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A contribution to the study of fuzzy metric spaces

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Abstract. We give some examples and properties of fuzzy metric spaces, in the sense of George and Veeramani, and characterize the T_0 topological spaces which admit a compatible uniformity that has a countable transitive base, in terms of the fuzzy theory.

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1. Introduction

One of the main problems in the theory of fuzzy topological spaces is to obtain an appropriate and consistent notion of a fuzzy metric space. Many authors have investigated this question and several notions of a fuzzy metric space have been defined and studied. In particular, and modifying the concept of metric fuzziness introduced by Kramosil and Michalek [9] (which is a generalization of the concept of probabilistic metric space introduced by K. Menger [10] to the fuzzy setting), George and Veeramani [4, 5], have studied a notion of fuzzy metric space. In a previous paper [7], Gregori and Romaguera proved that the class of fuzzy metric spaces, in George and Veeramani's sense, coincides with the class of metric spaces. In the light of the results obtained in [7], we think that the George and Veeramani's definition is an appropriate notion of metric fuzziness in the sense that it provides rich fuzzy topological structures which can be obtained, in many cases, from classical theorems. On the other hand, metric spaces can be studied from the point of view of fuzzy theory. Unfortunately, not much examples of such spaces have been given. In this paper we give new examples of fuzzy metric spaces and study some properties of these spaces. The structure of the paper is as follows. After preliminaries, in section 3, we construct new fuzzy metrics from a given one, and study some questions relative to boundedness. In Section 4 we give new examples of fuzzy metrics. In Section 5 we study a property of Cauchy sequences in standard fuzzy metric spaces, and finally, in Section 6, we define the concept of non-Archimedean fuzzy metric space and prove that the family of these spaces agrees with the class of non-Archimedean metric spaces, so it provides a characterization of the T_0 topological spaces which admit a compatible uniformity, that has a countable transitive base, in the fuzzy setting.

2. Preliminaries

Throughout this paper the letters $\mathbb{N}$ and $\mathbb{R}$ will denote the set of all positive integers and real numbers, respectively. Our basic reference for General Topology is [2].

According to [11] a binary operation $* : [0,1]\times[0,1] \longrightarrow [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is associative and commutative
(ii) $*$ is continuous
(iii) $a * 1 = a$ for every $a \in [0,1]$
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq c$, for all $a, b, c, d \in [0,1]$

According to [4],[5], a fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set of $X \times X \times]0,+\infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(i) $M(x,y,t) > 0$
(ii) $M(x,y,t) = 1$ if and only if $x = y$
(iii) $M(x,y,t) = M(y,x,t)$
(iv) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$ (triangular inequality)
(v) $M(x,y,\cdot) :]0,+\infty[\longrightarrow [0,1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$, or M (if it is not necessary to mention $*$), is a fuzzy metric on X.

Lemma 2.1. [6] *$M(x,y,\cdot)$ is nondecreasing for all $x, y \in X$.*

Lemma 2.2. [1] *Let $(X, M, *)$ be a fuzzy metric space.*

(i) *If $M(x,y,t) > 1 - r$ for $x, y \in X$, $t > 0$, $0 < r < 1$, we can find a t_0, $0 < t_0 < t$ such that $M(x,r,t_0) > 1 - r$.*
(ii) *For any $r_1 > r_2$, we can find a r_3 such that $r_1 * r_3 \geq r_2$, and for any r_4 we can find a r_5 such that $r_5 * r_5 \geq r_4$, ($r_1, r_2, r_3, r_4, r_5 \in]0,1[$).*

Let (X, d) be a metric space. Define $a * b = ab$ for every $a, b \in [0,1]$, and let M_d be the function on $X \times X \times]0,+\infty[$ defined by

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}$$

Then $(X, M_d, *)$ is a fuzzy metric space, and M_d is called the standard fuzzy metric induced by d (see [4]).

George and Veeramani proved that every fuzzy metric M on X generates a Hausdorff topology τ_M on X which has as a base the family of open sets of the form:

$$\{B_M(x,r,t) : x \in X, 0 < r < 1, t > 0\}$$

where

$$B_M(x,r,t) = \{y \in X : M(x,y,t) > 1-r\}$$

for every $r \in]0,1[$, and $t > 0$. (We will write $B(x,r,t)$ when confusion is not possible).

Definition 2.3. *A sequence $\{x_n\}$ in a fuzzy metric space $(X,M,*)$ is called a Cauchy sequence [5], if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1-\varepsilon$, for all $m,n \geq n_0$.*

A subset A of X is said to be F-bounded if there exist $t > 0$ and $r \in]0,1[$ such that $M(x,y,t) > 1-r$ for all $x,y \in A$.

Proposition 2.4 ([4]). *If (X,d) is a metric space, then:*

(i) The topology τ_d on X generated by d coincides with the topology τ_{M_d} generated by the standard fuzzy metric M_d.
(ii) $\{x_n\}$ is a $d-$Cauchy sequence (i.e., a Cauchy sequence in (X,d)) if and only if it is a Cauchy sequence in $(X,M_d,*)$.
(iii) $A \subset X$ is bounded in (X,d) if and only if it is F-bounded in $(X,M_d,*)$.

We say that a topological space (X,τ) is fuzzy metrizable if there exists a fuzzy metric M on X such that $\tau = \tau_M$. In [7] it is proved that a topological space is fuzzy metrizable if and only if it is metrizable.

Unless explicit mention we will suppose $\mathbb{R}$ endowed with the usual topology.

3. Some properties of fuzzy metric spaces

From now on we will denote by T_i $(i = 1,2,3)$ the following continuous t-norms:

$$\begin{aligned} T_1(x,y) &= \min\{x,y\} \\ T_2(x,y) &= xy \\ T_3(x,y) &= \max\{0, x+y-1\} \end{aligned}$$

The following inequalities are satisfied:

$$T_3(x,y) \leq T_2(x,y) \leq T_1(x,y)$$

and

$$T(x,y) \leq T_1(x,y)$$

for each continuous t-norm T.

In consequence the following lemma holds.

Lemma 3.1. *Let X be a non-empty set. If (M,T) is a fuzzy metric on X and T' is a continuous t-norm such that $T' \leq T$, then (M,T') is a fuzzy metric on X.*

Next two properties give methods for constructing F-bounded fuzzy metrics from a given fuzzy metric.

Proposition 3.2. *Let $(X, M, *)$ be a fuzzy metric space and $k \in]0,1[$. Define*

$$N(x,y,t) = \max\{M(x,y,t), k\}, \text{ for each } x, y \in X,\ t > 0.$$

Then $(N, *)$ is an F-bounded fuzzy metric on X, which generates the same topology that M.

Proof. It is straightforward. □

Proposition 3.3. *Let $i \in \{1,2,3\}$ and $k > 0$. Suppose that (X, M, T_i) is a fuzzy metric space, and define:*

$$N(x,y,t) = \frac{k + M(x,y,t)}{1+k} \text{ for all } x, y \in X, t > 0.$$

Then, (N, T_i) is an F-bounded fuzzy metric on X, which generates the same topology that M.

Proof. We prove this proposition for the case $i = 2$. For seeing that (N, T_i) is a fuzzy metric on X, we only show the triangular inequality.

Now, it is an easy exercise to verify that the following relation

$$\frac{k+a}{1+k} \cdot \frac{k+b}{1+k} \leq \frac{k+ab}{1+k}$$

holds, for all $a, b \in [0,1]$.

Therefore,

$$\begin{aligned} \frac{k+M(x,y,t)}{1+k} \cdot \frac{k+M(y,z,s)}{1+k} &\leq \frac{k+M(x,y,t)\cdot M(y,z,s)}{1+k} \\ &\leq \frac{k+M(x,z,t+s)}{1+k} \end{aligned}$$

Clearly $\frac{k}{1+k}$ is a lower bound of $N(x,y,t)$, for all $x, y \in X$, $t > 0$.

Finally, for $t > 0, r \in]0,1[$ it is satisfied that

$$B_M(x,r,t) = B_N(x, \frac{r}{1+k}, t)$$

and

$$B_N(x,r,t) = B_M(x, r(k+1), t),$$

and so $\tau_M = \tau_N$.

The cases $i = 1, 3$ are left as simple exercises. □

Problem 3.4. *If $(M, *)$ is a fuzzy metric on X and $k > 0$, then, is*

$$\left(\frac{k+M(x,y,t)}{1+k}, *\right)$$

a fuzzy metric on X?

Proposition 3.5. *Let $(M_1, *)$ and $(M_2, *)$ be two fuzzy metrics on X. Define:*

$$M(x,y,t) = M_1(x,y,t) * M_2(x,y,t)$$
$$N(x,y,t) = \min\{M_1(x,y,t), M_2(x,y,t)\}$$

Then:

(i) $(M, *)$ *is a fuzzy metric on* X *if* $a * b \neq 0$ *whenever* $a, b \neq 0$.
(ii) $(N, *)$ *is a fuzzy metric on* X.
(iii) The topologies generated by M *and* N *are the same.*

Proof. The proofs of (i) and (ii) are straightforward.
(iii) First we will prove that $\tau_N < \tau_M$
Let $A \in \tau_N$; then $\forall x \in A, \exists r \in]0,1[$ such that

$$B_N(x,r,t) = \{y \in X : N(x,y,t) > 1 - r\} \subset A$$

Consider

$$B_M(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}.$$

If $z \in B_M(x,r,t)$, then $M(x,z,t) > 1 - r$, i.e.,

$$M_1(x,z,t) * M_2(x,z,t) > 1 - r.$$

Notice that

$$M_1(x,z,t) \geq M_1(x,z,t) * M_2(x,z,t) > 1 - r,$$

and

$$M_2(x,z,t) \geq M_1(x,z,t) * M_2(x,z,t) > 1 - r$$

so,

$$N(x,z,t) = \min\{M_1(x,z,t), M_2(x,z,t)\} > 1 - r.$$

Then,

$$B_M(x,r,t) \subset B_N(x,r,t) \subset A,$$

thus $A \in \tau_M$ and hence $\tau_N < \tau_M$.
For seeing that $\tau_M < \tau_N$, let $A \in \tau_M$; then $\forall x \in A, \exists r \in]0,1[$ such that

$$B_M(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\} \subset A.$$

Let $s \in]0,1[$ such that $(1-s) * (1-s) > 1 - r$.
Consider

$$\begin{aligned} B_N(x,s,t) &= \{y \in X : N(x,y,t) > 1 - s\} \\ &= \{y \in X : \min\{M_1(x,y,t), M_2(x,y,t)\} > 1 - s\} \end{aligned}$$

If $z \in B_N(x,s,t)$, then $M_1(x,z,t) > 1 - s$ and $M_2(x,z,t) > 1 - s$.
So,

$$\begin{aligned} M_1(x,z,t) * M_2(x,z,t) &> (1-s) * (1-s) \\ &> 1 - r. \end{aligned}$$

Then $B_N(x,s,t) \subset B_M(x,r,t) \subset A$, and hence $\tau_M < \tau_N$. □

Remark 3.6. If we consider the fuzzy metric $(M_d, *)$ where d is the usual metric on $\mathbb{R}$ and $*$ is T_3, it is easy to verify that $M = M_d * M_d$ is not a fuzzy metric on $\mathbb{R}$ (compare with 2.10 of [5]).

Definition 3.7. [7] *A fuzzy metric space* $(X, M, *)$ *is called precompact if for each* $r \in]0,1[$, *and* $t > 0$, *there exists a finite subset* A *of* X *such that* $X = \bigcup\{B(a,r,t) : a \in A\}$. *In this case, we say that* M *is a precompact fuzzy metric on* X.

In [7] it is proved that a fuzzy metric space is precompact if and only if every sequence has a Cauchy subsequence. Using this fact, the proof of the following proposition is straightforward.

Proposition 3.8. *Let (X,d) be a metric space and let M_d be the standard fuzzy metric deduced from d. Then, d is a precompact metric if and only if M_d is a precompact fuzzy metric.*

Proposition 3.9. *Let $(X,M,*)$ be a precompact fuzzy metric space, and suppose $a*b \neq 0$ whenever $a,b \neq 0$. Then, $(M,*)$ is F-bounded.*

Proof. (Compare with the end of the proof of [4, Theorem 3.9].)

Let $r \in]0,1[$ and $t > 0$. By assumption there is a finite subset $A = \{a_1,\dots,a_n\}$ of X such that $X = \bigcup_{i=1}^{n} B(a_i,r,t)$. Let

$$\alpha = \min\{M(a_i,a_j,t) : i,j = 1,\dots,n\} > 0.$$

Let $x,y \in X$. Then $x \in B(a_i,r,t)$ and $y \in B(a_j,r,t)$ for some $i,j \in \{1,\dots,n\}$. Therefore $M(x,a_i,t) > 1-r$ and $M(y,a_j,t) > 1-r$. Now,

$$\begin{aligned} M(x,y,3t) &\geq M(x,a_i,t) * M(a_i,a_j,t) * M(a_j,y,t) \\ &\geq (1-r) * \alpha * (1-r) \\ &> 1-s \end{aligned}$$

for some $s \in]0,1[$ by the assumption on $*$, and so M is F-bounded □

Problem 3.10. *Is each precompact fuzzy metric space F-bounded?*

Remark 3.11. The converse of the last proposition is false. In fact, the subspace X of the Hilbert metric space $(\mathbb{R}^\infty, d)$, formed by the points of unit weight $(0,\dots,0,1,0,\dots,0)$, is not precompact and bounded (it has diameter $\sqrt{2}$), and then by (iii) of Proposition 2.2, $(X,M_d,*)$ is F-bounded, and by Proposition 3.8 M_d is not precompact.

4. Examples of fuzzy metric spaces

In this section we will see examples of fuzzy metrics where the t-norm is T_1, and other fuzzy metrics $(M,T_i),(i=2,3)$ which are not fuzzy metrics in considering (M,T_{i-1}). Before, we need the following lemma.

Lemma 4.1. *Let (X,d) be a metric space and $s,t > 0$. The following inequality holds, for all $n \geq 1$;*

$$\frac{d(x,z)}{(t+s)^n} \leq \max\left\{\frac{d(x,y)}{t^n}, \frac{d(y,z)}{s^n}\right\}$$

Proof. We distinguish three cases:

(1) $d(x,z) \leq d(x,y)$
(2) $d(x,z) \leq d(y,z)$
(3) $d(x,z) > d(x,y)$ and $d(x,z) > d(y,z)$

The inequality chosen is obvious in cases (1) and (2). Now, suppose (3) is satisfied and distinguish two possibilities:

(3.1) $d(x,z) = d(x,y) + d(y,z)$
(3.2) $d(x,z) < d(x,y) + d(y,z)$

Suppose (3.1) is satisfied. Put $d(x,y) = \beta d(x,z)$ with $\beta \in]0,1[$ and hence

$$d(y,z) = (1-\beta)d(x,z).$$

Now, to show the above inequality we have to prove that

$$\frac{1}{(t+s)^n} \leq \max\left\{\frac{\beta}{t^n}, \frac{1-\beta}{s^n}\right\}.$$

Therefore, consider the functions $f(\beta) = \frac{t^n}{\beta}$ and $g(\beta) = \frac{s^n}{1-\beta}$ which are strictly decreasing and increasing, respectively. Now, the largest value of $\min\left\{\frac{t^n}{\beta}, \frac{s^n}{1-\beta}\right\}$ is taken when $f(\beta) = g(\beta)$, that is, for $\beta = \frac{t^n}{t^n+s^n}$. Then,

$$\begin{aligned} (t+s)^n &\geq t^n + s^n \\ &= f\left(\frac{t^n}{t^n+s^n}\right) \\ &\geq \min\left\{\frac{t^n}{\beta}, \frac{s^n}{1-\beta}\right\} \end{aligned}$$

and the chosen inequality is stated.

The case (3.2) is a consequence of (3.1). □

Example 4.2. Let (X,d) be a metric space, and denote $B(x,r)$ the open ball centered in $x \in X$ with radius $r > 0$.

(i) For each $n \in \mathbb{N}$, (X, M, T_1) is a fuzzy metric space where M is given by

$$M(x,y,t) = \frac{1}{e^{\frac{d(x,y)}{t^n}}} \text{ for all } x,y \in X,\ t > 0,$$

and $\tau_M = \tau(d)$.

(This example when $n = 1$ has been given in [4].)

(ii) For each $k, m \in \mathbb{R}^+$, $n \geq 1$, (X, M, T_1) is a fuzzy metric space where M is given by

$$M(x,y,t) = \frac{kt^n}{kt^n + md(x,y)} \text{ for all } x,y \in X, t > 0,$$

and $\tau_M = \tau(d)$.

Proof. (i) It is easy to verify that (M, T_1) satisfies all conditions of fuzzy metrics; in particular the triangular inequality is a consequence of the previous lemma.

Now, for $x \in X, r \in]0,1[$ and $t > 0$ we have that

$$B_M(x,r,t) = B(x, -t^n \ln(1-r)),$$

and

$$B(x,r) = B_M(x, 1 - \frac{1}{e^{\frac{r}{t^n}}}, t),$$

and hence $\tau_M = \tau(d)$.

(ii) We will only give a proof of the triangular inequality. Indeed, by the previous lemma

$$1+\frac{md(x,z)}{k(t+s)^n} \leq \max\left\{1+\frac{md(x,y)}{kt^n}, 1+\frac{md(y,z)}{ks^n}\right\}$$

hence

$$\frac{k(t+s)^n}{k(t+s)^n+md(x,z)} \geq \min\left\{\frac{kt^n}{kt^n+md(x,y)}, \frac{ks^n}{ks^n+md(y,z)}\right\},$$

and the triangular inequality is stated.

Now, for $x \in X$, $t>0$ and $r \in]0,1[$ we have that

$$B_M(x,r,t) = B(x, \frac{kt^n r}{m(1-r)}),$$

and

$$B(x,r) = B_M(x, \frac{mr}{kt^n+mr}, t),$$

and hence $\tau_M = \tau(d)$. □

Remark 4.3. The above expression of M cannot be generalized to $n \in \mathbb{R}^+$ (take the usual metric d on $\mathbb{R}$, $k=m=1$, $n=1/2$). Nevertheless it is easy to verify that (M,T_2) is a fuzzy metric on X, for $n \geq 0$. (Compare with 2.9-2.10 of [5]).

Next, we will give fuzzy metrics which cannot be deduced from a metric, in the sense of last example, since they will not be fuzzy metrics for the t-norm T_1.

Example 4.4. Let X be the real interval $]0,+\infty[$ and $a>0$. It is easy to verify that (X,M,T_2) is a fuzzy metric space, where M is defined by

$$M(x,y,t) = \begin{cases} \left(\frac{x}{y}\right)^a & \text{if } x \leq y \\ \left(\frac{y}{x}\right)^a & \text{if } y \leq x \end{cases}$$

for all $x,y \in X$, $t>0$.

(We notice that this example for $X=\mathbb{N}$ and $a=1$ was given in [4]).

Now, for $x \in X$, $t>0$ and $r \in]0,1[$, we have

$$B(x,r,t) = \left](1-r)^{\frac{1}{a}}x, \frac{x}{(1-r)^{\frac{1}{a}}}\right[$$

and hence $B(x,r,t)$ is an open interval of $\mathbb{R}$, whose diameter converges to zero as $r \to 0$. In consequence, τ_M is the usual topology of $\mathbb{R}$ relative to X.

Finally, (X,M,T_1) is not a fuzzy metric space. Indeed, for $a=1$, if we take $x=1$, $y=2$ and $z=3$, then

$$\begin{aligned} M(x,z,t+s) &= \frac{1}{3} \\ &< \min\{\frac{1}{2}, \frac{2}{3}\} \\ &= \min\{M(x,y,t), M(y,z,s)\}. \end{aligned}$$

Next, we will give examples of fuzzy metric spaces for the t-norm T_3 which are not for the t-norm T_2.

Example 4.5. Let X be the real interval $]1,+\infty[$ and consider the mapping M on $X^2\times]0,+\infty[$ given by

$$M(a,b,t) = 1 - (\frac{1}{a \wedge b} - \frac{1}{a \vee b}) \text{ for all } a, b \in X, t > 0.$$

We will see that (X, M, T_3) is a fuzzy metric space and (X, M, T_2) is not. Further, the topology τ_M on X is the usual topology of $\mathbb{R}$ relative to X.

For seeing that (M, T_3) is a fuzzy metric we only prove the triangular inequality, which becomes (when the left side of the inequality is distinct of zero) (4.1)

$$\left[1-\left(\frac{1}{a\wedge b}-\frac{1}{a\vee b}\right)\right]+\left[1-\left(\frac{1}{b\wedge c}-\frac{1}{b\vee c}\right)\right]-1\leq 1-\left(\frac{1}{a\wedge c}-\frac{1}{a\vee c}\right)$$

For it, first, we distinguish 6 cases:

(1) Suppose $a < b < c$. In this case, the inequality 4.1 becomes an equality.

(2) Suppose $a < c < b$. In this case, the inequality 4.1 becomes:

$$\frac{1}{b}+\frac{1}{b}+\frac{1}{a}\leq\frac{1}{a}+\frac{1}{c}+\frac{1}{c}$$

which is true, since $\frac{1}{b} < \frac{1}{c}$.

(3) Suppose $c < a < b$. In this case, the inequality 4.1 becomes:

$$\frac{1}{b}+\frac{1}{b}+\frac{1}{c}\leq\frac{1}{a}+\frac{1}{c}+\frac{1}{a}$$

which is true, since $\frac{1}{b} < \frac{1}{a}$.

(4) Suppose $b < a < c$. In this case, the inequality 4.1 becomes:

$$\frac{1}{a}+\frac{1}{c}+\frac{1}{a}\leq\frac{1}{b}+\frac{1}{b}+\frac{1}{c}$$

which is true, since $\frac{1}{a} < \frac{1}{b}$.

(5) Suppose $b < c < a$. In this case, the inequality 4.1 becomes:

$$\frac{1}{a}+\frac{1}{c}+\frac{1}{c}\leq\frac{1}{b}+\frac{1}{b}+\frac{1}{a}$$

which is true, since $\frac{1}{c} < \frac{1}{b}$

(6) Suppose $c < b < a$. In this case, the inequality 4.1 becomes an equality.

Now, if $a = b$, or $a = c$, or $b = c$, the inequality 4.1 is obvious, and the triangular inequality is stated, so (M, T_3) is a fuzzy metric.

On the other hand, if we take $a = 2$, $b = 3$ and $c = 10$, then

$$M(a,b,t)\cdot M(b,c,s) > M(a,c,t+s)$$

and thus, (M, T_2) is not a fuzzy metric.

Finally, if we take $x \in X$, $r \in]0,1[$ with $r < \frac{1}{x}$, and $t > 0$, it is easy to verify that $B(x,r,t) = \left]\frac{x}{1+rx}, \frac{x}{1-rx}\right[$, then $B(x,r,t)$ is an open interval or $\mathbb{R}$ which diameter converges to zero as $r \to \infty$, and thus τ_M is the usual topology of $\mathbb{R}$ relative to X.

Example 4.6. Let X be the real interval $]2,+\infty[$ and consider the mapping M on $X^2\times]0,+\infty[$ defined as follows

$$M(a,b,t)=\begin{cases}1 & \text{if } a=b\\ \frac{1}{a}+\frac{1}{b} & \text{if } a\neq b,\, t>0.\end{cases}$$

It is easy to verify that (X,M,T_3) is a fuzzy metric space. On the other hand if we take $a=1000$, $b=3$ and $c=10000$, then

$$M(a,b,t)\cdot M(b,c,s)>M(a,c,t+s)$$

and so (X,M,T_2) is not a fuzzy metric space.

Finally, the topology τ_M is the discrete topology on X. Indeed, for $x\in X$, if we take $r<\frac{1}{2}-\frac{1}{x}$ then $B(x,r,t)=\{x\}$.

Next example is based in [8].

Example 4.7. Let $\{A,B\}$ be a nontrivial partition or the real interval $X=\,]2,+\infty[$. Define the mapping M on $X^2\times]0,+\infty[$ as follows

$$M(x,y,t)=\begin{cases}1-\left(\frac{1}{x\wedge y}-\frac{1}{x\vee y}\right) & \text{if } x,y\in A \text{ or } x,y\in B\\ \frac{1}{x}+\frac{1}{y} & \text{elsewhere.}\end{cases}$$

Then, imitating example 2 of [8], one can prove that (X,M,T_3) is a fuzzy metric space, and by example 4.4, clearly (X,M,T_2) is not a fuzzy metric space.

From examples 4.4 and 4.5 it is deduced that an open base for the neighborhood system of a point $x\in X$, is $\left]\frac{x}{1+rx},\frac{x}{1-rx}\right[\cap A$ if $x\in A$, and $\left]\frac{x}{1+rx},\frac{x}{1-rx}\right[\cap B$, with $0<r<\frac{1}{2}-\frac{1}{x}$, if $x\in B$.

5. Some properties of standard fuzzy metrics

In this section (X,d) will be a metric space, and M_d the standard fuzzy metric deduced of d.

Grosso modo, we can say that all properties of classical metrics can be translated to standard fuzzy metrics. Now, an interesting question is to know which of these properties can be generalized to any fuzzy metric. In this sense we will see a new property which is satisfied by standard fuzzy metrics. (Notice that there is no significative difference between the standard fuzzy metric M_d and the fuzzy metric $\frac{kt^n}{kt^n+md(x,y)}$ of example 4.2, unless M_d is the most simple expression depending of t).

Proposition 5.1. *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two Cauchy sequences in (X, M_d, T_2) and let $t>0$. Then, the sequence of real numbers $\{M_d(x_n,y_n,t)\}_{n=1}^{\infty}$ converges to some real number in $]0,1[$.*

Proof. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in (X,M_d,T_2). By (ii) of proposition 2.2, $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in (X,d) and then it is easy to verify that $\{d(x_n,y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$.

Now, let $\varepsilon > 0$, $t > 0$. Then, there exists $n_0 \in \mathbb{N}$ such that

$$|d(x_n, y_n) - d(x_m, y_m)| < \frac{\varepsilon}{t} \text{ for all } m, n \geq n_0.$$

Hence,

$$\begin{aligned} \left| \frac{1}{M_d(x_n, y_n, t)} - \frac{1}{M_d(x_m, y_m, t)} \right| &= \frac{1}{t} |d(x_n, y_n) - d(x_m, y_m)| \\ &< \varepsilon, \end{aligned}$$

for all $m, n \geq n_0$, and therefore $\left\{ \frac{1}{M_d(x_n, y_n, t)} \right\}$ is a Cauchy sequence in $\mathbb{R}$, which converges to some $k \in \mathbb{R}$, and then the sequence $\{M_d(x_n, y_n, t)\}_{n=1}^{\infty}$ converges to $\frac{1}{k} \in]0, 1[$, since $k \neq \infty$ and $M_d(x_n, y_n, t) \leq 1$, for all $n \in \mathbb{N}$. □

Corollary 5.2. *Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the fuzzy metric space (X, M_d, T_2) and $a \in X$. Then, the sequence of real numbers $\{M_d(a, x_n, t)\}_{n=1}^{\infty}$ converges to some real number in $]0, 1[$.*

Problem 5.3. *Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the fuzzy metric space $(X, M, *)$ and let $a \in X$, $t > 0$. Does the sequence of real numbers $\{M(a, x_n, t)\}_{n=1}^{\infty}$ converge to some real number in $]0, 1[$?*

The last proposition is not true for any fuzzy metric space as shows the following example.

Example 5.4. Let $\{A, B\}$ be a partition of the real interval $X =]2, +\infty[$, such that $\{2n - 1\}_{n=2}^{\infty} \subset A$ and $\{2n\}_{n=1}^{\infty} \subset B$, and consider the fuzzy metric space (X, M, T_3) of example 4.6. It is easy to verify that both sequences are Cauchy in (X, M, T_3). Now, if we put $a_n = 2n - 1$ and $b_n = 2n$, for $n = 2, 3, \ldots$ we have

$$M(a_n, b_n, t) = \left(\frac{1}{2n - 1} + \frac{1}{2n} \right) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

6. On non-Archimedean fuzzy metrics

Recall that a metric d on X is called non-Archimedean if

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}, \text{ for all } x, y, z \in X.$$

Now, we give the following definition.

Definition 6.1. *A fuzzy metric $(M, *)$ on X is called non-Archimedean if*

$$M(x, z, t) \geq \min\{M(x, y, t), M(y, z, t)\} \textit{ for all } x, y, z \in X,\ t > 0.$$

Clearly, if M is a non-Archimedean fuzzy metric on X, then (M, T_1) is a fuzzy metric on X.

Proposition 6.2. *Let d be a metric on X and M_d the corresponding standard fuzzy metric. Then, d is non-Archimedean if and only if M_d is non-Archimedean.*

Proof. Suppose d is non-Archimedean. Then,

$$\begin{aligned} M_d(x,z,t) &= \frac{t}{t+d(x,z)} \\ &\geq \frac{t}{t+\max\{d(x,y),d(y,z)\}} \\ &= \min\{M_d(x,y,t),M_d(y,z,t)\}. \end{aligned}$$

Conversely, if M_d is non-Archimedean then,

$$\begin{aligned} d(x,z) &= t\left(\frac{1}{M_d(x,y,t)}-1\right) \\ &\leq t\left(\frac{1}{\min\{M_d(x,y,t),M_d(y,z,t)\}}-1\right) \\ &= \max\{d(x,y),d(y,z)\}. \end{aligned}$$

□

Recall that a completely regular space is called strongly zero-dimensional if each zero-set is the countable intersection of sets that are open and closed, and that a T_0 topological space (X,τ) is strongly zero-dimensional and metrizable if and only if there is a uniformity $\mathcal{U}$ compatible with τ that has a countable transitive base ([3, Theorem 6.8]).

Theorem 6.3. *A topological space (X,τ) is strongly zero-dimensional and metrizable if and only if (X,τ) is non-Archimedeanly fuzzy metrizable.*

Proof. Suppose (X,τ) is strongly zero-dimensional and metrizable. Then, from [3, Theorem 6.8], (X,τ) is non-Archimedeanly metrizable and by Proposition 6.2 it is non-Archimedeanly fuzzy metrizable.

Conversely, suppose M is a compatible non-Archimedean fuzzy metric for (X,τ). Now, for a fuzzy metric space $(X,M,*)$ in [7] it is proved that the family $\{U_n : n\in\mathbb{N}\}$ where

$$U_n=\left\{(x,y)\in X\times X : M(x,y,\frac{1}{n})>1-\frac{1}{n}\right\}$$

is a base for a uniformity $\mathcal{U}$ on X which is compatible with τ_M. Now, we will see that $\{U_n : n\in\mathbb{N}\}$ is transitive.

Indeed, if $(x,y),(y,z)\in U_n$ then

$$\begin{aligned} M(x,z,\frac{1}{n}) &\geq \min\left\{M(x,y,\frac{1}{n}),M(y,z,\frac{1}{n})\right\} \\ &> 1-\frac{1}{n}, \end{aligned}$$

and thus $(x,z)\in U_n$, i.e., $U_n\circ U_n\subset U_n$.

Now, from the mentioned theorem of [3], the Hausdorff topological space (X,τ) is a strongly zero-dimensional and metrizable space. □

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