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Skew compact semigroups

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Abstract. Skew compact spaces are the best behaving generalization of compact Hausdorff spaces to non-Hausdorff spaces. They are those (X, τ) such that there is another topology τ^* on X for which $\tau \vee \tau^*$ is compact and (X, τ, τ^*) is pairwise Hausdorff; under these conditions, τ uniquely determines τ^*, and (X, τ^*) is also skew compact. Much of the theory of compact T_2 semigroups extends to this wider class. We show:

A *continuous skew compact semigroup* is a semigroup with skew compact topology τ, such that the semigroup operation is continuous $\tau^2 \to \tau$. Each of these contains a unique minimal ideal which is an upper set with respect to the specialization order.

A skew compact semigroup which is a continuous semigroup with respect to both topologies is called a *de Groot semigroup*. Given one of these, we show:

It is a compact Hausdorff group if either the operation is cancellative, or there is a unique idempotent and $S^2 = S$.

Its topology arises from its subinvariant quasimetrics.

Each *-closed ideal $\neq S$ is contained in a proper open ideal.

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1. Introduction

Skew compact spaces have been studied in some guise since at least 1948, when Nachbin introduced them as "compact ordered spaces"; his work is most conveniently found in [10]. They have recently become important in computer science, as well as topology, since they are the spaces needed to approximate compact Hausdorff spaces with finite T_0 (rarely T_2) spaces. The purpose of this paper is to show that many basic concepts of the theory of compact (Hausdorff) semigroups can be extended to these spaces with relatively little change. In the next section we give basic results, motivation, and references on skew compact spaces. In section 3 we extend the classically-known fact that compact (Hausdorff) semigroup topologies arise from subinvarant pseudometrics by observing that skew compact semigroup topologies arise from subinvarant quasimetrics. The central result of section 4 is that skew compact cancellative semigroups with de Groot continuous operations, are compact Hausdorff groups; related results can also be found there. The final section shows that much of the basic structure theory of compact (Hausdorff) semigroup can be extended to this situation.

In all cases, a key difference between the traditional Hausdorff situation and this non-Hausdorff situation is the need to pay attention to the specialization order $\leq_\tau$. For it, and any binary relation $\leq$, we adopt the conventions $\uparrow[S] = \{y \mid (\exists x \in S)(x \leq_\tau y)\}$, and $\downarrow[S] = \{y \mid (\exists x \in S)(x \geq_\tau y)\}$ (sometimes we may decorate the notation to indicate which relation we have in mind, e.g. $\uparrow_\leq [S]$).

Definition 1.1. For any topology, τ, the (*Alexandroff*) *specialization* (*order*), $\leq_\tau$ is defined by $x \leq_\tau y$ if $x \in \mathrm{cl}(y)$. Thus, $\mathrm{cl}(x) = \downarrow(x)$. The *saturation* of a set $S \subseteq X$ is $\uparrow[S]$, and a set S is $\leq_\tau$-*saturated* if it is a $\leq_\tau$ −*upper set*, that is, if $\uparrow[S] \subseteq S$.

It is easy to see that $x \leq_\tau y$ if and only if $\mathrm{cl}(x) \subseteq \mathrm{cl}(y)$, and so $\leq_\tau$ is transitive and reflexive; it is a partial order if and only if τ is T_0, and equality if and only if τ is T_1. The study of spaces in which $\leq_\tau$ is not assumed symmetric is called *asymmetric topology.*

Example 1.2. The upper topology on the unit interval $\mathbb{I} = [0, 1]$ is $\mathcal{U} = \{(a, 1] \mid 0 \leq a\} \cup \{\mathbb{I}\}$. Notice that:

- $\leq_\mathcal{U}$ is the usual order on $\mathbb{I}$,
- the saturated sets are the upper sets,
- the compact sets are those with a least element.

2. Skew compact spaces

Except as noted below, in this section our notation and results are from [7]. It is characteristic of asymmetric topology that we must construct and study auxiliary topologies on the same space. Our terminology is adapted as follows; topological terms (eg. open, cl(osure), continuous) when not modified, refer to τ; we use these notations with decorations to refer to auxiliary topologies (eg. *-open means open in τ^*, cl^S means the closure in the topology $\tau^S = \tau \vee \tau^*$, and *-continuous means continuous from (X, τ^*_X) to (Y, τ^*_Y)).

Definition 2.1. A T_0 topological space (X, τ) is *skew compact* if there is a topology τ^* on X such that:

$\leq_{\tau^*} = \leq_\tau^{-1}$ (that is, $y \in \mathrm{cl}(x) \Leftrightarrow x \in \mathrm{cl}^*(y)$),

(X, τ, τ^*) is *pseudoHausdorff,* that is: whenever $x \notin \mathrm{cl}(y)$ then there are disjoint open T and *-open T^* such that $x \in T$ and $y \in T^*$,

$\tau \vee \tau^*$ is compact.

Theorem 2.2. *This second topology, τ^* is uniquely determined by τ. It is its de Groot dual: the topology τ^G whose closed sets are generated by the saturations of the τ-compact subsets of X.*

As a result of our discussion in 1.2, $\mathcal{U}^G = \mathcal{L}$, that is, $\{[0, a) \mid 1 \geq a\} \cup \{\mathbb{I}\}$.

Notice that for topological spaces (X, τ_X), (Y, τ_Y), a function $f : X \to Y$ is continuous from (X, τ_X^G) to (Y, τ_Y^G) if and only if whenever $f(x) \notin K$, K compact saturated, then there is a compact, saturated L such that $x \notin L$ and $f^{-1}[K] \subseteq L$. A function is *de Groot* if it is continuous with respect to both the original and de Groot dual topologies.

There are several ways of saying that only symmetry has been sacrificed in Definition 2.1:

Theorem 2.3. *A topological space (X, τ) is compact Hausdorff if and only if it is skew compact, and any of the following equivalent conditions hold:*

(a) $\leq_\tau$ *is a symmetric relation,*
(b) $\leq_\tau$ *is equality (that is, τ is T_1),*
(c) *the second topology is equal to the first.*

What follows is a special case of the definition of continuity space in [6], which suffices for our uses:

Definition 2.4. A *continuity space*, is a set X together with two other sets, A, P and a function $d : X \times X \to A$, where for some index set J:

$A \subseteq [0, \infty]^J$ contains $0, \infty$, and is closed under these pointwise operations: $+$, truncated $-$, multiplication by $\frac{1}{2}$, arbitrary $\bigvee, \bigwedge$,

$d(x, x) = 0$ and d obeys the triangle inequality, and finally,

$P \subseteq A$ is an upper set closed under finite $\wedge$, multiplication by $\frac{1}{2}$, and such that if for $a, b \in A$, $a \leq b + r$ for each $r \in P$, then $a \leq b$ (P is called the *set of positives*).

From a continuity space, we get an *induced topology*, τ_d defined by $T \in \tau_d$ whenever for each $x \in T$ there is an $r \in P$ such that $N_r(x) = \{y \mid d(x, y) \leq r\} \subseteq T$, and an *induced quasiuniformity*, $\mathcal{Q}_d = \{U \subseteq X \times X \mid \text{for some } r \in P,\ N_r \subseteq U\}$, where $N_r = \{(x, y) \mid d(x, y) \leq r\}$. We also get a *dual*, (X, d^*, A, P), where $d^*(x, y) = d(y, x)$, and a *symmetrization* (X, d^S, A, P), where $d^S = d + d^*$. A continuity space is *symmetric* if for all $x, y \in X, d(x, y) = d(y, x)$.

Topological notions for continuity spaces are defined in terms of the induced topologies, and uniform notions for them are defined in terms of the induced quasiuniformities; however, it can be shown that these are equivalent to extensions of the usual metric notions (but replace $d(x, y) < r$ by $d(x, y) \leq r$ and $0 < r$ by $r \in P$). Note that a quasiuniformity $\mathcal{Q}$ also has a *dual*, $\mathcal{Q}^* = \{U^{-1} \mid U \in \mathcal{Q}\}$ and a *symmetrization*, the join $\mathcal{Q}^S = \mathcal{Q} \vee \mathcal{Q}^*$.

Theorem 2.5. *A T_0 topological space (X, τ) is skew compact if and only if any of the following (equivalent) conditions occurs:*

(a) *τ arises from a continuity space whose symmetrization is complete and totally bounded. Any two such continuity spaces are uniformly equivalent.*

(b) *τ arises from a quasiuniformity with complete, totally bounded symmetrization. There is exactly one such quasiuniformity.*

As is well-known, a topology is compact Hausdorff if and only if, it comes from a uniformity which is complete and totally bounded (and, like each uniformity, is its own symmetrization). A similar equivalent is that it arises from a symmetric, complete and totally bounded continuity space. Another useful characterization is given in topological ordered space terms:

Definition 2.6. Recall from Nachbin [10] that an *order-Hausdorff space*, $(X, \tau, \leq)$, is a topological space together with a partial order closed in its square, $X \times X$. For such, $\tau^{\leq}$ denotes the topology of open upper sets.

If $(X, \tau, \leq)$ is order-Hausdorff then τ is Hausdorff.

Theorem 2.7. *A topological space (X, τ) is skew compact if and only if either of the following (equivalent) conditions occurs:*

(a) *There is a compact topology τ^S and a partial order $\leq$, both on X, such that $(X, \tau^S, \leq)$ is order-Hausdorff and $\tau = (\tau^S)^{\leq}$. (This topology and partial order are uniquely determined by τ: $\tau^S = \tau \vee \tau^G$, and $\leq = \leq_\tau$).*

(b) *$(X, \tau \vee \tau^G)$ is a compact Hausdorff topological space.*

Again, the space is compact Hausdoff if and only if further, in (a), $\leq$ is equality, and in (b), $\tau = \tau^G$.

3. Lengths on Skew-compact Semigroups

Definition 3.1. A *length space* is a continuity space (S, d, A, P) together with a semigroup operation on S for which d is subinvariant; that is, whenever $a, b \in S \cup \{1\}$ (1 an identity element added to S unless S has one already) and $x, y \in X$ then $d(axb, ayb) \leq d(x, y)$.

The above results from the following theorem in [8]. For it we need for any $R \subseteq S \cup \{1\}$ the notation I_R for the diagonal of R, $\{(r, r) \mid r \in R\}$.

Theorem 3.2. *Let $\cdot$ be an associative operation on S, $\mathcal{Q}$ a quasiuniformity on S, and $L, R \subseteq S$ closed under $\cdot$ be such that if $U \in \mathcal{Q}$ there are $V, W \in \mathcal{Q}$ such that $I_L W \cup V I_R \subseteq U$. Then there is a continuity space structure on S for which $\mathcal{Q}$ is the induced quasiuniformity and such that whenever $a \in L, b \in R, x, y \in S$, we have $d(axb, ayb) \leq d(x, y)$. If further, $\mathcal{Q}$ is a uniformity, then d may be chosen symmetric.*

Proof. (Sketch; details in [8]): Usual (quasi)uniform techniques yield for each $U \in \mathcal{Q}$ a $V \in \mathcal{Q}$ (symmetric in the uniform case) such that $V \circ V \circ V \subseteq U$ and $I_L V \cup V I_R \subseteq V$. As in Kelley, p. 185, there is a quasimetric q on S (a pseudometric in the uniform case) such that $U \in \mathcal{Q}_q \subseteq \mathcal{Q}$ and whenever $a \in L, b \in R, x, y \in S$, we have $q(axb, ayb) \leq q(x,y)$. Let K be a set of such quasimetrics containing one having the above property for each $U \in \mathcal{Q}$. Then take our continuity space distance to be the product of the collection of quasimetric spaces so obtained. □

The theory of Hausdorff topological semigroups could have several possible generalizations to the asymmetric case. The two topologies τ, and τ^G are equal in the compact Hausdorff case, so the statement that $\cdot : \tau \times \tau \to \tau$ is continuous could be generalized by allowing ^{G} to appear on any of the τ's. But mixed forms, such as $\cdot : \tau \times \tau^G \to \tau$ are usually trivial since then $\cdot : \{1\} \times (S, \tau^G) \to (S, \tau)$, so $\tau \subseteq \tau^G$, and this forces $\tau = \tau^G$ for skew compact topologies, so it puts us back into the Hausdorff case. In our definition and below, if $(S, \tau, \cdot)$ is a skew compact space with semigroup operation, then we use the abbreviations S for $(S, \tau, \cdot)$ and S^G for $(S, \tau^G, \cdot)$.

Definition 3.3. A *skew compact (semi)group* is a skew compact space (S, τ) with a (semi)group operation $\cdot : S \times S \to S$. A skew compact (semi)group is:

continuous if $\cdot : S \times S \to S$ is continuous;

de Groot if S and S^G are continuous semigroups ($\cdot$ is a de Groot map).

left continuous if the translation $y \to xy$ is continuous for each $x \in S$.

left de Groot if S and S^G are left continuous (each $y \to xy$ is a de Groot map).

Corollary 3.4.

(a) *The associative operation $\cdot$ on S, is uniformly continuous with respect to the quasiuniformity $\mathcal{Q}$ on S, if and only if whenever $U \in \mathcal{Q}$ there are $V, W \in \mathcal{Q}$ such that $I_{S \cup \{1\}} W \cup V I_{S \cup \{1\}} \subseteq U$.*

(b) *De Groot semigroup topologies are induced by length spaces.*

Proof. (a) is a special case of Theorem 3.2, while (b) results from the fact that de Groot maps on skew compact spaces are uniformly continuous ([7], 3.8). □

4. Cancellative Semigroups

Some asymmetric topological semigroups in which the indicated operation is continuous, are:

(a) $(\mathbb{R}, +, \mathcal{U})$, $\mathcal{U}$ the upper topology, $\{(a, \infty) \mid -\infty \leq a \leq \infty\}$,

(b) The circle group $(\mathbb{T}, \cdot)$; the indiscrete topology strictly weakens the usual compact Hausdorff topology on this group.

(c) The circle group $(\mathbb{T}, \cdot)$; the Sorgenfrey topology strictly strengthens the usual compact Hausdorff topology on this group. Its symmetrization is the locally compact discrete topology.

Theorem 4.1. *The topology of a left continuous skew compact group is compact T_2.*

Proof. It will do to show that $\leq_\tau$ is equality, since then the topology is compact T_2. Toward this end, notice that any continuous function is specialization-preserving:

> if $x \leq_\tau y$ and $f : (X, \tau) \to (Y, \tau')$ then $f(x) \in f(\mathrm{cl}(y)) \subseteq \mathrm{cl}'(f(y))$, so $f(x) \leq_{\tau'} f(y)$.

Thus each left translation, $y \to xy$, is specialization-preserving. Now consider $\mathrm{cl}(e) = \downarrow_{\leq_\tau} e$; as a compact set, it contains a specialization-minimal element, g (in fact, this is the only element in the intersection of a maximal chain of closures of points). We show that $g = e$: otherwise, $gg \leq_\tau ge = g \leq_\tau e$, and $\cdot$ is cancellative, ruling out equality and contradicting the minimality of g. Thus $\leq_\tau$ is equality: for if $k \leq_\tau h$ then $h^{-1}k \leq_\tau h^{-1}h = e$, so $h^{-1}k = e$, thus $k = h$, as required. But then τ is compact T_2, since it is skew compact, and its specialization is equality. □

Theorem 4.2.

(a) *Any continuous skew compact group is a compact T_2 topological group.*
(b) *Any de Groot (two-sided) cancellative semigroup is a compact T_2 group.*

Proof. (a) is immediate from Theorem 4.1. For (b), note that De Groot maps are pairwise continuous, thus ^{S}-continuous, and the symmetrization topology τ^S is compact T_2. Each (two-sided) cancellative compact T_2 semigroup is well-known to be a topological group; in particular, $(S, \cdot, \tau^S)$ is a topological group, so S is a group. Thus by (a), $(S, \cdot, \tau)$ is a compact T_2 topological group. □

Comments 4.3. (a) The argument in 4.2 can be used to show that algebraic facts about compact Hausdorff semigroups hold about de Groot semigroups, since the operation is continuous with respect to the compact Hausdorff symmetrization topology. In particular, each de Groot semigroup has an idempotent, and has maximal and minimal ideals.

(b) If X is a de Groot semigroup, $X^2 = X$, and X has a unique idempotent, then X is a group. Also, the operation is continuous, so by Theorem 4.2 (a), it is a compact Hausdorff topological group.

5. Some structure results

Below, all semigroups will be assumed skew compact unless stated otherwise. Some implications hold with weaker assumptions: eg. 5.1 (c) and (d) do not require any compactness assumption. Also, each compact semigroup has minimal closed subsemigroups, left ideals, and right ideals, and a minimum closed ideal (simply adapt the first two paragraphs of the proof of 5.3 below). But $((0, .9], \cdot, \mathcal{L})$ is a compact space with continuous abelian semigroup operation, and has no idempotent.

Comments 5.1. (a) Compact Hausdorff topological semigroups are de Groot since $\cdot$ is assumed continuous and $\tau^G = \tau$.

(b) Let $S = [0,1]$ with the upper topology, $\mathcal{U}$, and define $\otimes : S \times S \to S$ by

$$x \otimes y = \begin{cases} 0 & \text{if } xy = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\otimes$ is continuous with respect to the upper topology, because the one nontrivial open set in the subspace $\{0,1\}$ is $\{1\}$, and $\otimes^{-1}[\{1\}] = (0,1] \times (0,1]$, an open set. But $\otimes$ is not continuous with respect to $\mathcal{U}^G = \mathcal{L}$, for if so, it would be continuous with respect to their join, the usual topology, and it clearly is not.

(c) Closures of subsemigroups (resp. left, right, two-sided ideals) in continuous semigroups are subsemigroups (resp. left, right, two-sided ideals). Here is the argument for subsemigroups, typical of the four: Suppose that G is a subsemigroup, and $x, y \in \mathrm{cl}(G)$. If $xy \in T$, T open, then for some open U, V, $x \in U$, $y \in V$, and $UV = \cdot[U \times V] \subseteq T$. But since $x, y \in \mathrm{cl}(G)$, there are $u \in G \cap U$, $v \in G \cap V$, and $uv \in (G \cap U) \cdot (G \cap V) \subseteq G \cap UV \subseteq G \cap T$, so the latter is nonempty. This shows that $xy \in \mathrm{cl}(G)$.

(d) If $\cdot$ is specialization-preserving, upper sets of subsemigroups (resp. left, right, two-sided ideals) in these semigroups are subsemigroups (resp. left, right, two-sided ideals). This is a special case of (c), since specialization-preserving functions are precisely $\geq$-Alexandroff-continuous functions, and $\geq$-Alexandroff closures are exactly upper sets. Further, whenever S (or S^G) is continuous, $\cdot$ is specialization-preserving.

Definition 5.2. An *upper left* (resp. *right, two-sided*) *ideal* is an upper set which is also a left (resp. right, two-sided) ideal. By an *ideal* we mean a two-sided ideal.

Theorem 5.3. *If S is a continuous skew compact semigroup, then S contains a unique minimal upper ideal, and it is *-closed. Further, it contains at least one minimal upper left ideal and at least one minimal upper right ideal; these are also *-closed. All minimal upper left ideals and all minimal upper right ideals are subsets of the minimal upper ideal.*

Proof. X is a *-closed ideal of X. If $M_1, \ldots, M_n$ is a finite number of *-closed ideals in X, then $\varnothing \neq M_1 \cdots M_n \subseteq M_1 \cap \cdots \cap M_n$; thus the set of *-closed ideals has the finite intersection property, and so has nonempty intersection in the compact (X, τ^*). This intersection is a *-closed, thus upper ideal, which we call M; it is clearly the smallest *-closed ideal. Now let $I \subseteq M$ be an upper ideal (not assumed *-closed). Let $m \in I$; by the continuity of S, $X\{m\}X$ is a compact ideal, so $\mathrm{cl}^G(X\{m\}X) = \uparrow(X\{m\}X)$ is a *-closed ideal which is a subset of I, thus of M. Since M is minimal among *-closed ideals, $\mathrm{cl}^G(X\{m\}X) = M$, so $I = M$. So M is the smallest upper ideal of X, and it is *-closed.

By Zorn's lemma, there are minimal *-closed upper left and right ideals (not necessarily unique). By an argument similar to that ending the last paragraph, these are minimal among the not necessarily *-closed upper left and upper right ideals, respectively. Let L be a minimal *-closed left ideal of X. The saturated $\uparrow(LM)$ is a subset of L, and by continuity it is also compact, so it is a *-closed left ideal, showing $\uparrow(LM) = L$. But $\uparrow(LM) \subseteq M$, since M is an upper (two-sided) ideal; thus $L \subseteq M$. A similar argument works for right ideals. □

Comments and Examples 5.4. (a) In the Hausdorff case, the $\leq_\tau$-upper ideals are simply the ideals; and so given a de Groot semigroup S, the compact Hausdorff topological semigroup $(S, \cdot, \tau^S)$ has a smallest ideal. Further, if M is this smallest ideal, then $\uparrow M$ is the smallest upper ideal of S: for if I is any upper ideal, then the ideal $M \subseteq I$, thus $\uparrow M \subseteq I$. Also, $\uparrow M$ is an ideal (by Comments 5.1 (d)), and surely an upper set, so it is the smallest such.

(b) Let $S = ([0,1], \mathcal{U}, \times)$, where $a \times b = b$. Then S is de Groot, its upper left ideals are the upper intervals, the *-closed ones are the closed upper intervals, and the minimal *-closed left ideal is $\{1\}$. But its only right or two-sided ideal is $[0,1]$. Thus in particular, the minimal upper ideal is not the disjoint union of the minimal *-closed left ideals.

(c) The following is an example of a compact Hausdorff semigroup (whose operation is not continuous) with no idempotent. Let $(\mathbb{N}, +)$ be the positive integers with the usual addition, and let τ be the topology in which each nontrivial sequence in $\mathbb{N} \setminus \{1\}$ has 1 as a cluster point (that is, use the map

$$n \to \begin{cases} 1/(n-1) & \text{if } n > 1, \\ 0 & \text{if } n = 1, \end{cases}$$

to identify $\mathbb{N}$ with $\{1/n \mid n \in \mathbb{N}\} \cup \{0\}$). Since $\leq_\tau$ is equality for Hausdorff spaces, the operation is specialization-preserving. No translation is continuous since $\lim_{k\to 1}(n+k) = 1 \neq n + \lim_{k \to 1} k$. Further, the only (*-) closed subsemigroup is the whole space, since if n is in our subsemigroup S, then each $kn \in S$, so $1 = \lim_{k\to 1} kn \in S$, and so each $k = k1 \in S$.

In [7] it is pointed out that for each skew compact topological space, the bitopological space (X, τ, τ^*) (τ^* as in Definition 2.1) is *normal* if $C \subseteq T$, where $T \in \tau$ and C is τ^*-closed, then for some $U \in \tau$ and τ^*-closed D, $C \subseteq U \subseteq D \subseteq T$.

Theorem 5.5.

(a) *If $A \subseteq X$, $B \subseteq Y$ are compact and $A \times B \subseteq T$, T open, then for some open $T_A \supseteq A$, $T_B \supseteq B$, $T_A \times T_B \subseteq T$.*

(b) *Suppose S is a de Groot skew compact semigroup. Then each proper *-closed (left, right, two-sided) ideal in S is contained in a proper open (left, right, two-sided) ideal.*

Proof. (a) This is a result of A. D. Wallace (see [4], page 142).

(b) We prove the two-sided case. For $A \subseteq S$ let $J(A) = \mathrm{cl}^*(A \cup SA \cup AS \cup SAS)$, certainly this is the smallest *-closed ideal containing A. Now let $I = I_0$

be a proper *-closed ideal. If $x \notin I$ then $T = S \setminus \mathrm{cl}(x)$ is a proper open subset of S containing I. This T will be kept fixed throughout the proof.

By the normality of (S, τ, τ^*), find $V \in \tau$ such that $I_0 \subseteq V$ and $\mathrm{cl}^*(V) \subseteq T$. By compactness of S thus (a), for each $y \in I$ there is an open U_y such that $y \in U_y$ and $J(U_y) \subseteq \mathrm{cl}^*(V) \subseteq T$; by compactness of I_0, there is a finite subcover of I_0, $\bigcup_{y\in F} U_y$. Let $I_1 = \bigcup_{y\in F} J(U_y)$. Thus $I_0 \subseteq \bigcup_{y\in F} U_y \subseteq \mathrm{int}(I_1)$ and I_1 is a *-closed ideal contained in T. Proceed in this manner to obtain $I_2, \ldots$; clearly $\bigcup_0^\infty I_n$ is an open upper ideal containing I which is contained in T, so is a proper subset of S. The other cases are similar. □

References

[1] P. Fletcher and W. F. Lindgren, *Quasi-uniform Spaces*, (Dekker, New York, 1982).

[2] J. de Groot, *An isomorphism principle in general topology* Bull. Amer. Math. Soc. **73** (1967), 465–467.

[3] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *A Compendium of Continuous Lattices*, (Springer-Verlag, Berlin, 1980).

[4] J. L. Kelley, *General Topology*, Van Nostrand, New York, 1955.

[5] J. C. Kelly, *Bitopological spaces*, Proc. London Math. Soc. **13** (1963), 71–89.

[6] R. D. Kopperman, *All topologies come from generalized metrics*, Amer. Math. Monthly **95** (1988), 89–97.

[7] R. D. Kopperman, *Asymmetry and duality in topology*, Topology and Appl. **66** (1995), 1–39.

[8] R. D. Kopperman, *Lengths on semigroups and groups*, Semigroup Forum **25** (1984), 345–360.

[9] J. D. Lawson, *Order and strongly sober compactifications*, Topology and Category Theory in Computer Science, G. M. Reed, A. W. Roscoe and R. F. Wachter, eds. (Oxford University Press, 1991), 179–205.

[10] L. Nachbin, *Topology and Order*, Van Nostrand, 1965.

[11] D. Robbie and S. Svetlichny, *An answer to A. D. Wallace's question about countably compact cancellative semigroups*, Proc. Amer. Math. Soc. **124 (1)** (1996), 325–330.

[12] S. Salbany, *Bitopological Spaces, Compactifications and Completions*, Math. Monographs **1** (University of Cape Town, 1974).

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