Applied General Topology
© Universidad Politécnica de Valencia
Volume 3, No. 2, 2002
pp. 113–131

Strengthening connected Tychonoff topologies

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Abstract. The problem of whether a given connected Tychonoff space admits a strictly finer connected Tychonoff topology is considered. We show that every Tychonoff space X satisfying $w(X) \leq \mathfrak{c}$ and $c(X) \leq \aleph_0$ admits a finer strongly σ-discrete connected Tychonoff topology of weight $2^{\mathfrak{c}}$. We also prove that every connected Tychonoff space is an open continuous image of a connected strongly σ-discrete submetrizable Tychonoff space. The latter result is applied to represent every connected topological group as a quotient of a connected strongly σ-discrete submetrizable topological group.

2000 AMS Classification: Primary 54D05, 54A25, 54B10, 54H11; Secondary 54C10, 54A10, 54E35.
Keywords: Connected, strongly σ-discrete, submetrizable, regular open set, dense subset, topological group, quotient group, free topological group

1. Introduction

All spaces we consider are assumed to be completely regular unless explicitly stated.

In general, it is difficult to construct a strictly finer connected Tychonoff topology on a connected space X. That is why several assumptions on X usually appear to make this possible. In [17] it was proved that if X is a connected space which is locally Čech-complete (or first countable or locally separable), then X admits a strictly finer Tychonoff connected topology. Sometimes a strictly finer connected topology on X can even be chosen strongly σ-discrete. It was proved in [20] that this is the case for every connected space X satisfying $w(X) \leq \mathfrak{c}$, $|X| = \mathfrak{c}$ and which has ω_1 as a precalibre. Actually, one can replace "ω_1 a precalibre for X" by the weaker condition "$c(X) \leq \omega$" almost without changing the original proof.

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†The research was supported by Mexican National Council of Science and Technology (CONACyT), grant number 400200-5-28411E.

In this paper we describe a class $\mathcal{P}$ of connected spaces that admit a finer connected strongly σ-discrete Tychonoff topology. It turns out that every connected space X with $w(X) \leq \mathfrak{c}$ and $c(X) \leq \aleph_0$ admits such a topology (Theorem 2.14), thus generalizing [20, Theorem 2.12]. If additionally $|X| = \mathfrak{c}$, then a finer connected strongly σ-discrete topology on X can be chosen to be submetrizable (Theorem 2.9).

In Section 3 we consider the problem of representation of a connected space as an open continuous image of a connected strongly σ-discrete one. The main result here is Theorem 3.4 which states that every connected space Z is an open continuous image of a connected strongly σ-discrete submetrizable space S. In addition, if $|Z| = \mathfrak{c}$, then the space S can be chosen to satisfy the same equality $|S| = \mathfrak{c}$ (Theorem 3.5).

As far as the authors know, no examples of non-trivial connected σ-discrete topological groups have been constructed previously. In Section 4 we show that every connected topological group G is a quotient of a connected strongly σ-discrete submetrizable group H (Theorem 4.2). If additionally $|G| = \mathfrak{c}$, then the group H can be chosen to satisfy $|H| = \mathfrak{c}$ (Corollary 4.3).

1.1. **Notation and terminology.** The interior of a closed subset of a space X is called *regular open* in X. As usual, $\mathcal{R}o(X)$ denotes the family of all regular open subsets of X. The interior and the boundary of a subset Y of X are denoted by $\mathrm{Int}_X Y$ and $\mathrm{Fr}_X Y$ respectively, or simply by $\mathrm{Int}\, Y$ and $\mathrm{Fr}\, Y$ if no confusion is possible.

We say that a space X is *strongly σ-discrete* if it is a union of countably many closed discrete subsets. If the space X admits a coarser metrizable topology, it is called *submetrizable.* Equivalently, X is submetrizable iff there exists a continuous bijection $f\colon X \to M$ onto a metrizable space M.

The weight, π-weight, density and cellularity of a space X are denoted by $w(X)$, $\pi w(X)$, $d(X)$ and $c(X)$ respectively. The space X is *countably cellular* if $c(X) \leq \aleph_0$. If every uncountable family of non-void open sets in X contains an uncountable subfamily with the finite intersection property, then we say that ω_1 is a *precalibre* for X. Clearly, every space for which ω_1 is a precalibre must be countably cellular.

We use $\mathbb{I}$ to denote the closed unit interval $[0,1]$ of the real line $\mathbb{R}$ endowed with the interval topology. The power of continuum is also denoted by $\mathfrak{c}$.

2. Finer connected strongly σ-discrete topologies

It is shown in [3] that there exists a dense connected strongly σ-discrete subspace Y of the Tychonoff cube $\mathbb{I}^{\mathfrak{c}}$ such that $|Y| = \mathfrak{c}$ and $Y \setminus A$ remains connected for every subset A of Y with $|A| < \mathfrak{c}$. Strengthening this result, Tkachuk showed in [20, Theorem 2.12] that every connected space X satisfying $w(X) \leq \mathfrak{c}$, $|X| = \mathfrak{c}$ and for which ω_1 is a precalibre admits a finer connected strongly σ-discrete topology. Actually, one can weaken "ω_1 a precalibre for X" to "$c(X) \leq \aleph_0$" with minimal changes in the original proof. Our aim is to strengthen the latter result by showing that the finer connected strongly σ-discrete topology on X can be chosen *submetrizable.*

We will also show that the conclusion of [20, Theorem 2.12] remains valid without the condition $|X| = \mathfrak{c}$ (see Theorem 2.14). It is not known, however, whether every connected space X with $w(X) \leq \mathfrak{c}$ and $c(X) \leq \aleph_0$ admits a finer connected submetrizable topology if no restriction on the cardinality of X is given (see Problem 5.5).

Our generalizations of [20, Theorem 2.12] require considerable improvements of the methods applied in [2], [3] and [20] when refining connected topologies. We start with two general lemmas on dense connected subspaces of products that do not require any axioms of separation.

Lemma 2.1. *A dense subset S of a connected space X is connected iff S intersects the boundary of every proper regular open set in X. In addition, a dense subspace S of a product $\Pi = \prod_{\alpha\in A} X_\alpha$ of connected spaces X_α satisfying $c(\Pi) \leq \aleph_0$ is connected iff $\pi_B(S)$ is connected for each countable subset $B \subseteq A$, where $\pi_B : \Pi \to \prod_{\alpha\in B} X_\alpha$ is the projection.*

Proof. I. Let us prove the first part of the lemma. The necessity of the condition is clear. Suppose that S is disconnected and intersects the boundary of every $O \in \mathcal{R}o(X)$ with $\varnothing \neq O \neq X$. Represent S as the union of two nonempty disjoint open subsets, say $S = U_1 \cup U_2$. There exist open subsets V_1 and V_2 of X such that $U_i = X \cap V_i$, $i = 1, 2$. Since S is dense in X, the sets V_1 and V_2 are also disjoint. Put $W_i = \operatorname{Int} \overline{V_i}$, $i = 1, 2$. Then $V_i \subseteq W_i$ $(i = 1, 2)$ and $W_1 \cap W_2 = \varnothing$. We have $S = U_1 \cup U_2 \subseteq V_1 \cup V_2 \subseteq W_1 \cup W_2$, so the set $F = \overline{W}_1 \cap \overline{W}_2$ is disjoint from S. The set $X \setminus F = W_1 \cup W_2$ is disconnected, so that $F = \operatorname{Fr} W_1 = \overline{W}_1 \setminus W_1 \neq \varnothing$ and, hence, $F \cap S \neq \varnothing$, which is a contradiction. This proves the connectedness of S.

II. If S is connected, then all projections $\pi_B(S)$ are connected. Suppose that S is disconnected. Then one can find two nonempty open sets U and V in Π such that $S \subseteq U \cup V$ and $U \cap S \cap V = \varnothing$. Since S is dense in Π, the sets U and V are disjoint. Therefore, the set $F = \overline{U} \cap \overline{V}$ is disjoint from S. Since $c(\Pi) \leq \aleph_0$, the sets $\overline{U}$ and $\overline{V}$ depend on countably many coordinates. In other words, there exists a countable subset $B \subseteq A$ such that $\overline{U} = \pi_B^{-1}\pi_B(\overline{U})$ and $\overline{V} = \pi_B^{-1}\pi_B(\overline{V})$. Then $\pi_B(U)$ and $\pi_B(V)$ are disjoint open subsets of $\prod_{\alpha\in B} X_\alpha$ and $\pi_B(S) \subseteq \pi_B(U) \cup \pi_B(V)$, thus implying the disconnectedness of $\pi_B(S)$. $\square$

The proof of the second auxiliary result can be found in [17].

Lemma 2.2. *Let X and Y be connected spaces and S be a dense subset of the product $\Pi = X \times Y$ with $\pi_X(S) = X$, where $\pi_X : X \times Y \to X$ is the projection. If U and V are nonempty disjoint open sets in $X \times Y$ with $S \subseteq U \cup V$ and $F = \operatorname{cl}_\Pi U \cap \operatorname{cl}_\Pi V$, then the set $\pi_X(F)$ has a nonempty interior in X. In fact, there exists a nonempty open subset W of $\pi_X(F)$ which is contained in $\pi_X(U) \cap \pi_X(V)$.*

When one works with connected spaces, regular open sets appear in a natural way. For example, a dense subspace S of a space X is connected iff S intersects the boundary of each nonempty proper regular open set in X (see Lemma 2.1). Therefore, it is important to have an upper bound for the number of regular open sets in a space. The following was found by Šapirovskiĭ [16]. Again, no axiom of separation is required.

Lemma 2.3. $|\mathcal{R}o(X)| \leq \pi w(X)^{c(X)}$ *for every space* X.

It is also important to know when the product of two countably cellular spaces remains countably cellular. One special case of such a stability is given below (see also Theorem 4.8 of [15]).

Lemma 2.4. *Let* X *be a countably cellular space and* Z *be a dense subspace of a product of separable spaces. Then* $c(X \times Z) \leq \aleph_0$.

Proof. Suppose that Z is dense in the product $\Pi = \prod_{i \in I} Z_i$, where $d(Z_i) \leq \aleph_0$ for each $i \in I$. Since $X \times Z$ is dense in $X \times \Pi$, it suffices to show that $c(X \times \Pi) \leq \aleph_0$. The latter follows, however, from the fact that ω_1 is a calibre for Π [15]. □

A general idea of how one can refine connected topologies is in the following result.

Proposition 2.5. *Let* X *and* Y *be connected regular spaces such that* $|U| = |V| = \kappa$ *for all nonempty open subsets* U *of* X *and* V *of* Y*, where* κ *is an infinite cardinal. Suppose that* $|\mathcal{R}o(X \times Y)| \leq \kappa$ *and that* $Y \setminus A$ *is connected for each* $A \subseteq Y$ *with* $|A| < \kappa$*. Then there exists a (necessarily discontinuous) bijection* $f\colon X \to Y$ *such that the graph* $S = \mathrm{Gr}(f) = \{(x, f(x)) : x \in X\} \subseteq X \times Y$ *of the map* f *is dense in* $X \times Y$ *and connected.*

Proof. Enumerate the points of X and Y, say $X = \{x_\nu : \nu < \kappa\}$ and $Y = \{y_\nu : \nu < \kappa\}$. Denote by $\mathcal{R}$ the family of all proper regular open sets O in $X \times Y$ such that $\mathrm{Int}_X\, \pi_X\, \mathrm{Fr}(O) \neq \varnothing$, where $\pi_X : X \times Y \to X$ is the projection. By our assumption, $|\mathcal{R}| \leq |\mathcal{R}o(X \times Y)| \leq \kappa$, so there exists an enumeration $\mathcal{R} = \{O_\alpha : \alpha < \kappa\}$.

We will define by recursion on $\alpha < \kappa$ subsets $X_\alpha \subseteq X$ and injective functions $f_\alpha\colon X_\alpha \to Y$ satisfying the following conditions for each $\alpha < \kappa$:

(1) $|X_\alpha| \leq |\alpha| \cdot \aleph_0$;
(2) $X_\beta \subseteq X_\alpha$ if $\beta < \alpha$;
(3) $\{x_\beta : \beta \leq \alpha\} \subseteq X_\alpha$;
(4) $\{y_\beta : \beta \leq \alpha\} \subseteq f_\alpha(X_\alpha)$;
(5) $f_\alpha|_{X_\beta} = f_\beta$ if $\beta < \alpha$;
(6) $\overline{O}_\alpha \cap \mathrm{Gr}(f_\alpha) \neq \varnothing$;
(7) $\mathrm{Fr}(O_\alpha) \cap \mathrm{Gr}(f_\alpha) \neq \varnothing$ provided that there exist $x \in X \setminus \bigcup_{\beta<\alpha} X_\beta$ and $y \in Y \setminus \bigcup_{\beta<\alpha} f_\beta(X_\beta)$ such that $(x, y) \in \mathrm{Fr}(O_\alpha)$.

Suppose that for some $\alpha < \kappa$ we have defined the sequences $\{X_\beta : \beta < \alpha\}$ and $\{f_\beta : \beta < \alpha\}$ satisfying (1)–(7). Let $P = \bigcup_{\beta<\alpha} X_\beta$, $Q = \bigcup_{\beta<\alpha} f_\beta(X_\beta)$ and $g = \bigcup_{\beta<\alpha} f_\beta$. Clearly, $|P| \leq |\alpha| \cdot \aleph_0 < \kappa$ by (1). Since all f_β are injections, from (2) and (5) it follows that g is a bijection from P onto Q. In particular,

$|Q| = |P| \leq |\alpha| \cdot \aleph_0 < \kappa$. If $\mathrm{Fr}(O_\alpha) \subseteq (P \times Y) \cup (X \times Q)$, we choose $p \in X \setminus P$ and $q \in Y \setminus Q$ such that $(p, q) \in O_\alpha$ (this is possible because $|P| < \kappa$, $|Q| < \kappa$ and nonempty open sets in X and Y have cardinality κ). Otherwise we pick $p \in X \setminus P$ and $q \in Y \setminus Q$ such that $(p, q) \in \mathrm{Fr}(O_\alpha)$. Let X_α be a subset of X such that $P \cup \{p, x_\alpha\} \subseteq X_\alpha$ and $|X_\alpha \setminus P| = 3$. We can extend g to an injective function $f_\alpha \colon X_\alpha \to Y$ such that $y_\alpha \in f_\alpha(X_\alpha)$ and $f_\alpha(p) = q$. It is easy to see that X_α and f_α satisfy (1)–(7). This finishes the construction.

Define $f = \bigcup_{\alpha<\kappa} f_\alpha$ and $S = \mathrm{Gr}(f)$. From (2), (3), (4) and (5) it follows that f is a bijection of X onto Y. Since the product $X \times Y$ is regular, the definition of f together with (5) and (6) imply that S is dense in $X \times Y$.

Let us show that S is connected. Assume to the contrary that there exist nonempty open disjoint sets U and V in $X \times Y$ such that $S \subseteq U \cup V$. Put $U^* = \mathrm{Int}\,\overline{U}$ and $V^* = \mathrm{Int}\,\overline{V}$. Then U^* and V^* are disjoint regular open sets and $S \subseteq U \cup V \subseteq U^* \cup V^*$. Therefore, the set $F = \mathrm{Fr}(U^*)$ is disjoint from S. Clearly, $F = \overline{U^*} \cap \overline{V^*}$. By Lemma 2.2, the set $\pi_X(F)$ contains a nonempty open set W such that $W \subseteq \pi_X(U^*) \cap \pi_X(V^*)$. In particular, $U^* \in \mathcal{R}$, and, hence, $U^* = O_\alpha$ for some $\alpha < \kappa$. Since nonempty open subsets of X have size κ and $|X_\alpha| < \kappa$ by (1), we can pick a point $x \in W \setminus X_\alpha \subseteq X \setminus \bigcup_{\beta<\alpha} X_\beta$. The space $K_x = \{x\} \times Y$ is homeomorphic to Y. The sets $U_x = U^* \cap K_x$ and $V_x = V^* \cap K_x$ are nonempty, open and disjoint in the connected space K_x. Therefore, $U_x \cup V_x$ is disconnected. Put $H = K_x \setminus (U_x \cup V_x)$. Since $Y \setminus A$ is connected for each $A \subseteq Y$ with $|A| < \kappa$, we conclude that $|H| = \kappa$. Note that $|\bigcup_{\beta<\alpha} f_\beta(X_\beta)| \leq |\alpha| \cdot \aleph_0 < \kappa$, and so there exists a point $y \in Y \setminus \bigcup_{\beta<\alpha} f_\beta(X_\beta)$ such that $(x, y) \in H \subseteq F = \mathrm{Fr}(O_\alpha)$. Item (7) now implies that $F \cap S = \mathrm{Fr}(O_\alpha) \cap \mathrm{Gr}(f) \supseteq \mathrm{Fr}(O_\alpha) \cap \mathrm{Gr}(f_\alpha) \neq \varnothing$, a contradiction with $S \cap F = \varnothing$. Thus, S is connected.

If f were a continuous map, $S = \mathrm{Gr}(f)$ would become a closed subset of $X \times Y$ (note that Y is regular, hence Hausdorff). Since S is dense in $X \times Y$, it would then follow that $S = X \times Y$, a contradiction. □

Remark 2.6. The above proposition remains valid for Hausdorff spaces X and Y under the additional assumption that $\pi w(X) \cdot \pi w(Y) \leq \kappa$. To see this, take a π-base $\{W_\alpha : \alpha < \kappa\}$ for $X \times Y$ and define the functions $\{f_\alpha : \alpha < \kappa\}$ and the sets $\{X_\alpha : \alpha < \kappa\}$ to satisfy conditions (1)–(7) of the above proof, and the following one:

(8) $\mathrm{Gr}(f_\alpha) \cap W_\alpha \neq \varnothing$ for each $\alpha < \kappa$.

It is not clear, however, whether there exists a Hausdorff space X with $|\mathcal{R}o(X)| \leq |X| < \pi w(X)$.

We remind the reader that from now on, all spaces are Tychonoff.

Corollary 2.7. *Let X and Y be connected spaces such that $w(X) \cdot w(Y) \leq \mathfrak{c}$, $|X| = |Y| = \mathfrak{c}$ and $c(X \times Y) \leq \aleph_0$. Suppose that $Y \setminus A$ is connected for each $A \subseteq Y$ with $|A| < \mathfrak{c}$. Then the product $X \times Y$ contains a dense connected subspace S such that the projections of S to the factors are one-to-one maps onto X and Y.*

Proof. Note that the spaces X, Y and the cardinal $\kappa = \mathfrak{c}$ satisfy the conditions of Proposition 2.5. Indeed, every nonempty open subset of an infinite connected Tychonoff space has cardinality greater than or equal to $\mathfrak{c}$, and $|\mathcal{R}o(X \times Y)| \leq w(X \times Y)^{c(X\times Y)} \leq \mathfrak{c}^{\aleph_0} = \mathfrak{c}$ by Lemma 2.3. □

Remark 2.8. The roles of X and Y in Corollary 2.7 (and in Proposition 2.5) are not symmetric: the space Y was assumed to have the property that $Y \setminus A$ is connected for each $A \subseteq Y$ with $|A| < \mathfrak{c}$. This condition is actually essential: the plane $\mathbb{R} \times \mathbb{R}$ does not contain a dense connected subset S whose projections to both factors are bijections. Indeed, let S be a dense subset of $\mathbb{R}^2$ whose projections to the factors are one-to-one, and suppose that S intersects the y-axis in the point $(0, b)$. Let $(0, c)$ be any other point on the y-axis and let $Q_1, \ldots, Q_4$ be the open quadrants determined by the lines $x = 0$, $y = c$. It is clear that one of these quadrants has a frontier which misses S.

We show now that many connected spaces admit finer connected strongly σ-discrete (submetrizable) topologies.

Theorem 2.9. *Let X be a connected space satisfying $w(X) \leq \mathfrak{c} = |X|$ and $c(X) \leq \aleph_0$. Then there exists a connected strongly σ-discrete submetrizable space T which satisfies the same cardinal restrictions and admits a continuous bijection $i\colon T \to X$.*

Proof. By Theorem 2.24 of [3], there exists a dense connected strongly σ-discrete subspace Y of $\mathbb{I}^{\mathfrak{c}}$ such that $Y \setminus A$ is connected for each $A \subseteq Y$ with $|A| < \mathfrak{c}$ (for the latter property of Y, see the proof of Theorem 2.12 of [20]). Note that $w(Y) = |Y| = \mathfrak{c}$. In addition, Lemma 2.4 implies that $c(X \times Y) \leq \aleph_0$. Apply Proposition 2.5 to find a dense connected subset S of the product $X \times Y$ such that the projections of S to the factors X and Y are bijections. Clearly, S is strongly σ-discrete and satisfies $w(S) = |S| = \mathfrak{c}$, $c(S) \leq \aleph_0$. It is easy to verify that the boundary of every proper regular open subset of $\mathbb{R}^2$ is of cardinality $\mathfrak{c}$, so that $\mathbb{R}^2 \setminus A$ is connected for each $A \subseteq \mathbb{R}^2$ with $|A| < \mathfrak{c}$. Apply Proposition 2.5 once again to find a dense connected subset T of the product $S \times \mathbb{R}^2$ whose projections to S and $\mathbb{R}^2$ are bijections. Clearly, T is strongly σ-discrete and submetrizable. Let $\pi_X\colon X \times Y \to X$ and $\pi_S\colon S \times \mathbb{R}^2 \to S$ be projections. Then $i = \pi_X \circ \pi_S|_T$ is a continuous bijection of T onto X. Note that $w(T) = |T| = \mathfrak{c}$ and $c(T) \leq \aleph_0$. □

The following result resembles Theorem 2.9, but deals with a more general situation.

Theorem 2.10. *Let X be a non-trivial connected space such that $|U| = |X|$ for every nonempty open subset U of X. Suppose that there exists a space Y with the following properties:*

(i) *$|V| = |X|$ for every nonempty open subset V of Y;*
(ii) *$Y \setminus A$ is connected for every set $A \subseteq Y$ with $|A| < |X|$, and*
(iii) *$|\mathcal{R}o(X \times Y)| \leq |X|$.*

Then there exists a strictly finer connected topology $\mathcal{T}$ on X such that $(X, \mathcal{T})$ admits a one-to-one continuous map onto the space Y.

Proof. Note that X, Y and $\kappa = |X|$ satisfy the assumptions of Proposition 2.5. Let $f\colon X \to Y$ be the bijection as in the conclusion of this proposition. Then $S = \mathrm{Gr}(f)$ is a connected space and $\pi_X|_S\colon S \to X$ and $\pi_Y|_S\colon S \to Y$ are continuous bijections, where $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ are the projections. Since f is discontinuous, the inverse map $(\pi_X|_S)^{-1} : X \to S$ is not continuous, i.e., the topology of S is strictly finer than that of X. It remains to define the topology $\mathcal{T}$ on X by

$$\mathcal{T} = \{\pi_X(U \cap S) : U \text{ is open in } X \times Y\}.$$

One easily verifies that the space $(X, \mathcal{T})$ is as required. □

Lemma 2.11. *Suppose that X and Y are spaces such that $w(X)^{c(X)} \le 2^\tau$ and $w(Y) \le \tau$. Then $w(X \times Y)^{c(X\times Y)} \le 2^\tau$.*

Proof. Let us consider two cases. If $c(X) \le \tau$, then $c(X \times Y) \le \tau$ because $w(Y) \le \tau$, and we have

$$w(X \times Y)^{c(X\times Y)} \le w(X \times Y)^\tau \le \max\{w(X)^\tau, \tau^\tau\} \le \max\{(2^\tau)^\tau, 2^\tau\} = 2^\tau.$$

Assume now that $c(X) > \tau$. In this case we have $c(X \times Y) \le c(X)$ and $\tau^{c(X)} \le c(X)^{c(X)} = 2^{c(X)} \le w(X)^{c(X)} \le 2^\tau$. Therefore,

$$w(X\times Y)^{c(X\times Y)} \le w(X\times Y)^{c(X)} \le \max\{w(X)^{c(X)}, \tau^{c(X)}\} \le \max\{2^\tau, 2^\tau\} = 2^\tau.$$

This proves the lemma. □

In what follows $J(\tau)$ denotes the metric hedgehog with τ many spines [10, Example 4.1.5].

Theorem 2.12. *Let X be a connected space such that $w(X)^{c(X)} \le 2^\tau$ and $|U| = |X| = 2^\tau$ for every nonempty open subset of X, where τ is an infinite cardinal. Then:*

- (i) *there exists a strictly finer connected topology $\mathcal{T}_0$ on X such that $(X, \mathcal{T}_0)$ admits a one-to-one continuous map onto the Tychonoff cube I^τ;*
- (ii) *if one additionally assumes that $\tau^\omega = 2^\tau$, then there exists a strictly finer connected topology $\mathcal{T}_1$ on X such that $(X, \mathcal{T}_1)$ admits a one-to-one continuous map onto the countable power $J(\tau)^\omega$ of the hedgehog $J(\tau)$.*

Proof. (i) Let $Y = I^\tau$. Then $|V| = |I^\tau| = 2^\tau = |X|$ for every nonempty open subset of Y. Furthermore, it is easy to see that $I^\tau \setminus A$ is connected for every $A \subseteq I^\tau$ such that $|A| < 2^\tau = |X|$. Thus, the conditions (i) and (ii) of Theorem 2.10 are satisfied. To check (iii), note that $w(Y) = \tau$, and so $w(X\times Y)^{c(X\times Y)} \le 2^\tau = |X|$ by Lemma 2.11. Therefore, $|\mathcal{R}o(X \times Y)| \le w(X \times Y)^{c(X\times Y)} \le |X|$ by Lemma 2.3. Now Theorem 2.10 applies.

(ii) Assume now that $\tau^\omega = 2^\tau$ holds. Let $Y = J(\tau)^\omega$. Then $|Y| = \tau^\omega = 2^\tau = |X|$. It is easy to check that Y satisfies the conditions (i) and (ii) of Theorem 2.10. Since $w(Y) \le \tau$, the argument of (i) shows that Y also satisfies (iii) of Theorem 2.10. Our result now follows from Theorem 2.10. □

The lemma that follows is the first step on the way to refining connected topologies on “large” connected spaces X with $\mathcal{R}o(X) \leq \mathfrak{c}$.

Lemma 2.13. *Let X be a non-trivial connected space with $w(X) \leq \mathfrak{c}$ and $c(X) \leq \aleph_0$. Then one can find a space X^*, a dense open connected subset U of X^* and a continuous bijection $i\colon X^* \to X$ satisfying*

(1) *$w(X^*) \leq \mathfrak{c}$ and $c(X^*) \leq \aleph_0$;*
(2) *$|U| = \mathfrak{c}$;*
(3) *$U = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed and discrete in X^*.*

Proof. If $|X| = \mathfrak{c}$, the conclusion follows from [20, Theorem 2.12]. Suppose that $|X| > \mathfrak{c}$. By Lemma 2.3, the family $\mathcal{R}o(X)$ satisfies $|\mathcal{R}o(X)| \leq w(X)^{c(X)} \leq \mathfrak{c}^{\aleph_0} = \mathfrak{c}$. For every proper $U \in \mathcal{R}o(X)$, pick a point $x_U \in \overline{U} \setminus U$ and put $S = \{x_U : U \in \mathcal{R}o(X)\}$. Then S is obviously dense in X, $|S| \leq \mathfrak{c}$, and Lemma 2.1 implies that S is connected. The latter enables us to conclude that $|S| = \mathfrak{c}$.

Let Y be a dense connected strongly σ-discrete subspace of $\mathbb{I}^{\mathfrak{c}}$ such that $|Y| = \mathfrak{c}$ and $Y \setminus A$ is connected for every subset A of Y with $|A| < \mathfrak{c}$ (see [3, Theorem 2.24] and [20, Theorem 2.12]). Pick a point $y^* \in Y$ and put $Y^* = Y \setminus \{y^*\}$. By Proposition 2.5 (with $\kappa = \mathfrak{c}$), there exists a dense connected subspace U of $S \times Y^*$ such that the restriction to U of the projections $\pi_S\colon S \times Y^* \to S$ and $\pi_{Y^*}\colon S \times Y^* \to Y^*$ are one-to-one maps of U onto S and Y^* respectively. Clearly, U is strongly σ-discrete. Put

$$X^* = U \cup (X \setminus S) \times \{y^*\}, \;\; X^* \subseteq X \times Y.$$

Since $U \subseteq X^*$, the set X^* is dense in $X \times Y$ and connected. In addition, $X^* \setminus U = (X \setminus S) \times \{y^*\}$ is closed in X^* and, hence, U is open in X^*. Clearly, we have $w(X^*) \leq w(X) \cdot w(Y) \leq \mathfrak{c}$. It remains to note that $c(X^*) \leq \aleph_0$. Indeed, since $c(X) \leq \aleph_0$, $d(\mathbb{I}^{\mathfrak{c}}) \leq \aleph_0$ and X^* is a dense subspace of $X \times \mathbb{I}^{\mathfrak{c}}$, Lemma 2.4 implies that $c(X^*) \leq \aleph_0$. □

Recall that a dense in itself space X is called *submaximal* if every dense subset of X is open. It is an unsolved problem dating back to Hewitt [14] whether there exists a submaximal connected Tychonoff space or even a submaximal connected Tychonoff topology on $\mathbb{R}$. Arhangel'skiĭ and Collins formulated a similar question for normal and paracompact spaces (see Problem 4.1 of [6]), but it has been shown in [1, Corollary 3.4] that at least consistently such spaces cannot be normal. It is also known that the topology of a submaximal space contains a base of a free ultrafilter [9]. Therefore, any submaximal connected (Tychonoff) topology on $\mathbb{R}$ (if such exists) has large weight. The following result shows that there exist connected Tychonoff topologies on $\mathbb{R}$ of weight $2^{\mathfrak{c}}$.

Theorem 2.14. *Let X be a non-trivial connected space with $w(X) \leq \mathfrak{c}$ and $c(X) \leq \aleph_0$. Then X admits a finer connected strongly σ-discrete topology $\mathcal{T}$ of weight $2^{\mathfrak{c}}$. In addition, the topology $\mathcal{T}$ can be chosen so that $(X, \mathcal{T})$ will be homeomorphic to a dense subspace S of $X \times \mathbb{I}^{2^{\mathfrak{c}}}$ and, hence, $\chi(y, S) = 2^{\mathfrak{c}}$ for each $y \in S$.*

Proof. Put $\tau = 2^{\mathfrak{c}}$. One can find a space X^*, a dense open connected subset U of X^* and a continuous bijection $i\colon X^* \to X$ satisfying (1)–(3) of Lemma 2.13. The idea of our construction is to define a dense connected subspace S of $\Pi = X^* \times \mathbb{I}^\tau$ such that the restriction to S of the projection $p^*\colon X^* \times \mathbb{I}^\tau \to X^*$ will be a one-to-one map of S onto X^*. In addition, the space S will be strongly σ-discrete.

To guarantee connectedness of S we have to make connected all projections of S to countable faces in Π (see Lemma 2.1). The latter necessarily implies that S will intersect the boundary of all regularly open sets in Π. We will construct S as the graph of a (discontinuous) map $f^*\colon X^* \to \mathbb{I}^\tau$. The problem we face is the fact that $|U| = \mathfrak{c}$ while there are $\tau = 2^{\mathfrak{c}}$ different regular open sets in $X \times \mathbb{I}^\tau$ and almost all projections of their boundary to X^* must intersect U. This obstacle makes our construction technically involved. Let us divide the construction of S into several steps.

I. *Definitions.* Let t be a Hausdorff topology on the index set τ such that $w(\tau, t) = \mathfrak{c}$ and every G_δ-set in (τ, t) is open. For example, we can identify (τ, t) with the space $\{0,1\}^{\mathfrak{c}}$ endowed with the $\aleph_0$-box topology. Denote by $\mathcal{B}$ a base of t with $|\mathcal{B}| = \mathfrak{c}$, say $\mathcal{B} = \{V_\alpha : \alpha < \mathfrak{c}\}$.

For every ordinal δ with $\omega \le \delta < \omega_1$, we fix a bijection $b_\delta\colon \omega \to \delta$ and define the homeomorphism $g_\delta\colon \mathbb{I}^\omega \to \mathbb{I}^\delta$ by the rule $g_\delta(x)(\alpha) = x(b_\delta^{-1}(\alpha))$ for all $x \in \mathbb{I}^\omega$ and $\alpha \in \delta$. In other words, the homomorphism g_δ is determined by the coordinate bijection b_δ.

Suppose that C is a countably infinite subset of τ. Denote by δ the order type of $(C, <_C)$, where $<_C$ is the order on C induced by the usual well-ordering of τ. Let $b_C\colon \delta \to C$ be the order preserving bijection of δ onto C. Define the homeomorphism $g_C\colon \mathbb{I}^\delta \to \mathbb{I}^C$ by the rule $g_C(x)(\alpha) = x(b_C^{-1}(\alpha))$ for all $x \in \mathbb{I}^\delta$ and $\alpha \in \delta$. Then $g_C^* = g_C \circ g_\delta\colon \mathbb{I}^\omega \to \mathbb{I}^C$ and $h_C = id_{X^*} \times g_C^*\colon X^* \times \mathbb{I}^\omega \to X^* \times \mathbb{I}^C$ are coordinate homeomorphisms.

Since $c(X^* \times \mathbb{I}^\omega) \le \aleph_0$, Lemma 2.3 implies that the cardinality of the family of regular open sets in $X^* \times \mathbb{I}^\omega$ does not exceed $\mathfrak{c}^\omega = \mathfrak{c}$. Let $\{O_\mu : \mu < \mathfrak{c}\}$ be an enumeration of all regular open subsets O of $X^* \times \mathbb{I}^\omega$ such that the projection of the boundary of O to the first factor X^* has a nonempty interior in X^*. Let us also fix an enumeration of points of U in a one-to-one way, say $U = \{x_\alpha : \alpha < \mathfrak{c}\}$.

Finally, we put

$$\mathcal{F} = \{p \in \mathfrak{c}^\delta : \omega \le \delta < \omega_1,\ \ \forall \nu, \mu < \delta\, (\nu \ne \mu \Rightarrow V_{p(\nu)} \cap V_{p(\mu)} = \varnothing)\}.$$

Clearly, $|\mathcal{F}| = \mathfrak{c}$.

II. *Recursive construction.* We will first define a map $f\colon U \to \mathbb{I}^\tau$ with the dense connected graph $\mathrm{Gr}(f) = \{(x, f(x)) : x \in U\} \subseteq \Pi$. To this end, we shall construct by recursion auxiliary map $\varphi\colon \mathfrak{c} \to \mathfrak{c}$ and $\psi\colon \mathfrak{c} \to \mathbb{I}^\tau$.

Let $\theta\colon \mathfrak{c} \to \mathfrak{c} \times \mathcal{F}$ be a bijection. Suppose that for some $\alpha < \mathfrak{c}$, we have defined $\varphi(\beta)$ and $\psi(\beta)$ for each $\beta < \alpha$. Consider $\theta(\alpha) = (\mu, p)$ and put $\delta = \mathrm{dom}(p)$. By definition of $\mathcal{F}$, the family $\{V_{p(\nu)} : \nu < \delta\}$ is disjoint. Choose a strictly increasing sequence $C_\alpha = \{\lambda_\nu : \nu < \delta\} \subseteq \tau$ such that $\lambda_\nu \in V_{p(\nu)}$ for each $\nu < \delta$.

This is possible because V_β is cofinal in τ for each $\beta < \mathfrak{c}$. By definition, the projection of $\operatorname{Fr} O_\mu \subseteq X^* \times \mathbb{I}^\omega$ to the first factor has a nonempty interior in X^*, say W. Therefore, W is contained in $\pi_C(\operatorname{Fr} h_C(O_\mu))$, where $C = C_\alpha$ and $\pi_C\colon X^* \times \mathbb{I}^C \to X^*$ is the projection. Note that $|W| \geq \mathfrak{c}$. Put

$$U_\alpha = \{x_{\varphi(\beta)} : \beta < \alpha\} \text{ and } \varphi(\alpha) = \min\{\lambda < \mathfrak{c} : x_\lambda \in W \setminus U_\alpha\}.$$

Then $x_{\varphi(\alpha)} \in W$ and we can choose a point $y_\alpha \in \mathbb{I}^C$ such that $(x_{\varphi(\alpha)}, y_\alpha) \in \operatorname{Fr}(h_C(O_\mu))$. We define a point $\tilde{y}_\alpha \in \mathbb{I}^\tau$ as follows:

$$\tilde{y}_\alpha(\gamma) = y_\alpha(\lambda_\nu) \text{ if } \gamma \in V_{p(\nu)} \text{ for some } \nu < \delta, \text{ and } \tilde{y}_\alpha(\gamma) = 0 \text{ otherwise.}$$

It remains to put $\psi(\alpha) = \tilde{y}_\alpha$. This finishes the construction of the map $\varphi\colon \mathfrak{c} \to \mathfrak{c}$ and $\psi\colon \mathfrak{c} \to \mathbb{I}^\tau$. It is easy to see that φ is injective. Put $U^* = \{x_{\varphi(\alpha)} : \alpha < \mathfrak{c}\}$. Denote by y^* the point in $\mathbb{I}^\tau$ all coordinates of which are equal to zero and define the map $f\colon U \to \mathbb{I}^\tau$ by the rule:

$$f(x_{\varphi(\alpha)}) = \tilde{y}_\alpha \text{ for every } \alpha < \mathfrak{c}, \text{ and } f(x) = y^* \text{ if } x \in U \setminus U^*.$$

Let us verify that the graph of f is dense in Π and connected. By Lemma 2.1, it suffices to show that $\operatorname{Gr}(f)$ intersects the boundary of every proper regular open set in Π. Consider a nonempty regular open set O in Π, $O \neq \Pi$, and put $F = \operatorname{Fr} O$. Since $c(\Pi) \leq \aleph_0$, there exists a countable subset D of τ such that $O = p_D^{-1} p_D(O)$, where $p_D\colon X^* \times \mathbb{I}^\tau \to X^* \times \mathbb{I}^D$ is the projection. Without loss of generality we can assume that D is infinite. Since p_D is an open map, we have $F = p_D^{-1} p_D(F)$ and, in addition, $p_D(O)$ is regular open in $X^* \times \mathbb{I}^D$. Therefore, the set $h_D^{-1}(p_D(O))$ is regular open in $X^* \times \mathbb{I}^\omega$ and, hence, $h_D^{-1}(p_D(O)) = O_\mu$ for some $\mu < \mathfrak{c}$. Denote by δ the order type of $(D, <_D)$ and enumerate the elements of D in increasing order, say $D = \{\gamma_\nu : \nu < \delta\}$. There exists a disjoint subfamily $\{V_{\alpha_\nu} : \nu < \delta\}$ of $\mathcal{B}$ such that $\gamma_\nu \in V_{\alpha_\nu}$ for each $\nu < \lambda$. Let us define a map $p\colon \delta \to \mathfrak{c}$ by the rule $p(\nu) = \alpha_\nu$ for each $\nu < \delta$. Clearly, $p \in \mathcal{F}$. By definition of θ, there exists $\alpha < \mathfrak{c}$ such that $\theta(\alpha) = (\mu, p)$. We claim that $z_\alpha = (x_{\varphi(\alpha)}, \tilde{y}_\alpha) \in \operatorname{Gr}(f) \cap F \neq \varnothing$, and this is the key point of the proof. The fact that $z_\alpha \in \operatorname{Gr}(f)$ follows directly from the definition of f, so it suffices to verify that $z_\alpha \in F$. At the step α of our construction, we had $(x_{\varphi(\alpha)}, y_\alpha) \in \operatorname{Fr} h_C(O_\mu)$, where $C = C_\alpha$. Since $\tilde{y}_\alpha|_C = y_\alpha$, we deduce that

$$p_C(z_\alpha) = (x_{\varphi(\alpha)}, \tilde{y}_\alpha|_C) = (x_{\varphi(\alpha)}, y_\alpha) \in \operatorname{Fr} h_C(O_\mu) \tag{1}$$

Let $C = \{\lambda_\nu : \nu < \delta\}$ be the enumeration of C in increasing order. By recursive definitions at step α, we have $\lambda_\nu \in V_{\alpha_\nu}$ for each $\nu < \delta$. Therefore, $\gamma_\nu, \lambda_\nu \in V_{\alpha_\nu}$ for each $\nu < \delta$. The latter immediately implies that

$$\tilde{y}_\alpha(\gamma_\nu) = \tilde{y}_\alpha(\lambda_\nu) \text{ for each } \nu < \lambda. \tag{2}$$

Let us define a coordinate homeomorphism $g_D^C\colon \mathbb{I}^C \to \mathbb{I}^D$ by $g_D^C(x)(\gamma_\nu) = x(\lambda_\nu)$ for all $x \in \mathbb{I}^C$ and $\nu < \delta$. Then $h_D^C = id_{X^*} \times g_D^C$ is also a coordinate homeomorphism of $X^* \times \mathbb{I}^C$ onto $X^* \times \mathbb{I}^D$. Clearly, from (2) it follows that

$$g_D^C(\tilde{y}_\alpha|_C) = \tilde{y}_\alpha|_D. \tag{3}$$

In addition, the definition of the coordinate homeomorphisms h_C and h_D implies the equality

$$h_D = h_D^C \circ h_C. \tag{4}$$

Applying (3), (4) and (1), we deduce that

$$\begin{aligned} p_D(z_\alpha) = (x_{\varphi(\alpha)}, \tilde{y}_\alpha|_D) = h_D^C(x_{\varphi(\alpha)}, \tilde{y}_\alpha|_C) &= h_D^C p_C(z_\alpha) \in h_D^C(\operatorname{Fr} h_C(O_\mu)) \\ &= \operatorname{Fr} h_D^C h_C(O_\mu) = \operatorname{Fr} h_D(O_\mu). \end{aligned}$$

Combining the latter fact with the equalities

$$F = \operatorname{Fr} O,\ O = p_D^{-1} p_D(O) \text{ and } p_D(O) = h_D(O_\mu),$$

we obtain

$$z_\alpha \in p_D^{-1}(\operatorname{Fr} h_D(O_\mu)) = \operatorname{Fr} p_D^{-1}(h_D(O_\mu)) = \operatorname{Fr} p_D^{-1}(p_D(O)) = \operatorname{Fr} O = F.$$

So, we have proved that $\mathrm{Gr}(f)$ is a dense connected subset of Π. It remains to extend f to a map $f^*\colon X^* \to \mathbb{I}^\tau$ in such a way that $\mathrm{Gr}(f^*)$ will be strongly σ-discrete.

III. *Final step.* Put $K = X^* \setminus U$. Then K is closed in X^* and $|K| \le |X^*| \le 2^{w(X^*)} \le 2^{\mathfrak{c}} = \tau$. Let $\{y_\alpha : \alpha < \kappa\}$ be an enumeration of K, where $\kappa = |K| \le \tau$. For every $\alpha < \kappa$, define the point $a_\alpha \in \mathbb{I}^\tau$ by the rule: $e_\alpha(\beta) = 1$ if $\alpha = \beta$, and $e_\alpha(\beta) = 0$ otherwise. Clearly, $\{e_\alpha : \alpha < \kappa\}$ is a discrete subset of $\mathbb{I}^\tau$ and, hence, $K' = \{(y_\alpha, e_\alpha) : \alpha < \kappa\}$ is discrete in $X^* \times \mathbb{I}^\tau$. The closure of K' is contained in $K \times \mathbb{I}^\tau$, so that K' is closed and discrete in $S = K' \cup \mathrm{Gr}(f)$.

By the choice of $U \subseteq X^*$, there exists a family $\{F_n : 1 \le n < \omega\}$ of closed discrete subsets of X^* such that $U = \bigcup_{n=1}^\infty F_n$. For every $n \ge 1$, put $F_n' = \{z \in S : p^*(z) \in F_n\}$, where $p^*\colon X^* \times \mathbb{I}^\tau$ is the projection. Then F_n' is closed discrete in S and $S = K' \cup \bigcup_{n=1}^\infty F_n'$. In other words, S is a strongly σ-discrete subspace of $X^* \times \mathbb{I}^\tau$. Note that S is the graph of the map $f^*\colon X^* \to \mathbb{I}^\tau$ defined by $f^*(x) = f(x)$ if $x \in U$ and $f^*(y_\alpha) = e_\alpha$ for each $\alpha < \kappa$. Since $\mathrm{Gr}(f) \subseteq \mathrm{Gr}(f^*)$, we conclude that S is dense in $X^* \times \mathbb{I}^\tau$ and connected. Clearly, the map $j = i \circ p^*$ is a continuous bijection of S onto X. This means that X admits a finer strongly σ-discrete connected Tychonoff topology. Finally, X^* is a dense subspace of $X \times \mathbb{I}^{\mathfrak{c}}$, so that S is dense in $X \times \mathbb{I}^{\mathfrak{c}} \times \mathbb{I}^\tau \cong X \times \mathbb{I}^\tau$. This finishes the proof. □

Corollary 2.15. *Every separable connected space (in particular, the Čech-Stone compactification $\beta\mathbb{R}$ of the reals) admits a finer strongly σ-discrete connected topology.*

Corollary 2.16. *The real line $\mathbb{R}$ admits a finer connected strongly σ-discrete Tychonoff topology of weight $2^{\mathfrak{c}}$.*

3. Open preimages of connected spaces

By Theorem 2.6 of [20], for every Tychonoff connected space X one can find a Tychonoff connected strongly σ-discrete space Y which admits an open continuous onto map $f\colon Y \to X$. Our aim is to show that the space Y can be additionally chosen submetrizable. This improvement will enable us to represent every connected topological group as a quotient of a connected strongly σ-discrete group (see Theorem 4.2). Primarily, we show that the preimage Y for X can always be chosen of the same cardinality as X. In fact, our proof is a simplified version of that given in [20].

Theorem 3.1. *Let X be a connected space of size τ. Then there exists a dense connected subspace Y of $X \times \mathbb{I}^\tau$ with $|Y| = |X|$ which openly and continuously maps onto X under the projection $\pi\colon X \times \mathbb{I}^\tau \to X$ and satisfies:*

(i) *Y is strongly σ-discrete;*
(ii) *$Y_x = \pi^{-1}(x) \cap Y$ is connected and dense in $\{x\} \times \mathbb{I}^\tau$ for each $x \in X$;*
(iii) *for every nonempty open subset U of $X \times \mathbb{I}^\tau$ and every point $x \in \pi(U)$ one has $|U \cap Y_x| = \tau$;*
(iv) *if $B \subseteq Y$ and $|B| < \tau$, then B is closed in Y and discrete.*

Proof. Without loss of generality we can assume that X is infinite, so that $\tau = |X| \geq \mathfrak{c}$. For every $\alpha < \tau$, choose a set $T_\alpha \subseteq \tau$ such that $|T_\alpha| = \tau$, $\bigcup_{\alpha<\tau} T_\alpha = \tau$ and $T_\alpha \cap T_\beta = \varnothing$ if $\alpha \neq \beta$. Consider the set

$$\mathcal{F} = \{\mathbb{I}^F : F \subseteq \tau,\ |F| < \omega\}.$$

Clearly, $|\mathcal{F}| = \tau \cdot \mathfrak{c} = \tau$. Since the set $X \times \mathcal{F}$ has the same cardinality τ, we can enumerate it in order type τ, say, $X \times \mathcal{F} = \{e_\alpha : \alpha < \tau\}$. For every $\alpha < \tau$ we have $e_\alpha = (x_\alpha, p_\alpha)$, where $x_\alpha \in X$ and $p_\alpha \in \mathcal{F}$. Denote by S_α the finite subset of τ which determines the face the point p_α belongs to, i.e., $p_\alpha \in \mathbb{I}^{S_\alpha}$.

For every $\alpha < \tau$, let y_α be a point of $X \times \mathbb{I}^\tau$ defined by

$$y_\alpha(\beta) = \begin{cases} p_\alpha(\beta), & \text{if } \beta \in S_\alpha; \\ 1, & \text{if } \beta \in T_\alpha \setminus S_\alpha; \\ 0, & \text{if } \beta \in \tau \setminus (T_\alpha \cup S_\alpha). \end{cases}$$

We claim that the subspace $Y = \{y_\alpha : \alpha < \tau\}$ of the product space $X \times \mathbb{I}^\tau$ is as required.

Denote by f the restriction to Y of the projection π. Clearly, $f(Y) = X$, because for any $x \in X$ there exists an $\alpha < \tau$ such that $x_\alpha = x$ and, hence, $f(y_\alpha) = x$. Let us verify that the set $Y_x = f^{-1}(x)$ is dense in $\{x\} \times \mathbb{I}^\tau$ and connected for each $x \in X$. The density is immediate, because Y_x covers all finite faces in $\{x\} \times \mathbb{I}^\tau$. Since this holds for all $x \in X$, the map $f\colon Y \to X$ is open (see [7, Chap. 2, Problem 340]). By Lemma 2.1, to prove connectedness of Y_x it suffices to verify that the projection of Y_x to every countable face $\mathbb{I}^A$ of $\mathbb{I}^\tau$ is connected. Let $A \subseteq \tau$ be countable and $\pi_A\colon \{x\} \times \mathbb{I}^\tau \to \mathbb{I}^A$ be the corresponding projection. We will show that $\pi_A(Y_x)$ contains the σ-product

$$\sigma(A) = \{z \in \mathbb{I}^A : |\{\alpha \in A : z(\alpha) \neq 0\}| < \omega\}.$$

Fix a point $z \in \sigma(A)$ and put $F = \{\alpha \in A : z(\alpha) \neq 0\}$. Then $|F| < \omega$ and, hence, the set $B = \{\beta < \tau : p_\beta = z|_F\}$ has cardinality τ. Since the sets T_α are disjoint, there exists $\beta \in B$ such that $A \cap T_\beta = \varnothing$. Then $y_\beta(\alpha) = 0$ for each $\alpha \in A \setminus F = A \setminus S_\beta$, whence it follows that $\pi_A(y_\beta) = z$. Therefore,

$\sigma(A) \subseteq \pi_A(F_x)$. Since $\sigma(A)$ is a dense connected subspace of $\mathbb{I}^A$, the above inclusion implies connectedness of $\pi_A(Y_x)$. This implies (ii).

We can now conclude that Y is a connected space being an open monotone preimage of the connected space X [10, Theorem 6.1.29].

Note that for every $x \in X$ and every $p \in \mathcal{F}$, there exist τ many indices $\alpha < \tau$ such that $x_\alpha = x$ and $p_\alpha = p$. Since $f(y_\alpha) = x_\alpha$ and $y_\alpha(\nu) = p_\alpha(\nu)$ for each $\nu \in S_\alpha$, the property (iii) is immediate.

Let H be a subset of Y with $|H| < \tau$. Then $H = \{y_\beta : \beta \in B\}$, where $B \subseteq \tau$ and $|B| < \tau$. To show that H is closed in Y and discrete, take any $\alpha < \tau$ and put $S = S_\alpha \cup \bigcup_{\beta \in B} S_\beta$. Then $|S| < \tau$, so there exists $\gamma \in T_\alpha \setminus S$. The open subset $U = \{y \in Y : y(\gamma) > 0\}$ of Y contains y_α and $|U \cap H| \leq 1$, i.e., (iv) holds.

Finally, for every non-negative integer n consider the set $Y_n = \{y_\beta : \beta < \tau,\ |S_\beta| = n\}$. We claim that Y_n is closed in Y and discrete for each n. Indeed, take any $\alpha < \tau$ and choose distinct ordinals $\beta_1, \dots, \beta_{n+1} \in T_\alpha \setminus S_\alpha$. The set

$$V = \{y \in Y : y(\beta_i) > 0 \text{ for each } i \leq n+1\}$$

is an open neighborhood of y_α in Y and $|V \cap Y_n| \leq 1$, whence the conclusion follows. Since $Y = \bigcup_{n \in \omega} Y_n$, this proves (i). □

The next set-theoretic lemma is well known (see [8]), so we just give a brief outline of its proof here.

Lemma 3.2. *For every cardinal σ there exists a cardinal τ such that $\sigma < \tau$ and $\tau^\omega = 2^\tau$.*

Proof. Define a sequence $\{\tau_n : n \in \omega\}$ of cardinals by $\tau_0 = \sigma \cdot \aleph_0$ and $\tau_{n+1} = 2^{\tau_n}$. Let $\tau = \sup\{\tau_n : n \in \omega\}$. Then $\sigma < \tau$ and $\tau^\omega = 2^\tau$ by a theorem of [8]. □

Recall that $J(\tau)$ is the metric hedgehog with τ many spines.

Lemma 3.3. *For every infinite cardinal τ there exists a partition $J(\tau)^\omega = \bigcup\{Y_\nu : \nu < \tau^\omega\}$ of $J(\tau)^\omega$ into pairwise disjoint sets satisfying the following conditions for each $\nu < \tau^\omega$:*

(i) *Y_ν is dense in $J(\tau)^\omega$;*
(ii) *$|V| = \tau^\omega$ for every nonempty open subset V of Y_ν;*
(iii) *$Y_\nu \setminus A$ is connected for each $A \subseteq Y_\nu$ with $|A| < \tau^\omega$.*

Proof. For $x, y \in J(\tau)^\omega$, put

$$\operatorname{diff}(x, y) = \{n \in \omega : x(n) \neq y(n)\}.$$

Then define an equivalence relation $\sim$ on $J(\tau)$ by $x \sim y$ if $\operatorname{diff}(x, y)$ is finite. An easy verification shows that the family of equivalence classes of $(J(\tau)^\omega, \sim)$ has cardinality τ^ω. Pick one point in every equivalence class and enumerate the corresponding set of representatives, say $\{z_\nu : \nu < \tau^\omega\}$. For every $\nu < \tau^\omega$, denote by E_ν the equivalence class containing z_ν. It is clear that E_ν is a dense connected subspace of $J(\tau)^\omega$. For $\nu < \tau^\omega$, set $Y_\nu = E_\nu \times J(\tau)^\omega \subseteq J(\tau)^\omega \times J(\tau)^\omega$. Since $J(\tau)^\omega$ is homeomorphic to $J(\tau)^\omega \times J(\tau)^\omega$, it is easy to verify that $\{Y_\nu : \nu < \tau^\omega\}$ is the required partition. □

Our next result considerably strengthens Theorem 2.6 of [20] by making the preimage submetrizable:

Theorem 3.4. *Let Z be a connected T_i-space ($i = 2, 3, 3\frac{1}{2}$). Then there exists a connected strongly σ-discrete T_i-space S which openly and continuously maps onto Z and admits a one-to-one continuous map onto the countable power of a hedgehog (in particular, S is submetrizable). Moreover, every set $B \subseteq S$ with $|B| < |S|$ is closed in S and discrete.*

Proof. Let $|Z| = \sigma$ and use Lemma 3.2 to find a cardinal $\tau > \sigma$ such that $\tau^\omega = 2^\tau$. By Theorem 3.1, there exists a subspace $X \subseteq Z^\tau \times \mathbb{I}^{2^\tau}$ which satisfies the following conditions:

(i) X is strongly σ-discrete;

(ii) $X_z = \pi^{-1}(z) \cap X$ is connected and dense in $\{z\} \times \mathbb{I}^{2^\tau}$ for every $z \in Z^\tau$, where $\pi\colon Z^\tau \times \mathbb{I}^{2^\tau} \to Z^\tau$ is the projection;

(iii) for every nonempty open subset U of $Z^\tau \times \mathbb{I}^{2^\tau}$ and every point $z \in \pi(U)$ one has $|U \cap X_z| = 2^\tau$;

(iv) if $B \subseteq X$ and $|B| < 2^\tau$, then B is closed in X and discrete.

Since $\tau^\omega = 2^\tau = |Z^\tau|$, we can fix a bijection $j : Z^\tau \to \tau^\omega$. Let $J(\tau)^\omega = \bigcup\{Y_\nu : \nu < \tau^\omega\}$ be the partition constructed in Lemma 3.3.

Fix $z \in Z^\tau$. From (iii) it follows that $|U| = 2^\tau$ for every nonempty open subset of X_z. In particular, $|X_z| = 2^\tau$. Note that $w(X_z) \leq w(\mathbb{I}^{2^\tau}) = 2^\tau$ and $c(X_z) = \omega$ because X_z is dense in $\{z\} \times \mathbb{I}^{2^\tau}$. Therefore, $w(X_z)^{c(X_z)} \leq (2^\tau)^\omega = 2^\tau$. Since $w(Y_{j(z)}) \leq \tau$, by Lemmas 2.3 and 2.11 we have $|\mathcal{R}o(X_z \times Y_{j(z)})| \leq w(X_z \times Y_{j(z)})^{c(X_z \times Y_{j(z)})} \leq 2^\tau$. Applying Proposition 2.5 with X_z as X, $Y_{j(z)}$ as Y and 2^τ as κ we can find a bijection $f_z\colon X_z \to Y_{j(z)}$ such that the graph $S_z = \mathrm{Gr}(f_z)$ is dense in $X_z \times Y_{j(z)}$ and connected.

Define now $S = \bigcup\{S_z : z \in Z^\tau\}$. Let $p_1\colon Z^\tau \times \mathbb{I}^{2^\tau} \times J(\tau)^\omega \to Z^\tau \times \mathbb{I}^{2^\tau}$ and $p_2\colon Z^\tau \times \mathbb{I}^{2^\tau} \times J(\tau)^\omega \to J(\tau)^\omega$ be the projections. By our construction, $p_1|_S : S \to Z^\tau \times \mathbb{I}^{2^\tau}$ and $p_2|_S\colon S \to J(\tau)^\omega$ are one-to-one (continuous) maps and $p_1(S) = X$. From (i) it follows that S is strongly σ-discrete. Since each S_z is dense in $X_z \times Y_{j(z)}$, X_z is dense in $\{z\} \times \mathbb{I}^{2^\tau}$ (by (ii)) and $Y_{j(z)}$ is dense in $J(\tau)^\omega$, we conclude that S is dense in $Z^\tau \times \mathbb{I}^{2^\tau} \times J(\tau)^\omega$. Let $q\colon Z^\tau \times \mathbb{I}^{2^\tau} \times J(\tau)^\omega \to Z^\tau$ be the projection. For every $z \in Z^\tau$, the set $S \cap q^{-1}(z) = \{z\} \times S_z$ is connected and dense in $\{z\} \times \mathbb{I}^{2^\tau} \times J(\tau)^\omega$. Therefore, $q|_S\colon S \to Z^\tau$ is monotone (i.e., an open continuous map with connected fibers). Since Z^τ is connected, so is S [10, 6.1.29]. □

The conclusion of Theorem 3.4 can be strengthened in the case $|Z| = \mathfrak{c}$.

Theorem 3.5. *Let Z be a connected space of size $\mathfrak{c}$. Then Z is an open continuous image of a connected strongly σ-discrete submetrizable space of the same size.*

Proof. Apply Theorem 3.1 to find a strongly σ-discrete subspace Y of $Z \times \mathbb{I}^\mathfrak{c}$ such that $|Y| = \mathfrak{c}$ and $p_Z^{-1}(x) \cap Y$ is dense in $\mathbb{I}^\mathfrak{c}$ and connected for each $x \in Z$, where $p_Z\colon Z \times \mathbb{I}^\mathfrak{c}$ is the projection. Then the restriction of p_Z to Y is an open monotone map onto Z, so that Y is a dense connected subspace of $Z\times\mathbb{I}^{\mathfrak{c}}$. Our idea is to define a dense connected submetrizable subspace S of the product $Y\times\mathbb{I}^{\omega}$ whose projections to the factors are one-to-one map onto Y and $\mathbb{I}^{\omega}$. For every $r\in\mathbb{I}$, put

$$\sigma(r)=\{z\in\mathbb{I}^{\omega}: z(n)=r \text{ for almost all } n\in\omega\}.$$

It is clear that $\sigma(r)$ is a dense connected subspace of $\mathbb{I}^{\omega}$. In addition, $\sigma(r)\cap\sigma(r')=\varnothing$ whenever $r\neq r'$. Note that $\sigma(r)\setminus A$ is connected for each $A\subseteq\sigma(r)$ with $|A|<\mathfrak{c}$. Put $\sigma^*(0)=\mathbb{I}^{\omega}\setminus\bigcup_{0<r\leq 1}\sigma(r)$ and $\sigma^*(r)=\sigma(r)$ for each $r>0$. Since $|Z|=\mathfrak{c}=|\mathbb{I}|$, we can fix a bijection $f\colon Z\to\mathbb{I}$ and put $r_x=f(x)$ for each $x\in Z$. For every $x\in Z$, put $Y_x=p_Z^{-1}(x)\cap Y$ and consider the product $Y_x\times\sigma^*(r_x)$. By definition, Y_x is dense in $\mathbb{I}^{\mathfrak{c}}$, whence $w(Y_x)\leq\mathfrak{c}$ and $c(Y_x)\leq\aleph_0$. Apply Proposition 2.5 to find a dense connected subspace S_x of $Y_x\times\sigma^*(r_x)$ whose projections to the factors are bijections onto Y_x and $\sigma^*(r_x)$. Clearly, $|S_x|=|Y_x|=\mathfrak{c}$ for each $x\in Z$. We now put $S=\bigcup_{x\in Z}S_x$. Then $|S|\leq|Z|\cdot\mathfrak{c}=\mathfrak{c}$. Let us verify that S is as required.

From the definition of S it follows that the projections $\pi_1\colon Y\times\mathbb{I}^{\omega}\to Y$ and $\pi_2\colon Y\times\mathbb{I}^{\omega}\to\mathbb{I}^{\omega}$ restricted to S are continuous bijections. Therefore, S is strongly σ-discrete and submetrizable. Since S_x is dense in $Y_x\times\mathbb{I}^{\omega}$ for each $x\in Z$, we conclude that S is dense in $Z\times\mathbb{I}^{\mathfrak{c}}\times\mathbb{I}^{\omega}$. In addition, $S_x=\pi_Z^{-1}(x)\cap S$ is dense in $\mathbb{I}^{\mathfrak{c}}\times\mathbb{I}^{\omega}$ for each $x\in Z$, where $\pi_Z\colon Z\times\mathbb{I}^{\mathfrak{c}}\times\mathbb{I}^{\omega}\to Z$ is the projection. Therefore, the restriction $f=\pi_Z|_S\colon S\to Z$ is an open map of S onto Z. In other words, f is an open monotone map of S onto the connected space Z, so S is connected. □

4. Quotients of connected strongly σ-discrete groups

By Arhangel'skiĭ's result, every topological group is a quotient of a strongly σ-discrete topological group (see Theorem on page 137 of [5]). It was not known whether a similar result remains valid in the class of connected groups. However, Theorem 3.4 helps us to answer this question positively. We start with an auxiliary lemma.

Lemma 4.1. *Let X be a connected, submetrizable, strongly σ-discrete space. Then the free Graev topological group $F(X)$ is connected, submetrizable, and strongly σ-discrete.*

Proof. First, $F(X)$ is connected by Assertion A) of [11, Section 6]. Let us show that $F(X)$ is strongly σ-discrete and submetrizable.

Since X is submetrizable, there exists a continuous bijection h of X onto a metrizable space M. Denote by ϱ a metric on M generating its topology. Extend h to a continuous isomorphism $\widehat{h}\colon F(X)\to F(M)$. There exists an extension of ϱ to a continuous invariant metric $\widehat{\varrho}$ on $F(M)$ [11, Section 3]. Let $F_\varrho(M)$ be the abstract group $F(M)$ endowed with the topology generated by $\widehat{\varrho}$. Then $F_\varrho(M)$ is a Hausdorff topological group [11] and, hence, the topology of $F_\varrho(M)$ is coarser than the topology of the free topological group $F(M)$.

In particular, the isomorphism $\widehat{h}\colon F(X) \to F_\varrho(M)$ is continuous, so that the group $F(X)$ is submetrizable.

For every integer $n \geq 0$, denote by B_n the subspace of $F_\varrho(M)$ consisting of all elements in $F(M)$ of reduced length $\leq n$, and put $A_{n+1} = B_{n+1} \setminus B_n$. Then B_n is closed in $F_\varrho(M)$ for each $n \geq 0$ (see (α_6) on page 133 of [5]) and, hence, A_n is a union of countably many closed subsets of B_n. Let us now consider the group $F(X)$. Again, for every $n \geq 0$, define B_n^* as the subspace of $F(X)$ consisting of all elements of length $\leq n$ with respect to the basis X, and put $A_{n+1}^* = B_{n+1}^* \setminus B_n^*$. It is clear that $\widehat{h}(B_n^*) = B_n$ and $\widehat{h}(A_n^*) = A_n$ for each $n \in \omega$. It is well known that the sets B_n^* are closed in $F(X)$ (see (α_1) on page 133 of [5]). In addition, for every positive n, the multiplication map $j_n\colon (X \oplus X^{-1})^n \to B_n^*$ is continuous and the restriction $j_n|_{j_n^{-1}(A_n^*)}\colon j_n^{-1}(A_n^*) \to A_n^*$ is a homeomorphism (see (α_2) on page 133 of [5]). Note that the spaces $X \oplus X^{-1}$, $(X \oplus X^{-1})^n$ and $j_n^{-1}(A_n^*) \subseteq (X \oplus X^{-1})^n$ are strongly σ-discrete, so that A_n^* is strongly σ-discrete as well. Let $A_n^* = \bigcup_{i \in \omega} K_{n,i}$, where each $K_{n,i}$ is closed in A_n^* and discrete. Since $\widehat{h}(B_n^*) = B_n$, $\widehat{h}(A_n^*) = A_n$ and A_n is an F_σ-set in B_n, we conclude that A_n^* is an F_σ-set in B_n^*. Therefore, we can represent A_n^* as a union of countably many closed subsets of B_n^*, say $A_n^* = \bigcup_{j \in \omega} L_{n,j}$. For $i, j \in \omega$, put $A_{n,i,j} = K_{n,i} \cap L_{n,j}$. Then the sets $A_{n,i,j}$ are closed in B_n (and, hence, in $F(X)$) and discrete. Clearly, $A_n^* = \bigcup_{i,j \in \omega} A_{n,i,j}$. Since $F(X)$ is the union of $B_0^* = \{e\}$ and the sets A_n^*, $n \geq 1$, we conclude that $F(X)$ is strongly σ-discrete. □

Theorem 4.2. *Every connected topological group G is a quotient group of a connected, submetrizable, strongly σ-discrete group H.*

Proof. By Theorem 3.4, we can find a Tychonoff connected strongly σ-discrete submetrizable space X which admits an open continuous map onto G. Let $f\colon X \to G$ be such a map. Extend f to a continuous homomorphism $\hat{f}\colon F(X) \to G$, where $F(X)$ is the free topological group on X in the sense of Graev [11]. The homomorphism $\hat{f}$ is open by a theorem of [4]. Lemma 4.1 implies that the group $H = F(X)$ is as required. □

Note that the cardinality of the group H in the above theorem can be considerably bigger than that of its quotient G because our choice of the space X and the map $f\colon X \to G$ involves an application of Lemma 3.2. However, things change if $|G| = \mathfrak{c}$.

Corollary 4.3. *Every connected topological group G of cardinality $\mathfrak{c}$ is a quotient of a connected, submetrizable, strongly σ-discrete topological group H of the same cardinality.*

Proof. By Theorem 3.5, one can find a connected strongly σ-discrete submetrizable space X with $|X| = |G| = \mathfrak{c}$ and an open continuous onto map $f\colon X \to G$. As in Theorem 4.2, extend f to an open continuous homomorphism $\hat{f}\colon F(X) \to G$ and apply Lemma 4.1 to conclude that $H = F(X)$ is a connected, submetrizable, strongly σ-discrete topological group. □

Remark 4.4. It is easy to check that every strongly σ-discrete space X is left-separated, i.e., X admits a well-ordering $<$ such that the left ray $X_x = \{y \in X : y < x\}$ is closed in X for each $x \in X$. Therefore, the space X in Theorem 3.4 and the group H in Theorem 4.2 are automatically left-separated.

5. Open problems

The question below has been a motivation of the paper. Actually, Theorems 2.9, 2.10, 2.12, and 2.14 answer it positively in the special case of a space X with a "small" number of regular open sets.

Problem 5.1. *Let X be a connected space. Does X admit a finer connected strongly σ-discrete Tychonoff topology? What if X is compact?*

The case of a metrizable space X deserves a special mentioning.

Problem 5.2. *Does every connected metrizable space X admit a finer connected strongly σ-discrete Tychonoff topology?*

Note that by Corollary 7 of [17], every infinite connected metrizable space admits a strictly finer connected Tychonoff topology, so the problem is to choose such a topology to be strongly σ-discrete.

Recall that a connected space X is called *maximal connected* [19], [12] if every strictly finer topology on X is disconnected. Several examples of maximal connected Hausdorff spaces were constructed in [13] and [18]. It is not known, however, whether maximal connected Tychonoff spaces exist [6]. We conjecture that such spaces (if exist) must be strongly σ-discrete:

Problem 5.3. *Is it true that if there exists a maximal connected Tychonoff space X, then X is strongly σ-discrete?*

By Theorem 4.2, there is a lot of connected strongly σ-discrete topological groups. Our suspicion is that many connected topological groups admit finer connected strongly σ-discrete group topologies.

Problem 5.4. *Does every conected topological group admit a finer connected strongly σ-discrete group topology?*

The next problem is closely related to Theorem 2.14.

Problem 5.5. *Let X be a connected space satisfying $w(X) \leq \mathfrak{c}$ and $c(X) \leq \aleph_0$. Does X admit a finer connected submetrizable Tychonoff topology? Can such a topology additionally be chosen strongly σ-discrete?*

The following two problems arise in an attempt to strengthen conclusions of Theorems 3.4 and 4.2 and choose an open preimage as small as possible.

Problem 5.6. *Let Z be a connected space. Does there exist a connected, submetrizable, strongly σ-discrete space S which admits an open continuous map onto Z and satisfies $|S| = |Z|$?*

Problem 5.7. *Is it true that every connected topological group is a quotient of a connected strongly σ-discrete topological group of the same cardinality?*

Note that the positive answer to Problem 5.6 would imply the same to Problem 5.7.

We do not know whether an analog of Theorem 4.2 is valid for algebraic structures different from groups:

Problem 5.8. *Let R be a connected topological ring (field). Does there exist a connected strongly σ-discrete topological ring (field) S which admits an open continuous ring (field) homomorphism $f\colon S \to R$?*

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Received April 2001

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