \subset F(z)$.*

Theorem 3.2. *Let (X, d) be a left K-sequentially complete space and let $\{F_i : X \to W^*(X)\}_{i=1}^{\infty}$ be a countable family of fuzzy mappings. If there exists a constant h, $0 \le h < 1$, such that for each $x, y \in X$,*

$$\begin{aligned} D(F_i(x), F_{i+1}(y)) \quad \le \quad h \max\{ \quad & (d \wedge d^{-1})(x, y), \\ & p(x, F_i(x)), \\ & p(y, F_{i+1}(y)), \\ & \tfrac{p(x, F_{i+1}(y)) + p(y, F_i(x))}{2} \quad \}, \quad i = 1, 2, 3, \ldots \end{aligned}$$

$$\begin{aligned} D(F_i(x), F_1(y)) \quad \le \quad h \max\{ \quad & (d \wedge d^{-1})(x, y), \\ & p(x, F_i(x)), \\ & p(y, F_1(y)), \\ & \tfrac{p(x, F_1(y)) + p(y, F_i(x))}{2} \quad \}, \quad i = 2, 3, 4, \ldots, \end{aligned}$$

then there exists $z \in X$ such that $\{z\} \subset F_i(z)$, $i = 1, 2, 3, ldots$

Proof. Assume $\alpha = \sqrt{h}$. Let $x_{01} \in X$ and suppose $x_{11} \in (F_1(x_{01}))_1$. By Lemma 2.5 there exists $x_{12} \in (F_2(x_{11}))_1$ such that $d(x_{11}, x_{12}) = d(x_{11}, (F_2(x_{11}))_1)$ since $(F_2(x_{11}))_1$ is d^{-1}-countably compact. We have

$$d(x_{11}, x_{12}) = d(x_{11}, (F_2(x_{11}))_1) \le D_1(x_{11}, F_2(x_{11})) \le D(F_1(x_{01}), F_2(x_{11}))$$

Again, we can find $x_{21} \in X$ such that $x_{21} \in (F_1(x_{12}))_1$ and $d(x_{12,}x_{21}) \le D(F_2(x_{11}), F_1(x_{12}))$. Continuing in this manner we produce a sequence

$$\{x_{11}, x_{12}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, x_{34}, \ldots, x_{n1}, x_{n2}, \ldots, x_{n(n+1)}, \ldots\}$$

in X such that

$$\begin{aligned} & x_{n1} \in (F_1(x_{(n-1)n}))_1, \quad d(x_{(n-1)n}, x_{n1}) \le D(F_n(x_{(n-1)(n-1)}), F_1(x_{(n-1)n})), \\ & x_{n2} \in (F_2(x_{n1}))_1, \qquad\quad d(x_{n1}, x_{n2}) \le D(F_1(x_{(n-1)n}), F_2(x_{n1})), \end{aligned}$$

$n = 1, 2, \ldots$ and

$$x_{ni} \in (F_i(x_{n(i-1)}))_1, \quad d(x_{n(i-1)}, x_{ni}) \le D(F_{(i-1)}(x_{n(i-2)}), F_i(x_{n(i-1)})),$$

$i = 3, 4, \ldots, (n+1)$, $n = 2, 3, \ldots$

We will prove that (x_{rs}) is a left- K-Cauchy sequence. Firstly

$$\begin{aligned} d(x_{11}, x_{12}) \quad & \le \quad D(F_1(x_{01}), F_2(x_{11})) \\ & < \quad \alpha \max\{(d \wedge d^{-1})(x_{01}, x_{11}), p(x_{01}, F_1(x_{01})), p(x_{11}, F_2(x_{11})), \\ & \qquad \frac{p(x_{01}, F_2(x_{11})) + p(x_{11}, F_1(x_{01}))}{2}\} \\ & \le \quad \alpha \max\{(d \wedge d^{-1})(x_{01}, x_{11}), d(x_{01}, x_{11}), d(x_{11}, x_{12}), \\ & \qquad \frac{d(x_{01}, x_{12}) + d(x_{11}, x_{11})}{2}\} \\ & \le \quad \alpha \max\{d(x_{01}, x_{11}), d(x_{11}, x_{12}), \frac{d(x_{01}, x_{11}) + d(x_{11}, x_{12})}{2}\} \\ & = \quad \alpha \max\{d(x_{01}, x_{11}), d(x_{11}, x_{12})\} \end{aligned}$$

If $d(x_{11},x_{12}) > d(x_{01},x_{11})$, then $d(x_{11},x_{12}) < \alpha\, d(x_{11},x_{12})$, a contradiction. Thus, $d(x_{11},x_{12}) \leq d(x_{01},x_{11})$, and $d(x_{11},x_{12}) < \alpha\, d(x_{01},x_{11})$. Similarly

$$\begin{aligned} d(x_{12},x_{21}) &\leq D(F_2(x_{11}),F_1(x_{12})) \\ &< \alpha \max\{(d\wedge d^{-1})(x_{11},x_{12}), p(x_{11},F_2(x_{11})), p(x_{12},F_1(x_{12})), \\ &\quad \frac{p(x_{11},F_1(x_{12}))+p(x_{12},F_2(x_{11}))}{2}\} \\ &\leq \alpha \max\{d(x_{11},x_{12}), d(x_{12},x_{21})\} \end{aligned}$$

and $d(x_{12},x_{21}) < \alpha d(x_{11},x_{12}) < \alpha^2 d(x_{01},x_{11})$;

$$\begin{aligned} d(x_{21},x_{22}) &\leq D(F_1(x_{12}),F_2(x_{21})) \\ &< \alpha \max\{d(x_{12},x_{21}), d(x_{21},x_{22})\} \end{aligned}$$

and $d(x_{21},x_{22}) < \alpha d(x_{12},x_{21}) < \alpha^3 d(x_{01},x_{11})$;

$$\begin{aligned} d(x_{22},x_{23}) &\leq D(F_2(x_{21}),F_3(x_{22})) \\ &< \alpha \max\{d(x_{21},x_{22}), d(x_{22},x_{23})\} \end{aligned}$$

and so $d(x_{22},x_{23}) < \alpha d(x_{21},x_{22}) < \alpha^4 d(x_{01},x_{11})$.

Let $y_0 = x_{01}$. Now, we rename the constructed sequence (x_{rs}) as follows:

$$y_1 = x_{11}, y_2 = x_{12}, y_3 = x_{21}, y_4 = x_{22}, \ldots$$

and so, we obtain the sequence (y_n) of points of X such that

$$y_n = x_{ij} \in (F_j(y_{n-1}))_1 \text{ for } n = \tfrac{(i+1)i}{2} + j - 1$$

where $i = 1,2,\ldots,\ j = 1,\ldots,i+1$. By the above relations, one can verify that $d(y_n,y_{n+1}) < \alpha d(y_{n-1},y_n) < \alpha^n d(y_0,y_1)$ $n = 1,2,\ldots$ and for $m > n$ it is easy to see that $d(y_n,y_m) \leq \frac{\alpha^n}{1-\alpha} d(y_0,y_1)$. Then, from [6], (y_n) is a left K-Cauchy sequence in X, so there exists $z \in X$ such that $d(z,y_n) \to 0$ (and $d(z,x_{i(i+1)}) \to 0$, $d(z,x_{ii}) \to 0$, as $i \to \infty$).

Next, we show by induction that $p(z,F_j(z)) = 0$, $j = 1,2,3,\ldots$By lemmas 2.3, 2.4 we have:

$$\begin{aligned} p(z,F_1(z)) &\leq d(z,x_{12}) + p(x_{12},F_1(z)) \\ &\leq d(z,x_{12}) + D(F_2(x_{11}),F_1(z)). \end{aligned}$$

Similarly

$$\begin{aligned} p(z,F_1(z)) &\leq d(z,x_{23}) + p(x_{23},F_1(z)) \\ &\leq d(z,x_{23}) + D(F_3(x_{22}),F_1(z)) \end{aligned}$$

$$\begin{aligned} p(z,F_1(z)) &\leq d(z,x_{34}) + p(x_{34},F_1(z)) \\ &\leq d(z,x_{34}) + D(F_4(x_{33}),F_1(z)) \end{aligned}$$

and in general, for $i = 1,2,3,\ldots$

$$p(z,F_1(z)) \leq d(z,x_{i(i+1)}) + D(F_{i+1}(x_{ii}),F_1(z)) \tag{3.1}$$

But

$$
\begin{aligned}
D(F_{i+1}(x_{ii}),F_1(z)) &\leq h\,\max\{(d\wedge d^{-1})(x_{ii},z),p(x_{ii},F_{i+1}(x_{ii})),p(z,F_1(z)),\\
&\quad \frac{p(x_{ii},F_1(z))+p(z,F_{i+1}(x_{ii}))}{2}\}\\
&\leq h\,\max\{(d\wedge d^{-1})(x_{ii},z),d(x_{ii},x_{i(i+1)}),\\
&\quad d(z,x_{i(i+1)})+D(F_{i+1}(x_{ii}),F_1(z)),\\
&\quad \frac{d(x_{ii},x_{i(i+1)})+D(F_{i+1}(x_{ii}),F_1(z))+d(z,x_{i(i+1)}))}{2}\}. \qquad (3.2)
\end{aligned}
$$

In the sequel, the expression (2) will be denoted by $h\max\{C\}$. Now, there are four cases:

Case I: If $\max\{C\}=(d\wedge d^{-1})(x_{ii},z)$, then the inequality (3.1) becomes

$$
\begin{aligned}
p(z,F_1(z)) &\leq d(z,x_{i(i+1)}))+h\,(d\wedge d^{-1})(x_{ii},z)\\
&\leq d(z,x_{i(i+1)})+h\,d(z,x_{ii})\to 0,\ \text{as } i\to\infty.
\end{aligned}
$$

The other three cases **II-IV** coincide with the corresponding ones in [8], and $p(z,F_1(z))=0$ in all them. Thus, $p(z,F_1(z))=0$.

Suppose $p(z,F_j(z))=0$. Then, by lemma 2.2 $\{z\}\subset F_j(z)$ and by lemma 2.4 we have

$$
\begin{aligned}
p(z,F_{j+1}(z)) &\leq D(F_j(z),F_{j+1}(z))\\
&\leq h\max(d\wedge d^{-1})(z,z),p(z,F_j(z)),p(z,F_{j+1}(z)),\\
&\quad \frac{p(z,F_{j+1}(z))+p(z,F_j(z))}{2}\}\\
&= hp(z,F_{j+1}(z))
\end{aligned}
$$

Thus $(1-h)p(z,F_{j+1}(z))\leq 0$,and therefore $p(z,F_{j+1}(z))=0$. Hence, by lemma 2.2 it follows that $\{z\}\subset F_j(z)$, for each $j\in\mathbb{N}$. □

Theorem 3.3. *Let (X,d) be a left K-sequentially complete space and let $\{F_i:X\to W^*(X)\}_{i=1}^{\infty}$ be a countable family of fuzzy mappings. If there exists a constant $h\in]0,1[$, such that for each $x,y\in X$*

$$D(F_i(x),F_{i+1}(y))\leq k\,[p(x,F_i(x))\cdot p(y,F_{i+1}(y))]^{1/2},\qquad i=1,2,3,...$$

$$D(F_i(x),F_1(y))\leq k\,[p(x,F_i(x))\cdot p(y,F_1(y))]^{1/2},\qquad i=2,3,4,...,$$

then there exists $z\in X$ such that $\{z\}\subset F_i(z)$, $i=1,2,3,...$

Proof. Let $x_{01}\in X$. Let (x_{rs}) be the sequence in the proof of theorem 3.2.

Now,

$$
\begin{aligned}
d(x_{11},x_{12}) &\leq D(F_1(x_{01}),F_2(x_{11}))\leq\frac{1}{\sqrt{h}}D(F_1(x_{01}),F_2(x_{11}))\\
&\leq \frac{h}{\sqrt{h}}[p(x_{01},F_1(x_{01}))\cdot p(x_{11},F_2(x_{11}))]^{1/2}\\
&\leq h^{1/2}[d(x_{01},x_{11})\cdot d(x_{11},x_{12})]^{1/2}
\end{aligned}
$$

So, $d(x_{11}, x_{12}) \leq hd(x_{01}, x_{11})$. Similarly

$$\begin{aligned} d(x_{12}, x_{21}) &\leq \frac{1}{\sqrt{h}} D(F_2(x_{11}), F_1(x_{12})) \\ &\leq h^{1/2}[d(x_{11}, x_{12}) \cdot d(x_{12}, x_{21})]^{1/2} \end{aligned}$$

and $d(x_{12}, x_{21}) \leq hd(x_{11}, x_{12}) < h^2 d(x_{01}, x_{11})$;

$$\begin{aligned} d(x_{21}, x_{22}) &\leq \frac{1}{\sqrt{h}} D(F_1(x_{12}), F_2(x_{21})) \\ &\leq h^{1/2}[d(x_{12}, x_{21}) \cdot d(x_{21}, x_{22})]^{1/2} \end{aligned}$$

and $d(x_{21}, x_{22}) \leq hd(x_{12}, x_{21}) \leq h^3 d(x_{01}, x_{11})$;

$$\begin{aligned} d(x_{22}, x_{23}) &\leq \frac{1}{\sqrt{h}} D(F_2(x_{21}), F_3(x_{22})) \\ &\leq h^{1/2}\,[d(x_{21}, x_{22}) \cdot d(x_{22}, x_{23})]^{1/2} \end{aligned}$$

and $d(x_{22}, x_{23}) \leq hd(x_{21}, x_{22}) \leq h^4 d(x_{01}, x_{11})$.

Let $y_0 = x_{01}$. Now, we rename the constructed sequence (x_{rs}) as theorem 3.2. By the above relations one can verify that $d(y_n, y_{n+1}) \leq h\, d(y_{n-1}, y_n) \leq h^n\, d(y_0, y_1)$, $\quad n = 1, 2, ...$and from [6], (y_n) is a left K-Cauchy sequence in X. Then, there exists $z \in X$ such that $d(z, y_n) \to 0$.

Next we will show by induction that $p(z, F_j(z)) = 0$, $j = 1, 2, 3, ...$ By lemmas 2.3 and 2.4 it follows that for $i = 1, 2, 3, ...$

$$\begin{aligned} p(z, F_1(z)) &\leq d(z, x_{i(i+1)}) + p(x_{i(i+1)}, F_1(z)) \\ &\leq d(z, x_{i(i+1)}) + D(F_{i+1}(x_{ii}), F_1(z)) \\ &\leq d(z, x_{i(i+1)}) + h[d(x_{ii}, x_{i(i+1)}) \cdot p(z, F_1(z))]^{1/2} \to 0 \text{ as } i \to \infty. \end{aligned}$$

Then, $p(z, F_1(z)) = 0$. Now, suppose $p(z, F_j(z)) = 0$. Then, by lemmas 2.2 and 2.4 we have

$$\begin{aligned} p(z, F_{j+1}(z)) &\leq D(F_j(z), F_{j+1}(z)) \\ &\leq h[p(z, F_j(z)) \cdot p(z, F_{j+1}(z))]^{1/2} = 0. \end{aligned}$$

It follows that $p(z, F_{j+1}(z)) = 0$ and $\{z\} \subset F_j(z)$, for each $j \in \mathbb{N}$. □

Since $D(A, B) \leq \delta(A, B)$, $\forall A, B \in W'(X)$, then we deduce the following corollary.

Corollary 3.4. *Let (X, d) be a left K-sequentially complete space and let $\{F_i : X \to W^*(X)\}_{i=1}^{\infty}$ be a countable family of fuzzy mappings. If there exists a constant $h \in]0, 1[$, such that for each $x, y \in X$*

$$\begin{aligned} &\delta(F_i(x), F_{i+1}(y)) \leq k[p(x, F_i(x)) \cdot p(y, F_{i+1}(y))]^{1/2}, \quad i = 1, 2, 3, \ldots \\ &\delta(F_i(x), F_1(y)) \leq k[p(x, F_i(x)) \cdot p(y, F_1(y))]^{1/2}, \qquad i = 2, 3, 4, \ldots, \end{aligned}$$

then there exists $z \in X$ such that $\{z\} \subset F_i(z)$, $i = 1, 2, 3, \ldots$

Theorem 3.5. *Let (X,d) be a left K-sequentially complete space and let $\{F_i : X \to W^*(X)\}_{i=1}^{\infty}$ be a countable family of fuzzy mappings. If there exist constants $h,\ k > 0$, with $h+k<1$, such that for each $x,y \in X$*

$$\begin{array}{ll} D(F_i(x),F_{i+1}(y)) \le \frac{h\,p(y,F_{i+1}(y))[1+p(x,F_i(x))]}{1+d(x,y)} + kd(x,y), & i=1,2,3,\dots \\ D(F_i(x),F_1(y)) \le \frac{h\,p(y,F_1(y))[1+p(x,F_i(x))]}{1+d(x,y)} + kd(x,y), & i=2,3,4,\dots \\ D(F_1(x),F_i(y)) \le \frac{h\,p(y,F_i(y))[1+p(x,F_1(x))]}{1+d(x,y)} + kd(x,y), & i=3,4,\dots, \end{array}$$

then there exists $z \in X$ such that $\{z\} \subset F_i(z)$, $i=1,2,3,\dots$.

Proof. Let $x_{01} \in X$. Let (x_{rs}) be the sequence in the proof of theorem 3.2.

Now, $d(x_{11},x_{12}) \le D(F_1(x_{01}),F_2(x_{11}))$ and using one of the two boundary conditions for D, it is proved that

$$d(x_{11},x_{12}) \le \frac{k}{1-h} d(x_{01},x_{11}) \text{ and } d(x_{12},x_{21}) \le \frac{k}{1-h} d(x_{11},x_{12}).$$

Similarly we have

$$d(x_{21},x_{31}) \le \frac{k}{1-h} d(x_{12},x_{21}), d(x_{32},x_{31}) \le \frac{k}{1-h} d(x_{21},x_{31}),\dots.$$

Let $y_0 = x_{01}$. Now, we rename the constructed sequence (x_{rs}) as theorem 3.2 and we can see that

$$d(y_n,y_{n+1}) \le \frac{k}{1-h}\, d(y_{n-1},y_n) \le \left(\frac{k}{1-h}\right)^n d(y_0,y_1).$$

Furthermore, taking $t = \frac{k}{1-h}$, for $m>n$ the following relation is satisfied

$$d(y_n,y_m) \le \frac{t^n}{1-t}\, d(y_0,y_1).$$

In consequence (y_n) is a left K-Cauchy sequence and hence converges to z in X. We will see that $p(z,F_j(z))=0$, $j=1,2,3,...$ First,

$$p(z,F_1(z)) \le d(z,x_{i(i+1)}) + \frac{hd(x_{ii},x_{i(i+1)})[1+p(z,F_1(z))]}{1+d(z,x_{ii})} + kd(z,x_{ii}) \to 0,$$

as $i \to \infty$.

Then we have $p(z,F_1(z)) = 0$. Now, suppose $p(z,F_j(z)) = 0$. Then by lemmas 2.2 and 2.4 we have $p(z,F_{j+1}(z)) \le h\,p(z,F_{j+1}(z))$ and it follows that $p(z,F_{j+1}(z)) = 0$. Hence, by lemma 2.2 it follows that $\{z\} \subset F_j(z)$, for each $j \in \mathbb{N}$. □

We consider the following theorem for complete metric spaces.

Theorem 3.6 (Bose and Sahani [1]). *Let (X,d) be a complete linear space and let F_1 and F_2 be fuzzy mappings from X to $W(X)$ satisfying the following condition: For any x,y in X,*

$$\begin{aligned} D(F_1(x),F_2(y)) \quad \le \quad & a_1p(x,F_1(x)) + a_2p(y,F_2(y)) + a_3p(y,F_1(x)) \\ & + a_4p(x,F_2(y)) + a_5d(x,y) \end{aligned}$$

where a_1, a_2, a_3, a_4, a_5, are non-negative real numbers, $a_1+a_2+a_3+a_4+a_5<1$ and $a_1=a_2$ or $a_3=a_4$. Then there exists $z \in X$ such that $\{z\} \subset F_i(z)$, $i=1,2$.

We will present two similar theorems for a countable family of fuzzy mappings in a quasi-pseudo-metric space (X, d).

Theorem 3.7. *Let (X, d) be a left K-sequentially complete space and let $\{F_i : X \to W^*(X)\}_{i=1}^{\infty}$ be a countable family satisfying the following condition: For any $x,\ y \in X$,*

$$\begin{aligned} D(F_i(x), F_{i+1}(y)) &\leq a_1 p(x, F_i(x)) + a_2 p(y, F_{i+1}(y)) + a_3 p(y, F_i(x)) \\ &\quad + a_4 p(x, F_{i+1}(y)) + a_5 (d \wedge d^{-1})(x, y), i = 1, 2, ... \end{aligned}$$

$$\begin{aligned} D(F_i(x), F_1(y)) &\leq a_1 p(x, F_i(x)) + a_2 p(y, F_1(y)) + a_3 p(y, F_i(x)) \\ &\quad + a_4 p(x, F_1(y)) + a_5 (d \wedge d^{-1})(x, y),\ \ i = 2, 3, ... \end{aligned}$$

where $a_1,\ a_2,\ a_3,\ a_4,\ a_5$, are non-negative real numbers and $a_1+a_2+2a_4+a_5 < 1$. Then there exists $z \in X$ such that $\{z\} \subset F_i(z)$, $i = 1, 2, 3, ...$

Proof. Let $x_{01} \in X$. Let (x_{rs}) be the sequence in the proof of theorem 3.2. Now

$$\begin{aligned} d(x_{11}, x_{12}) &\leq D(F_1(x_{01}), F_2(x_{11})) \\ &\leq a_1 p(x_{01}, F_1(x_{01})) + a_2 p(x_{11}, F_2(x_{11})) + a_3 p(x_{11}, F_1(x_{01})) \\ &\quad + a_4 p(x_{01}, F_2(x_{11})) + a_5 (d \wedge d^{-1})(x_{01}, x_{11}) \\ &\leq a_1 d(x_{01}, x_{11}) + a_2 d(x_{11}, x_{12}) + a_4 (d(x_{01}, x_{11}) + d(x_{11}, x_{12})) \\ &\quad + a_5 d(x_{01}, x_{11}), \end{aligned}$$

i.e.,

$$d(x_{11}, x_{12}) \leq \frac{a_1 + a_4 + a_5}{1 - a_2 - a_4} d(x_{01}, x_{11}).$$

Let $r = \dfrac{a_1 + a_4 + a_5}{1 - a_2 - a_4}$. Then $0 < r < 1$ and $d(x_{11}, x_{12}) \leq r d(x_{01}, x_{11})$. Again

$$\begin{aligned} d(x_{12}, x_{21}) &\leq D(F_2(x_{11}), F_1(x_{12})) \\ &\leq a_1 d(x_{11}, x_{12}) + a_2 d(x_{12}, x_{21}) + a_4 (d(x_{11}, x_{12}) + d(x_{12}, x_{21})) \\ &\quad + a_5 d(x_{11}, x_{12}), \end{aligned}$$

i.e.,

$$d(x_{12}, x_{21}) \leq r d(x_{11}, x_{12}) \leq r^2 d(x_{01}, x_{11}).$$

Let $y_0 = x_{01}$. Now, we rename the constructed sequence (x_{rs}) as theorem 3.2. By the above relations one can verify that $d(y_n, y_{n+1}) \leq r^n\, d(y_0, y_1)$, $\quad n = 1, 2, ...$ and there exists $z \in X$ such that $d(z, y_n) \to 0$.

We will show by induction that $p(z, F_j(z)) = 0$, $j = 1, 2, 3, ...$ By lemmas 2.3 and 2.4 it follows that for $i = 1, 2, 3, ...$

$$\begin{aligned} p(z, F_1(z)) &\leq d(z, x_{i(i+1)}) + p(x_{i(i+1)}, F_1(z)) \\ &\leq d(z, x_{i(i+1)}) + D(F_{i+1}(x_{ii}), F_1(z)) \end{aligned}$$

But

$$\begin{aligned} D(F_{i+1}(x_{ii}),F_1(z)) &\le a_1p(x_{ii},F_{i+1}(x_{ii}))+a_2p(z,F_1(z)) \\ &\quad +a_3p(z,F_{i+1}(x_{ii}))+a_4p(x_{ii},F_1(z)) \\ &\quad +a_5(d\wedge d^{-1})(x_{ii},z) \\ &\le a_1d(x_{ii},x_{i(i+1)}) \\ &\quad +a_2\left\{d(z,x_{i(i+1)})+D(F_{i+1}(x_{ii}),F_1(z))\right\} \\ &\quad +a_3d(z,x_{i(i+1)}) \\ &\quad +a_4\left\{d(x_{ii},x_{i(i+1)})+D(F_{i+1}(x_{ii}),F_1(z))\right\} \\ &\quad +a_5d(z,x_{ii}). \end{aligned}$$

Thus

$$\begin{aligned} D(F_{i+1}(x_{ii}),F_1(z)) &\le \tfrac{a_1+a_4}{1-a_2-a_4}d(x_{ii},x_{i(i+1)}) \\ &\quad +\tfrac{a_2+a_3}{1-a_2-a_4}d(z,x_{i(i+1)}) \\ &\quad +\tfrac{a_5}{1-a_2-a_4}d(z,x_{ii}). \end{aligned}$$

So

$$\begin{aligned} p(z,F_1(z)) \le\ & d(z,x_{i(i+1)})+\frac{a_1+a_4}{1-a_2-a_4}d(x_{ii},x_{i(i+1)}) \\ & +\frac{a_2+a_3}{1-a_2-a_4}d(z,x_{i(i+1)})+\frac{a_5}{1-a_2-a_4}d(z,x_{ii})\to 0 \\ & \text{as } i\to\infty. \end{aligned}$$

Then, $p(z,F_1(z))=0$. Now, suppose $p(z,F_j(z))=0$. Then, by lemma 2.2 $\{z\}\subset F_j(z)$ and by lemma 2.4 we have

$$\begin{aligned} p(z,F_{j+1}(z)) &\le D(F_j(z),F_{j+1}(z)) \\ &\le a_1p(z,F_j(z))+a_2p(z,F_{j+1}(z)) \\ &\quad +a_3p(z,F_j(z))+a_4p(z,F_{j+1}(z))+(d\wedge d^{-1})(z,z) \\ &= (a_2+a_4)p(z,F_{j+1}(z)). \end{aligned}$$

Thus $(1-a_2-a_4)p(z,F_{j+1}(z))\le 0$, and it follows that $p(z,F_{j+1}(z))=0$. By lemma 2.2 it follows that $\{z\}\subset F_j(z)$, for each $j\in\mathbb{N}$.

□

We notice the above theorem is not a generalization of theorem 3.6. Now we present a partial generalization of this theorem.

Theorem 3.8. *Let (X,d) be a left K-sequentially complete space and let $\{F_i : X\to W^*(X)\}_{i=1}^{\infty}$ be a countable family of fuzzy mappings, satisfying the following condition: For any x, $y\in X$,*

$$\begin{aligned} D(F_i(x),F_{i+1}(y)) &\le a_1p(x,F_1(x))+a_2p(y,F_2(y))+a_3p(y,F_1(x)) \\ &\quad +a_4p(x,F_2(y))+a_5(d\wedge d^{-1})(x,y),\ \ i=1,2,... \end{aligned}$$

$$\begin{aligned} D(F_1(x),F_i(y)) &\le a_1p(x,F_1(x))+a_2p(y,F_i(y))+a_3p(y,F_1(x)) \\ &\quad +a_4p(x,F_i(y))+a_5(d\wedge d^{-1})(x,y),\ \ i=3,4, \end{aligned}$$

where a_1, a_2, a_3, a_4, a_5, are non-negative real numbers and $a_1+a_2+2a_3+a_5<1$, $a_1+a_2+2a_4+a_5<1$. Then there exists $z\in X$ such that $\{z\}\subset F_i(z)$, $i=1,2,3,\dots$ (compare with 3.6).

Proof. Let $x_{01} \in X$. Let (x_{rs}) be the sequence in the proof of theorem 3.2.

Now

$$d(x_{11}, x_{12}) \leq D(F_1(x_{01}), F_2(x_{11}))$$

and, as in the proof of the above theorem, we have

$$d(x_{11}, x_{12}) \leq \frac{a_1 + a_4 + a_5}{1 - a_2 - a_4} d(x_{01}, x_{11}).$$

Again

$$d(x_{12}, x_{21}) \leq \frac{a_2 + a_3 + a_5}{1 - a_1 - a_3} d(x_{11}, x_{12}).$$

Let $r = \dfrac{a_1 + a_4 + a_5}{1 - a_2 - a_4}$, and $s = \dfrac{a_2 + a_3 + a_5}{1 - a_1 - a_3}$. Then $0 < r, s < 1$. Take $t = \max\{r, s\} < 1$. So, we have

$$\begin{array}{lllllll} d(x_{11}, x_{12}) & \leq & rd(x_{01}, x_{11}) & \leq & td(x_{01}, x_{11}), & & \\ d(x_{12}, x_{21}) & \leq & sd(x_{11}, x_{12}) & \leq & td(x_{11}, x_{12}) & \leq & t^2 d(x_{01}, x_{11}). \end{array}$$

Let $y_0 = x_{01}$. Now, we rename the constructed sequence (x_{rs}) as theorem 3.2. By the above relations one can verify that $d(y_n, y_{n+1}) \leq t^n\, d(y_0, y_1)$, $n = 1, 2, \ldots$ Then there exists $z \in X$ such that $d(z, y_n) \to 0$ and as in the proof of the above theorem it can be shown that $\{z\} \subset F_j(z)$, for each $j \in \mathbb{N}$. □

References

[1] R. K. Bose and D. Sahani, *Fuzzy mappings and fixed point theorems*, Fuzzy Sets and Systems, **21** (1987), 53–58.

[2] V. Gregori and S. Romaguera, *Common fixed point theorems for pairs of fuzzy mappings*, Indian Journal of Mathematics, **41**, N° 1 (1999), 43–54.

[3] V. Gregori and S. Romaguera, *Fixed point theorems for fuzzy mappings in quasi-metric spaces*, Fuzzy Sets and Systems, **115** (2000), 477–483.

[4] S. Heilpern, *Fuzzy mappings and fixed point theorem*, J. Math. Anal. Appl., **83** (1981), 566–569.

[5] J. C. Kelly, *Bitopological spaces*, Proc. London Math. Soc., **13** (1963), 71–89.

[6] H. P. A. Künzi, M. Mrsevic, I. L. Reilly, M. K. Vamanamurthy, *Convergence, precompactness and symmetry in quasi-uniform spaces*, Math. Japonica, **38** (1993), 239–253.

[7] S. B. Nadler, *Multivalued contraction mappings*, Pacific J. Math., **30** (1969), 475–488.

[8] J. Y. Park and J. U. Jeong, *Fixed point theorems for fuzzy mappings*, Fuzzy Sets and Systems, **87** (1997), 111–116.

[9] I. L. Reilly, P. V. Subrahmanyam and M. K. Vamanamurthy, *Cauchy sequences in quasi-pseudometric spaces*, Mh. Math., **93** (1982), 127–140.

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