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Cofinitely and co-countably projective spaces

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Abstract. We show that X is cofinitely projective if and only if it is a finite union of Alexandroff compactifications of discrete spaces. We also prove that X is co-countably projective if and only if X admits no disjoint infinite family of uncountable cozero sets. It is shown that a paracompact space X is co-countably projective if and only if there exists a finite set $B \subset X$ such that $B \subset U \in \tau(X)$ implies $|X \setminus U| \leqslant \omega$. In case of existence of such a B we will say that X is concentrated around B. We prove that there exists a space Y which is co-countably projective while there is no finite set $B \subset Y$ around which Y is concentrated. We show that any metrizable co-countably projective space is countable. An important corollary is that every co-countably projective topological group is countable.

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1. Introduction

Given an arbitrary class $\mathcal{F}$ of continuous mappings, a space X is called $\mathcal{F}$-projective if each surjective continuous mapping $f : X \to Y$ is an element of $\mathcal{F}$ whenever Y belongs to the class $\mathcal{U}$ of second countable regular spaces. The paper [6] was the first one where $\mathcal{F}$-projective properties were studied systematically. One of the results of [6] says that each continuous mapping of X with a second countable image is almost compact, i.e., its fibers are compact except finitely many if and only if $|\beta X \setminus X| \leqslant n$ for some $n \in \omega$. This motivates the following problem: is it true that for each almost compact projective second countable space X there exists a decomposition $X = K \cup Z$ such that K is compact and Z is countable? It was proved in [2] that under CH the answer is negative.

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A space X is said to be cofinitely projective if, for any surjective continuous mapping $f : X \to Y \in \mathcal{U}$, there exists a finite set $A \subset Y$ such that $|f^{-1}(y)| < \omega$ for every $y \in Y \setminus A$. We prove that X is cofinitely projective if and only if X is a finite union of primary compact spaces, i.e., the Alexandroff compactifications of discrete spaces. A space X is co-countably projective if, for any surjective continuous function $f : X \to Y \in \mathcal{U}$, there exists a finite set $A \subset Y$ such that $|f^{-1}(y)| \leqslant \omega$ for every $y \in Y \setminus A$. We establish that X is co-countably projective if and only if X does not admit a disjoint infinite family of uncountable cozero sets. In case when X is paracompact it is shown that X is co-countably projective if and only if it is concentrated around a finite set, i.e., there exists a finite $B \subset X$ such that, for any open $U \in \tau(X)$, if $B \subset U$ then $|X \setminus U| \leqslant \omega$. We also show that not all co-countably projective spaces are concentrated around a finite set. Our last group of results shows that there are some important classes of spaces in which every co-countably projective space is countable; we prove that this happens in metrizable spaces as well as in topological groups.

2. Notation and terminology.

All spaces are assumed to be Tychonoff. If X is a space, then $\tau(X)$ is its topology. Given $B \subset X$, let $\tau(B, X) = \{U \in \tau(X) : B \subset U\}$. We write $\tau(x, X)$ instead of $\tau(\{x\}, X)$. We say that a space X is concentrated around $B \subset X$ if, for every $U \in \tau(B, X)$, the set $X \setminus U$ is countable. All mappings are assumed to be continuous. A subspace $A \subset X$ is called *C^*-embedded in X* if for each bounded continuous function $f : A \to \mathbb{R}$ there exists a continuous function $F : X \to \mathbb{R}$ such that $F|A = f$.

Suppose that we are given a space X, a family $\{Y_s\}_{s \in S}$ of spaces and a family of continuous mappings $\{f_s\}_{s \in S}$, where $f_s : X \to Y_s$. The mapping that assigns to any point $x \in X$ the point $\{f_s(x)\} \in \Pi_{s \in S} Y_s$ is called the *diagonal product* $\triangle_{s \in S} f_s$ of the mappings $\{f_s\}_{s \in S}$. The symbol ω stands for the first infinite cardinal. Denote by I the interval $[0, 1]$ with the usual topology. If $f : X \to Y$ and $y \in Y$, the *fiber of y* is the set $f^{-1}(y) = \{x \in X : f(x) = y\}$. A surjective continuous mapping $f : X \to Y$ is *closed* if $f(F)$ is closed in Y for each closed $F \subset X$. A space X is *almost injectively projective* if each surjective continuous function $f : X \to Y \in \mathcal{U}$ is almost injective, i.e., $|\{y \in Y : |f^{-1}(y)| > 1\}| \leqslant \omega$.

A space X is *zero-dimensional* if it has a base that consists of clopen sets. We will use the symbol $A(X)$ for the Alexandroff compactification of a locally compact space X. *A primary compact space* is the space $A(D)$ for some discrete D.

3. Cofinitely projective spaces and dimension.

In this section we give complete characterizations of cofinite projectivity and projective n-dimensionality.

Definition 3.1. If $n \in \omega$ then a space X is called projectively n-dimensional if, for any continuous onto map $f : X \to Y \in \mathcal{U}$, we have $\dim Y \leqslant n$.

The following lemma is well known as a folklore but we give its proof here for the sake of completeness.

Lemma 3.2. *If a space X is not zero-dimensional then there exists a continuous onto map $f : X \to I$.*

Proof. As X it is not zero-dimensional, we can choose an $x \in X$ and a neighbourhood U of x such that $U \neq X$ and there is no clopen W for which $x \in W \subset U$. Since X is completely regular there exists $f : X \to [0,1]$ such that $f(x) = 1$ and $f|(X \backslash U) \equiv 0$. If there is $t \in (0,1) \backslash f(X)$ then the set $W = f^{-1}((t,1])$ is clopen and $x \in W \subset U$ which is a contradiction. This proves that $f(X) = [0,1]$. □

Theorem 3.3. *The following conditions are equivalent for any space X:*
(1) X is projectively zero-dimensional;
(2) X is projectively n-dimensional for some $n \in \omega$;
(3) X can not be continuously mapped onto I;
(4) X can not be continuously mapped onto an infinite-dimensional second countable space.

Proof. It is evident that (1) $\Rightarrow$ (2). Assume that (2) is true and there is a continuous onto map $f : X \to I$. Take a continuous onto map $g : I \to I^\omega$ and observe that $h = g \circ f$ maps X continuously onto I^ω which is a contradiction with the fact that any continuous second countable image of X must have dimension $\leqslant n$. This proves (2) $\Rightarrow$ (3).

Suppose that X can be continuously mapped onto a non-zero-dimensional space Y. Apply Lemma 3.2 to conclude that Y can be continuously mapped onto I. The relevant composition of maps shows that X can be continuously mapped onto I and hence we proved that (3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (1).

To finish our proof it suffices show that (4) $\Rightarrow$ (3). Assume that $f : X \to I$ is a continuous onto map. There exists a continuous onto map $g : I \to I^\omega$, so the map $h = g \circ f : X \to I^\omega$ is continuous and onto. Since the space I^ω is infinite-dimensional, this settles (4) $\Rightarrow$ (3) and our theorem is proved. □

Corollary 3.4. *If a space X is almost injectively projective then it is projectively zero-dimensional.*

Proof. By Theorem 3.3 it suffices to show that X can not be continuously mapped onto I. But if $f : X \to I$ is continuous and onto then I has to be almost injectively projective [2] which it is not; this has also been proved in [2]. □

Corollary 3.5. *Let X be projectively zero-dimensional. Then*
(1) every continuous image of X is projectively zero-dimensional;
(2) if Y is C^*-embedded in X then Y is projectively zero-dimensional. In particular, if X is normal and projectively zero-dimensional then so is every closed subspace of X.

Proof. That (1) holds is an immediate consequence of the definition. To prove (2) take any surjective continuous map $f : Y \to I$. Since Y is C^*-embedded in X there exists a continuous map $g : X \to I$ such that $g|Y = f$. Hence X can also be mapped continuously onto I and (2) holds. □

Corollary 3.6. *A compact space X is projectively zero-dimensional if and only if it is scattered.*

Proof. To prove the necessity we take a compact space X which is projectively zero-dimensional. If X is not scattered, there exists a surjective continuous function $f : X \to I$ (see [5]), which is a contradiction with Theorem 3.3(3).

For the sufficiency suppose that X is scattered. It follows from [2, Proposition 2.14], that X is almost injectively projective. By Corollary 3.4 the space X is projectively zero-dimensional. □

Example 3.7. There exists a pseudocompact scattered space which is not projectively zero-dimensional.

Proof. The following construction can be found in [3]. Let γ be a maximal almost disjoint family of infinite subsets of ω. For each $A \in \gamma$ take a point $x_A \notin \omega$. We introduce a topology τ on the set $\omega \cup \{x_A : A \in \gamma\}$ in the following way: if $x \in \omega$ then $\{x\} \in \tau$; if $x = x_A$ then the base at x consists of the sets $\{x_A\} \cup (A \backslash B)$ where $B \subset \omega$ is finite. Let M_γ be the set $\omega \cup \{x_A : A \in \gamma\}$, with the topology described above; then M_γ is a pseudocompact scattered Tychonoff space [3]. It was also proved in [3] that a maximal almost disjoint family γ can be chosen so that M_γ can be mapped continuously onto $[0, 1]$. The space M_γ is as promised because it is pseudocompact, scattered and not projectively zero-dimensional. □

The proofs of the following two statements are easy.

Proposition 3.8. *If X is cofinitely projective then any continuous image of X is cofinitely projective.*

Proposition 3.9. *If X is cofinitely projective then it is zero-dimensional.*

Lemma 3.10. *If X is cofinitely projective then it is pseudocompact.*

Proof. If $f : X \to Y \in \mathcal{U}$, then there exists a finite set $A \subset Y$ such that $|f^{-1}(y)| < \omega$ for every $y \in Y \backslash A$. This proves that $|\{y \in Y : f^{-1}(y)$ is not compact$\}| \leqslant |A| < \omega$. Therefore each map of X onto a space with countable base is not compact at only a finite number of points and by Proposition 3.14 of [6] we can conclude that X is pseudocompact. □

Example 3.11. The Alexandroff one-point compactification $A(D)$ of an arbitrary discrete space D, is a cofinitely projective space.

Proof. Let $A(D) = D \cup \{a\}$ where a is the only non-isolated point of $A(D)$. Take a surjective continuous mapping $f : A(D) \to Y \in \mathcal{U}$, and an arbitrary $y \in Y \backslash \{f(a)\}$. As f is continuous the subspace $f^{-1}(y)$ is compact and discrete in $A(D)$. Therefore $f^{-1}(y)$ is finite. □

Lemma 3.12. *If X is regular and $F \subset X$ is a closed infinite set then there exists a family $\{V_i : i \in \mathbb{N}\} \subset \tau(X)$ such that $\overline{V_i} \cap \overline{V_j} = \varnothing$ and $F \cap V_j \neq \varnothing$ for all $i, j \in \omega$ with $i \neq j$.*

Proposition 3.13. *Any cofinitely projective space has only a finite number of non-isolated points.*

Proof. Let Y be the set of non-isolated points of a cofinitely projective space X. Clearly Y is a closed subset of X. If Y is infinite, Lemma 3.12 implies that there exists a family $\{V_i : i \in \mathbb{N}\} \subset \tau(X)$ such that $\overline{V_i} \cap \overline{V_j} = \varnothing$ and $Y \cap V_j \neq \varnothing$ for all $i, j \in \mathbb{N}$ with $i \neq j$. Take $x_j \in Y \cap V_j$ for each $j \in \mathbb{N}$ and consider the set $Z = \{x_n : n \in \mathbb{N}\}$ which is discrete. As X is zero-dimensional, for each $n \in \mathbb{N}$ there exists a clopen set U_n such that $x_n \in U_n \subset V_n$. Define a function $f : X \to I$ by $f|U_n \equiv \frac{1}{n}$ for any $n \in \mathbb{N}$ and $f(x) = 0$ for all $x \notin \bigcup\{U_n : n \in \mathbb{N}\}$. Clearly $f|U_n$ is continuous for each $n \in \mathbb{N}$. Take any $x \notin \bigcup\{U_n : n \in \mathbb{N}\}$ and a neighbourhood $(-\epsilon, \epsilon)$ of the point $f(x) = 0$. For any n with $\frac{1}{n} < \epsilon$ we have $f(W) \subset (-\frac{1}{n}, \frac{1}{n}) \subset (-\epsilon, \epsilon)$ for $W = X \setminus \bigcup_{i=1}^{n} U_i$. Therefore f is continuous at x. Observe that $U_n \subset f^{-1}(\frac{1}{n})$ and U_n is infinite since x_n is not isolated. Therefore we have an infinite number of infinite fibers of f whence X is not cofinitely projective. □

Lemma 3.14. *If X is pseudocompact and has only finitely many non-isolated points then X is compact.*

Proposition 3.15. *If X is cofinitely projective then X is compact.*

Proof. As X is cofinitely projective, Proposition 3.10 says that X is pseudocompact. By Proposition 3.13, the space X has a finite number of non-isolated points and applying Lemma 3.14 we conclude that X is compact. □

Lemma 3.16. *Any finite union of cofinitely projective spaces is a cofinitely projective space.*

Proof. Suppose that $X = \bigcup_{i=1}^{n} X_i$ and each X_i is cofinitely projective. Given any $f : X \to Z \in \mathcal{U}$, consider the maps $f_i = f|X_i$ where $f_i : X_i \to Z_i$ and $Z_i = f(X_i)$. As X_i it is cofinitely projective there exists $A_i \subset Z_i$ such that $|f_i^{-1}(z)| < \omega$ for all $z \in Z_i \setminus A_i$. Observe that the set $A = \bigcup_{i=1}^{n} A_i$ is finite and $f^{-1}(x) = \bigcup_{i=1}^{n} f_i^{-1}(x)$ for all $x \in X$. If $x \notin A$, the set $f^{-1}(x)$ is finite because $f_i^{-1}(x)$ is finite for every $i < n$. □

Now we can give a complete characterization of cofinitely projective spaces.

Theorem 3.17. *The following conditions are equivalent for any space X:*
(1) X is cofinitely projective;
(2) X is a compact space with a finite number of non-isolated points;
(3) X is a discrete union of a finitely many primary compact spaces.

Proof. To show that (1) ⇒ (2), assume that X is cofinitely projective. Proposition 3.15 says that X is compact and, by Proposition 3.13, the space X has only a finite number of non-isolated points.

Suppose that (2) holds, and take an enumeration $\{x_1, ..., x_n\}$ of the set of the non-isolated points of X. Choose a disjoint family $\{U_1, ..., U_n\}$ of clopen sets of X such that $x_i \in U_i$ for all $i \leqslant n$. If $U = U_1 \cup U_2 \cup ... \cup U_n$ then $X \backslash U$ is finite and each U_i is a primary compact space. Therefore $X = X_1 \oplus X_2 \oplus ... \oplus X_n$ where $X_1 = U_1 \cup (X \backslash U)$ and $X_i = U_i$ for all $i \in \{2, ..., n\}$. It is clear that every X_i is also a primary compact space and the implication (2) $\Rightarrow$ (3) is proved.

Now if (3) holds, then $X = X_1 \oplus X_2 \oplus .. \oplus X_n$ where each X_i is a primary compact space. Example 3.11 shows that each X_i is cofinitely projective. Applying Lemma 3.16 we see that X is cofinitely projective and therefore (3) $\Rightarrow$ (1). □

Corollary 3.18. *If X is finite union of convergent sequences then X is cofinitely projective.*

Proof. Each convergent sequence is a cofinitely projective space, so X is cofinitely projective by Lemma 3.16. □

Example 3.19. There exists a space X which is cofinitely projective, while $X \times X$ is not cofinitely projective.

Proof. If X is a convergent sequence, the space $X \times X$ has an infinite number of non-isolated points. By Proposition 3.13 the square of X is not cofinitely projective. □

4. Co-countably projective spaces.

The concept of a co-countably projective space is an evident generalization of the notion of a cofinitely projective space. However we will see that these classes have very different properties.

Theorem 4.1. *A space X is co-countably projective if and only if there exists no infinite disjoint family of uncountable cozero subsets of X.*

Proof. To prove the necessity, assume that X is co-countably projective and there exists a disjoint infinite family $\{U_n : n \in \mathbb{N}\}$ of uncountable cozero subsets of X. For each n take a continuous function $f_n : X \to [0, 1]$ such that $f_n^{-1}(0) = X \backslash U_n$. For each $n \in \mathbb{N}$, the set $U_n = \bigcup\{f_n^{-1}([1/k, 1]) : k \in \mathbb{N}\}$ is uncountable and hence there exists $k_n \in \mathbb{N}$ such that $|f_n^{-1}([1/k_n, 1])| > \omega$. The sets $P_n = f_n^{-1}([1/k_n, 1])$ and $X \backslash U_n$ are functionally separated, so there exists a continuous function $g_n : X \to [0, \frac{1}{n}]$ such that $g_n|(X \backslash U_n) \equiv 0$ and $g_n|P_n \equiv 1/n$. It is easy to see that the function $g : X \to [0, 1]$ given by $g = \sum\{g_n : n \in \omega\}$ is continuous and, for each $n \in \mathbb{N}$, we have $P_n \subset g^{-1}(1/n)$. Therefore, $|g^{-1}(1/n)| > \omega$ and g has an infinite number of uncountable fibers which is a contradiction because X is co-countably projective.

To prove the sufficiency suppose that such a family does not exist and X is not co-countably projective. Then some surjective $f : X \to Y \in \mathcal{U}$ has an infinite number of uncountable fibers, i.e., there is an infinite $A \subset Y$ such that $f^{-1}(a)$ is uncountable for any $a \in A$. Let us take an infinite discrete $B = \{x_n : n \in \omega\} \subset A$. We can choose a disjoint family $\{W_n\}_{n \in \omega} \subset \tau(Y)$ such that $x_n \in W_n$ for each $n \in \omega$. If $U_n = f^{-1}(W_n)$ for all $n \in \omega$ then

$\{U_n\}_{n\in\omega} \subset \tau(X)$ is a disjoint infinite family of uncountable cozero sets which is a contradiction. □

Definition 4.2. Say that a space X is locally countable at a point $x \in X$ if there exists a countable $U \in \tau(x, X)$. The space X is locally countable if it is locally countable at all points.

Theorem 4.3. *Let X be a co-countably projective space. Then there exists a finite set $A \subset X$ such that X is locally countable at all points of $X \backslash A$.*

Proof. We claim that the set $A = \{x \in X : |U| > \omega$ for any $U \in \tau(x, X)\}$ is finite. Indeed, if A is infinite then there exists a discrete infinite set $D = \{d_n : n \in \omega\} \subset A$. Take a disjoint family $\{U_n\}_{n\in\omega} \subset \tau(X)$ such that $d_n \in U_n$. It is clear that there exists a disjoint family $\{V_n\}_{n\in\omega}$ of cozero sets such that $d_n \in V_n \subset U_n$ for all $n \in \omega$. Since every V_n is uncountable we obtain a contradiction with Theorem 4.1. Therefore $|A| < \omega$ and for each $z \in X\backslash A$ there exists $V_z \in \tau(z, X)$ such that $|V_z| \leqslant \omega$. □

Proposition 4.4. *If X is co-countably projective then any continuous image of X is co-countably projective.*

Proof. Suppose that Y is a continuous image of X, i.e., there exists a surjective continuous function $f : X \to Y$. Given a function $g : Y \to Z \in \mathcal{U}$, consider the composition $g \circ f : X \to Z \in \mathcal{U}$. As X is co-countably projective there is a finite set $A \subset Z$ such that $|(g \circ f)^{-1}(y)| \leqslant \omega$ for every $y \in Z\backslash A$. As a consequence $|f^{-1}(g^{-1}(y))| \leqslant \omega$ and hence $|g^{-1}(y)| \leqslant \omega$ for all $y \in Z\backslash A$. This proves that Y is co-countably projective. □

Example 4.5. The space $[0, 1]$ is not co-countably projective.

Proof. As $[0, 1] \times [0, 1]$ is a continuous image of $[0, 1]$, Proposition 4.4 implies that the space $[0, 1] \times [0, 1]$ is co-countably projective if so is $[0, 1]$. But the projection $[0, 1] \times [0, 1] \to [0, 1]$ has an infinite number of uncountable fibers. Therefore none of the spaces $[0, 1]$ and $[0, 1] \times [0, 1]$ is co-countably projective.
□

Proposition 4.6. *Any co-countably projective space X is zero-dimensional.*

Proof. If the space X is not zero-dimensional then there exists a continuous onto map $f : X \to [0, 1]$ (Lemma 3.2). Proposition 4.4 implies that $[0, 1]$ is co-countably projective which contradicts the statement of Example 4.5. □

Corollary 4.7. *If X is a compact metrizable space, then X is co-countably projective if and only if it is countable.*

Example 4.8. There exists a space X which is compact and scattered but not co-countably projective.

Proof. Given a cardinal κ, denote by $D(\kappa)$ the discrete space of cardinality κ and consider the space $X = A(D(\omega)) \times A(D(\omega_1))$. It is compact and scattered while the projection map $A(D(\omega)) \times A(D(\omega_1)) \to A(D(\omega))$ has an infinite number of uncountable fibers. Therefore X is not co-countably projective. □

Proposition 4.9. *If X is co-countably projective and Y is a C^*-embedded subspace of X then Y is co-countably projective.*

Proof. Let $f : Y \to Z$ be a continuous onto map for some $Z \in \mathcal{U}$. We can assume that $Z \subset I^\omega$ and therefore $f = \triangle_{n\in\omega} f_n$ where $f_n : Y \to I$ is a continuous map for each $n \in \omega$. As Y is C^*-embedded, there exists a continuous $F_n : X \to I$ such that $F_n|Y = f_n$ for each $n \in \omega$. The function $F = \triangle_{n\in\omega} F_n : X \to I^\omega$ maps X onto some $Z' \subset I^\omega$ with $Z' \supset Z$. The space X being co-countably projective, there exists a finite $A \subset Z'$ such that $|F^{-1}(y)| \leqslant \omega$ for all $y \in Z'\backslash A$. As $f^{-1}(y) \subset F^{-1}(y)$ we have $|f^{-1}(y)| \leqslant \omega$ for all $y \in Z\backslash A$. Therefore Y is co-countably projective. □

Theorem 4.10. *If X is pseudocompact then βX is co-countably projective if and only if X is co-countably projective.*

Proof. It is clear that X is C^*-embedded in βX, so we can apply Proposition 4.9 to see that X is co-countably projective if so is βX. Now assume that X is co-countably projective and βX is not. By Theorem 4.1 we can find a disjoint family $\{O_n : n \in \omega\}$ of uncountable cozero subsets of βX. Since the family $\mathcal{O} = \{O_n \cap X : n \in \omega\}$ is disjoint and consists of cozero subsets of X, all elements of $\mathcal{O}$ except finitely many, must be countable by Theorem 4.1. Take any $n \in \omega$ such that $O_n \cap X$ is countable. Since the uncountable set O_n is a countable union of compact sets, there is an uncountable compact $F \subset O_n$. By normality of βX we can find an open $U \subset \beta X$ such that $F \subset U \subset \overline{U} \subset O_n$ (the bar denotes the closure in βX). Observe that $\overline{U} = \overline{U \cap X}$ because X is dense in βX. However, $P = \mathrm{cl}_X(U \cap X) \subset O_n \cap X$ is compact being a pseudocompact countable subset of X. As a consequence, the set $U \subset \overline{U \cap X} \subset \overline{P} = P$ is countable which is a contradiction. □

Corollary 4.11. *If X is countably compact then βX is co-countably projective if and only if so is X.*

Corollary 4.12. *Any pseudocompact co-countably projective space is scattered.*

Proof. If X is pseudocompact and co-countably projective then βX is also co-countably projective by Theorem 4.10. If βX is not scattered then it can be mapped onto I [5]. By Propostion 4.4 the space I has to be co-countably projective which it is not (see Example 4.5). This contradiction shows that βX is scattered and hence so is X. □

Definition 4.13. We say that a space X is concentrated around a set $B \subset X$ if, for every $U \in \tau(B, X)$, the set $X\backslash U$ is countable.

Theorem 4.14. *The following conditions are equivalent for any space X:*
(1) X es Lindelöf and co-countably projective;
(2) X is paracompact and co-countably projective;
(3) X is concentrated around a finite set B.

Proof. The implication (1) $\Rightarrow$ (2), is immediate because each Lindelöf is paracompact.

Suppose that (2) holds, i.e., X is paracompact and co-countably projective. By Theorem 4.3 the space X is locally countable at all points of $X \backslash Y$ for some finite $Y \subset X$. For any $U \in \tau(Y, X)$, the set $X \backslash U$ is paracompact and co-countably projective by Proposition 4.9. Suppose that $D \subset X \backslash U$ is an uncountable closed discrete set with $|D| = \omega_1$. As $X \backslash U$ is collectionwise normal there is a discrete family $\{U_d : d \in D\} \subset \tau(X \backslash U)$ such that U_d is a cozero set and $d \in U_d$ for all $d \in D$. It is easy to find a disjoint family $\{D_n : n \in \omega\}$ such that $D = \bigcup_{n \in \omega} D_n$ and $|D_n| = \omega_1$ for each $n \in \omega$. If $U_n = \bigcup\{U_d : d \in D_n\}$ for all $n \in \omega$ then $\{U_n : n \in \omega\}$ is a disjoint infinite family of uncountable cozero sets which contradicts Theorem 4.1. This shows that each closed discrete subset of $X \backslash U$ is countable, i.e., $e(X \backslash U) = \omega$. Being a paracompact space of countable extent the space $X \backslash U$ has to be Lindelöf. For each point $z \in X \backslash U$ fix $V_z \in \tau(z, X)$ such that $|V_z| \leqslant \omega$. As $X \backslash U$ is Lindelöf, the open cover $\{V_z : z \in X \backslash U\}$ of the space $X \backslash U$ has a countable subcover $\{V_{z_n}\}_{n \in \omega}$. Since $X \backslash U = \bigcup_{n \in \omega} V_{z_n}$ and every V_{z_n} is countable, we have $|X \backslash U| \leqslant \omega$ and the implication (2) $\Rightarrow$ (3) is proved.

Now suppose that X is concentrated around a finite set $B = \{x_1, \ldots, x_n\}$. If $\mathcal{U}$ is an open cover of X then choose, for each $i \leqslant n$, a set $U_i \in \mathcal{U}$ such that $x_i \in U_i$. Then $B \subset U = U_1 \cup \ldots \cup U_n$ and hence $X \backslash U$ is countable. It is evident that there exists a countable $\mathcal{U}' \subset \mathcal{U}$ such that $X \backslash U \subset \bigcup \mathcal{U}'$. The family $\mathcal{U}' \cup \{U_1, \ldots, U_n\}$ is a countable subcover of $\mathcal{U}$ which proves that X is Lindelöf. It is easy to see that any space concentrated around a finite set, is co-countably projective so (3) $\Rightarrow$ (1) is established. □

Example 4.15. The space ω_1 with the usual order topology is co-countably projective and not concentrated around a finite set.

Proof. Each continuous real-valued function f on ω_1 is eventually constant, that is, there exists $\alpha_0 < \omega_1$ such that $f(\alpha) = f(\alpha_0)$, for every $\alpha \geqslant \alpha_0$. An easy consequence is that any continuous $f : \omega_1 \to Z \in \mathcal{U}$ is eventually constant. If $A = \{f(\alpha_0)\}$, and $y \in Z \backslash A$ then $f^{-1}(y) \subset \alpha_0$. Therefore $|f^{-1}(y)| \leqslant \omega$ for any $y \in Z \backslash A$ and ω_1 is co-countably projective. If ω_1 is concentrated around a finite set $B = \{\alpha_1, ..., \alpha_n\}$, where $\alpha_1 < \ldots < \alpha_n$, then, for the open set $U = \{\alpha : \alpha < \alpha_n + 1\}$, we have $B \subset U$ and $|X \backslash U| \leqslant \omega$ which is false. □

Example 4.16. There exist models of ZFC in which there is a perfectly normal space X which is co-countably projective without being concentrated around a finite set.

Proof. The space X of Ostaszewsky [4], is perfectly normal, uncountable, countably compact, and for any closed $F \subset X$ we have $|F| \leqslant \omega$ or $|X \backslash F| \leqslant \omega$. Let $f : X \to Y$ where $Y \in \mathcal{U}$. If $|f^{-1}(y)| \leqslant \omega$ for every $y \in Y$ then there is nothing to prove. If $|f^{-1}(y_0)| > \omega$ for some $y_0 \in Y$ then $|X \backslash \{f^{-1}(y_0)\}| \leqslant \omega$ so $f^{-1}(y) \subset X \backslash f^{-1}(y_0)$ and $|f^{-1}(y)| \leqslant \omega$ for every $y \in Y \backslash \{y_0\}$.

If X is concentrated around a finite set B then, by perfect normality of X we have $X \setminus B = \bigcup\{F_n : n \in \omega\}$ where each F_n is closed in X and hence countable because $X \setminus F_n \in \tau(B, X)$. As a consequence, $X = \bigcup\{F_n : n \in \omega\} \cup B$ is countable which is a contradiction. □

Proposition 4.17. *If X is second countable and co-countably projective then X is countable.*

Proof. As X es Lindelöf, Theorem 4.14 implies that there exists a finite set $B = \{x_1, ..., x_n\} \subset X$ around wich X is concentrated. Let us choose disjoint sets $U_1, ..., U_n \in \tau(X)$ such that $x_i \in U_i$ for each i. For each $i \leqslant n$ there exists a family $\{U_i^k\}_{k \in \omega} \subset \tau(X)$ such that $U_i^k \subset U_i$ for all $k \in \omega$ and $\bigcap_{k \in \omega} U_i^k = \{x_i\}$. The set $A_k = X \setminus \bigcup_{i \leqslant n} U_i^k$ is countable for all $k \in \omega$ so we have $X \setminus B = \bigcup_{i \in \omega} A_i$ and therefore $|X| \leqslant \omega$. □

Corollary 4.18. *If X is metrizable and co-countably projective then X is countable.*

Proof. Any metrizable space is paracompact and every paracompact co-countably projective space is Lindelöf by Theorem 4.14. Thus X is second countable. Applying Proposition 4.17, we can conclude that X is countable. □

Corollary 4.19. *If G is a co-countably projective topological group then G is countable.*

Proof. The set $Y = \{x \in G : |U| > \omega \text{ for any } U \in \tau(x, G)\}$ is finite by Theorem 4.3. For any $U \in \tau(Y, G)$ and any point $z \in G \setminus U$ there exists $V_z \in \tau(z, G)$ such that $|V_z| \leqslant \omega$. But topological groups are homogeneous spaces and hence each point of G has a countable neighbourhood. This shows that $\psi(G) \leqslant \omega$. Any topological group with countable pseudocharacter admits a continuous bijection onto a metrizable space M [1]. Applying Corollary 4.18 and Proposition 4.4, we can see that M is countable. Since $|G| = |M|$, we conclude that G is also countable. □

5. Open questions.

The following questions outline a natural development of the research done in this paper.

Problem 5.1. *Suppose that X is a metacompact co-countably projective space. Must X be concentrated around a finite set?*

Problem 5.2. *Is there a ZFC example of a perfectly normal co-countably projective space which is not concentrated around a finite set?*

Problem 5.3. *Is there a realcompact co-countably projective space which is not concentrated around a finite set?*

Problem 5.4. *Let X be a projectively zero-dimensional space. Must $X \times X$ be projectively zero-dimensional?*

Problem 5.5. *Let G be almost injectively projective second countable topological group. Must G be countable?*

Problem 5.6. *Let X be a homogeneous co-countably projective space. Must X be countable?*

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