Beltran, A.; Felipe Román, MJ.; Melchor, C. (2016). Normal subgroups whose conjugacy class graph has diameter three. Bulletin of the Australian Mathematical Society. 94(2):266-272. doi:10.1017/S0004972715001860.

Normal subgroups whose conjugacy class graph has diameter three.

Antonio Beltrán
Departamento de Matemáticas,
Universidad Jaume I, 12071 Castellón, Spain
e-mail: abeltran@mat.uji.es

María José Felipe
Instituto Universitario de Matemática Pura y Aplicada,
Universidad Politécnica de Valencia, 46022 Valencia, Spain
e-mail: mfelipe@mat.upv.es

Carmen Melchor
Departamento de Educación,
Universidad Jaume I, 12071 Castellón, Spain
e-mail: cmelchor@uji.es

Abstract

Let G be a finite group and N a normal subgroup of G. We determine the structure of N when the graph associated to the G-conjugacy classes contained in N has diameter three.

Keywords. Finite groups, conjugacy classes, normal subgroups, graphs.
Mathematics Subject Classification (2010): 20E45, 20D15.

1 Introduction

Let G be a finite group and let N be a normal subgroup of G and let $x \in N$. We denote by $x^G = \{x^g \mid g \in G\}$ the G-conjugacy class of x. Let $\Gamma_G(N)$ be the graph associated to these G-conjugacy classes, which was defined in [2] as follows: its vertices are the G-conjugacy classes of N of cardinality bigger than 1, that is, G-classes of elements in $N \setminus (\mathbf{Z}(G) \cap N)$, and two of them are joined by an edge if their sizes are not coprime. It was proved in [2] that $d(\Gamma_G(N)) \leq 3$ where $d(\Gamma_G(N))$ denotes the diameter of the graph. In this paper we analyze the structure properties of N when $d(\Gamma_G(N)) = 3$.

The above graph extends the ordinary graph, $\Gamma(G)$, which was formerly defined in [3], whose vertices are the non-central conjugacy classes of G and two vertices are joined by an edge if their sizes are not coprime. The graph $\Gamma_G(N)$ can be viewed as the subgraph of $\Gamma(G)$ induced by those vertices of $\Gamma(G)$ which are vertices in $\Gamma_G(N)$. This fact does not allow to obtain directly properties of the graph of G-classes.

Concerning ordinary classes, L.S. Kazarin characterizes in [7] the structure of a group G having two "isolated classes". Remember that a group G has isolated conjugacy classes if there exist elements $x, y \in G$ with coprime conjugacy class sizes such that every element of G has conjugacy class size coprime to either $|x^G|$ or $|y^G|$. Particularly Kazarin determined the structure of the groups G with $d(\Gamma(G)) = 3$. It should be noted that similar results have also been tested for other graphs. In [5], Dolfi defines the graph $\Gamma'(G)$ whose vertices are the elements of the set of all primes which occur as divisors of the lengths of the conjugacy classes of G, and two vertices p, q are joined by an edge if there exists a conjugacy class in G whose length is a multiple of pq. In [6] Dolfi and Casolo describe all finite groups G for which $\Gamma'(G)$ is connected and has diameter three.

We have to remark that the primes dividing the G-conjugacy class sizes not necessarily divide $|N|$, it can occur the case when N is abelian and it is non-central in G and consequently we have not control on these primes. For this reason, we observe that new cases appear when we work with G-classes which are not contemplated in the ordinary case. The main result of this paper is the following theorem. From now on, if H is a subgroup of a finite group G we denote by $\pi(H)$ the set of primes dividing $|H|$.

Theorem A. *Let G be a finite group and $N \trianglelefteq G$. Suppose that x^G and y^G are two non-central G-conjugacy classes of N such that any G-conjugacy of G has size coprime with $|x^G|$ or $|y^G|$. Let $\pi_x = \pi(|x^G|)$, $\pi_y = \pi(|y^G|)$ and $\pi = \pi_x \cup \pi_y$. Then, $N = \mathbf{O}_{\pi'}(N) \times \mathbf{O}_\pi(N)$ with $x, y \in \mathbf{O}_\pi(N)$ which is a quasi-Frobenius group with abelian kernel and complement or $\mathbf{O}_\pi(N) = P \times A$ with $A \leq \mathbf{Z}(N)$ and P a p-group for a prime p.*

Notice that in the conditions of Theorem A if $d(\Gamma_G(N)) \leq 2$ it follows that the graph is disconnected and the structure of N is determined in Theorem E of [2]. Consequently, $d(\Gamma_G(N)) = 3$ and we obtain the following result.

Corollary. *Let G be a finite group and $N \trianglelefteq G$. Suppose that $\Gamma_G(N)$ is connected and $d(\Gamma_G(N)) = 3$. Let us consider $x, y \in N$ such that $d(x^G, y^G) = 3$. Set $\pi = \pi(|x^G|) \cup \pi(|y^G|)$. We have that x, $y \in \mathbf{O}_\pi(N)$, $N = \mathbf{O}_{\pi'}(N) \times \mathbf{O}_\pi(N)$ with $\mathbf{O}_\pi(N)$ a quasi-Frobenius group with abelian kernel and complement or $\mathbf{O}_\pi(N) = P \times A$ with $A \leq \mathbf{Z}(N)$ and P a p-group for a prime p.*

Proof. It follows immediately by Theorem A. □

Proofs of these results are based on the techniques appeared in [7] although we do not use them in ours. When $N = G$ we obtain the result of Kazarin.

2 Proof of Theorem A

First, we show three elementary results necessary to prove the main theorem.

Lemma 1. *Let G a π-separable group. Then the conjugacy class length of every π-element of G is a π-number if and only if $G = H \times K$, where H and K are a Hall π-subgroup and a π-complement of G, respectively.*

Proof. This is Lemma 8 of [1].

In the particular case in which $\pi = p'$, the complement of some prime p, the above Lemma is true without assuming p-separability (which is equivalent to p-solvability).

Lemma 2. *If, for some prime p, every p'-element of a group G has index prime to p, then the Sylow p-subgroup of G is a direct factor of G.*

Proof. This is Lemma 1 of [4].

Lemma 3. *Let G be a finite group and $N \trianglelefteq G$. Let $B = b^G$ and $C = c^G$ be two non-central G-conjugacy classes of N. If $(|B|, |C|) = 1$. Then*

a. $\mathbf{C}_G(b)\mathbf{C}_G(c) = G$.

b. *$BC = CB$ is a non-central G-class of N and $|BC|$ divides $|B||C|$.*

c. *Suppose that $d(B, C) \geq 3$ and $|B| < |C|$. Then $|BC| = |C|$ and $CBB^{-1} = C$. Furthermore, $C\langle BB^{-1}\rangle = C$, $\langle BB^{-1}\rangle \subseteq \langle CC^{-1}\rangle$ and $|\langle BB^{-1}\rangle|$ divides $|C|$.*

Proof. This is Lemma 2.1 of [2].

Proof of Theorem A. We proceed by induction on $|N|$. Notice that the hypotheses are inherited by every normal subgroup in G which is contained in N and contains x and y. By using the primary decomposition we can assume that x and y have order a power of two primes, say p and q, respectively.

Step 1. $q = p$ if and only if $xy = yx$.

Suppose that $xy = yx$ and that $p \neq q$. Observe that $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$ and consequently, $|x^G|$ divides $|(xy)^G|$ and $|y^G|$ divides $|(xy)^G|$. Thus, we obtain a G-conjugacy class connected with x^G and y^G, which is a contradiction by hypotheses. Conversely, suppose that $p = q$. We know that p cannot divide either $|x^G|$ or $|y^G|$. Furthermore, the hypotheses imply that $(|x^G|, |y^G|) = 1$, so we have $G = \mathbf{C}_G(x)\mathbf{C}_G(y)$ and $|x^G| = |G : \mathbf{C}_G(x)| = |\mathbf{C}_G(y) : \mathbf{C}_G(x) \cap \mathbf{C}_G(y)|$. Now, since y is a p-element in $\mathbf{Z}(\mathbf{C}_G(y))$, we deduce that $y \in \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$ and hence $xy = yx$.

Step 2. $p, q \in \pi$.

We define $K = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$. First, we assume that $p \neq q$ and $xy \neq yx$. We have $|G : K| = |G : \mathbf{C}_G(x)||\mathbf{C}_G(x) : \mathbf{C}_G(x) \cap \mathbf{C}_G(y)| = |x^G||y^G|$, which is a π-number. Since $x \in \mathbf{Z}(\mathbf{C}_G(x))$ and x is a p-element but $x \notin K$, we know that p divides $|\mathbf{C}_G(x) : K| = |y^G|$. This means that $p \in \pi_y$. Similarly we obtain that q divides $|x^G|$, that is, $q \in \pi_x$. Consequently, $p, q \in \pi$.

Suppose now that $p = q$ and $xy = yx$. Let us see that $p \in \pi$. We denote $X = x^G$ and $Y = y^G$ and we assume for instance that $|X| > |Y|$. By hypothesis, the distance between X and Y in $\Gamma_G(N)$ is 3. We can apply Lemma 3(c) and we obtain $X\langle YY^{-1}\rangle = X$, $\langle YY^{-1}\rangle \subseteq \langle XX^{-1}\rangle$ and $|\langle YY^{-1}\rangle|$ divides $|X|$. On the other hand, since $G = \mathbf{C}_G(x)\mathbf{C}_G(y)$ we have $X \subseteq \mathbf{C}_G(y)$. As a result, $\langle YY^{-1}\rangle \subseteq \langle XX^{-1}\rangle \subseteq \mathbf{C}_G(y)$. In particular, if we take $z = y^g \neq y$, for some $g \in G$, we have $w = zy^{-1} \in \langle YY^{-1}\rangle \subseteq \mathbf{C}_G(y)$, so $[z, y] = 1$. We obtain that w is a non-trivial p-element and, since p divides $|\langle YY^{-1}\rangle|$, which divides $|X|$, we conclude that $p \in \pi_x$. If $|Y| > |X|$ we can argue similarly to get $p \in \pi_y$.

Step 3. We can assume that $N/\mathbf{Z}(N)$ is neither a p-group nor a q-group (particularly, we can assume that N is not abelian).

As we have said at the beginning, x is a p-element and y is a q-element. Suppose that $N/\mathbf{Z}(N)$ is a p-group (the reasoning is analogous if we suppose that it is a q-group). Hence we can write $N = P \times A$ where $A \leq \mathbf{Z}(N)$ and A is a p'-group. If $p \neq q$, it follows that $x \in P$ and $y \in A$, which leads to a contradiction with Step 1. Thus, $p = q$ and $x, y \in P$, so the theorem is proved.

Step 4. We can suppose that N is not a π-group.

Let us see that if N is a π-group, then N is a quasi-Frobenius group with abelian kernel and complement or $N = P \times A$ with $A \leq \mathbf{Z}(N)$ and A a p'-group. Assume that N is a π-group. As N is non-abelian by Step 3, there exists a conjugacy class z^Nsuch that $|z^N| \neq 1$, Since $|z^N|$ divides $|z^G|$, then either $(|z^N|, |x^G|) = 1$ or $(|z^N|, |y^G|) = 1$. Thus, $|z^N|$ either is a π_x-number or a π_y-number. If $\Gamma(N)$ is disconnected, we know by Theorem 2 of [3] that N is quasi-Frobenius group with abelian kernel and complement. Moreover, $\Gamma(N)$ cannot be empty since by Step 3, we can assume that N is not abelian. Consequently, we can assume that $\Gamma(N)$ is connected and this forces to either $|x^N| = 1$ or $|y^N| = 1$. Suppose for instance that $|x^N| = 1$, that is, $x \in \mathbf{Z}(N)$.

By Step 3 we can take w an s-element of $N \setminus \mathbf{Z}(N)$ with $s \neq p$. Observe that $|w^N|$ must be a π_y-number, so w^G is connected to y^G in $\Gamma_G(N)$. Since x and w have coprime orders and $x \in \mathbf{Z}(N)$ we have that $|w^G|$ and $|x^G|$ both divide $|(wx)^G|$. As a consequence, we have a contradiction because $|(wx)^G|$ has primes in π_x and π_y. Then we can suppose that N is not a π-group.

Step 5. Conclusion in case $p \neq q$.

Let z be a π'-element of $K \cap N$ and let us prove that $|z^G|$ is a π'-number. Suppose that $s \in \pi$ is a prime divisor of $|z^G|$. We can assume for instance that $s \in \pi_y$, otherwise we proceed analogously. Since $|z^G|$ divides $|(zx)^G|$ we obtain that s divides $|(zx)^G|$. On the other hand, we know by the proof of Step 2 that $q \in \pi_x$. Therefore, $|(zx)^G|$ is divisible by primes in π_x and π_y, a contradiction. Consequently, $s \notin \pi$ and $|z^G|$ is a π'-number, as wanted.

Let M be the subgroup generated by all π'-elements of $K \cap N$. Note that $M \neq 1$, otherwise $K \cap N$ would be a π-group and, since $|N : K \cap N| = |KN : K|$ divides $|G : K|$, which is a π-number too, then N would be a π-group, a contradiction with Step 2. Let us prove that $M \trianglelefteq G$. Let α be a generator of M, so $|\alpha^G|$ is π'-number. Since $(|G : K|, |\alpha^G|) = 1$ we have $G = K\mathbf{C}_G(\alpha)$ and hence, $\alpha^G = \alpha^K \subseteq K \cap N$. Therefore $\alpha^G \subseteq M$, as wanted.

Let $D = \langle x^G, y^G \rangle$. Notice that $D \trianglelefteq G$ and $D \subseteq N$. Let α be a generator of M. As we have proved that $|\alpha^G|$ is π'-number, then $(|\alpha^G|, |x^G|) = 1$, so $G = \mathbf{C}_G(x)\mathbf{C}_G(\alpha)$. Thus, $x^G = x^{\mathbf{C}_G(\alpha)} \subseteq \mathbf{C}_G(\alpha)$ because $\alpha \in K$. The same happens for y, that is, $y^G \subseteq \mathbf{C}_G(\alpha)$, so we conclude that $[M, D] = 1$.

We define $L = MD$ and we distinguish two cases. Assume first that $L < N$. Note that $x, y \in L \trianglelefteq G$ and L trivially satisfies the hypotheses of the theorem. By applying induction to L we have in particular $L = \mathbf{O}_\pi(L) \times \mathbf{O}_{\pi'}(L)$. Observe that the fact that $M \neq 1$ implies that $\mathbf{O}_{\pi'}(L) > 1$. Now, by definition of M, we have that $|K \cap N : M|$ is a π-number. As $|N : K \cap N|$ is also a π-number, it follows that $|N : \mathbf{O}_{\pi'}(L)|$ is a π-number too. Then, $\mathbf{O}_{\pi'}(L) = \mathbf{O}_{\pi'}(N)$ is a Hall π'-subgroup of N. We can apply Lemma 1 so as to conclude that $N = \mathbf{O}_\pi(N) \times \mathbf{O}_{\pi'}(N)$ with $x, y \in \mathbf{O}_\pi(N)$. Since $\mathbf{O}_{\pi'}(N) > 1$, we apply the inductive hypotheses to $\mathbf{O}_\pi(N) < N$ and we deduce that $\mathbf{O}_\pi(N)$ is a quasi-Frobenius group with abelian kernel and complement or $\mathbf{O}_\pi(N) = P \times A$ with $A \leq \mathbf{Z}(N)$ and P is a p-group so the theorem is finished.

From now on, we assume that $L = N$ and let us see that $\mathbf{Z}(N) = 1$. Otherwise, we take $\overline{N} = N/\mathbf{Z}(N)$ and $\overline{G} = G/\mathbf{Z}(N)$. If $|\overline{x}^{\overline{G}}| = 1$, then $[\overline{x}, \overline{y}] = 1$, and thus $[x, y] \in \mathbf{Z}(N)$. Since $(o(x), o(y)) = 1$, it is easy to prove that $[x, y] = 1$, a contradiction. Analogously, we have $|\overline{y}^{\overline{G}}| \neq 1$. Consequently, $\overline{N}$ satisfies the assumptions of the theorem. By induction, we have $\overline{N} = \mathbf{O}_{\pi'}(\overline{N}) \times \mathbf{O}_\pi(\overline{N})$ with $\overline{x}, \overline{y} \in \mathbf{O}_\pi(\overline{N})$ and $\mathbf{O}_\pi(\overline{N})$ is either a quasi-Frobenius group with abelian kernel and complement or $\overline{N} = \overline{P} \times \overline{A}$ with $\overline{A} \leqslant \mathbf{Z}(\overline{N})$ and $\overline{P}$ a p-group. In the latter case, $[\overline{y}, \overline{x}] = 1$ which leads to a contradiction as we have seen before. So we are in the former case. It follows that $N = \mathbf{O}_{\pi'}(N) \times \mathbf{O}_{\pi}(N)$ with $x, y \in \mathbf{O}_{\pi}(N)$ and by applying induction to $\mathbf{O}_{\pi}(N) < N$, we have the result. Therefore, $\mathbf{Z}(N) = 1$. On the other hand, we have proved that $[M, D] = 1$. Thus $M \cap D \subseteq \mathbf{Z}(N) = 1$ and $N = M \times D$ with $x, y \in D$. Since $M \neq 1$, we can apply induction to D and we get $D = \mathbf{O}_{\pi'}(D) \times \mathbf{O}_{\pi}(D)$ with $x, y \in \mathbf{O}_{\pi}(D)$ and $\mathbf{O}_{\pi}(D)$ is a Frobenius group with abelian kernel and complement (notice that $\mathbf{Z}(\mathbf{O}_{\pi}(D)) = 1$ because $\mathbf{Z}(N) = 1$). The p-group case cannot occur because x and y do not commute. Notice that if M is π'-group then the theorem is proved. Assume then that M is not a π'-group and we will obtain a contradiction. Let $s \in \pi$ such that s divides $|M|$. We can assume that $s \in \pi_x$ (we proceed analogously if $s \in \pi_y$). Suppose that there exists an s'-element $z \in M$ such that $|z^M|$ is divisible by s. Since N is the direct product of M and D, we have that $(zy)^N = z^N y^N$ is a non-trivial class of N whose size is divisible by s and by some prime of $|y^N| \neq 1$. This is not possible because $|(zy)^G|$ would have primes in π_x and π_y. Thus, the class size of every s'-element of M is a s'-number. It is known that $M = M_1 \times S$ with $S \in \mathrm{Syl}_s(M)$. In this case, $\mathbf{Z}(S) \subseteq \mathbf{Z}(N) = 1$, a contradiction.

Step 6. Conclusion in case $p = q$.

Let $K = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$ as in Step 2. Let z be a p'-element of $K \cap N$ and let us prove that $|z^G|$ is a π'-number. Suppose that $s \in \pi$ is a prime divisor of $|z^G|$. We can assume for instance that $s \in \pi_y$, otherwise we proceed analogously. Since $|z^G|$ divides $|(zx)^G|$ we obtain that s divides $|(zx)^G|$. On the other hand, we know by the proof of Step 2 that $q \in \pi_x$. Therefore, $|(zx)^G|$ is divisible by primes in π_x and π_y, a contradiction. Consequently, $s \notin \pi$ and $|z^G|$ is a π'-number, as wanted.

Let T be the subgroup generated by all p'-elements of $K \cap N$. We have that $T \neq 1$ because otherwise $K \cap N$ would be a π-group and this implies that N is a π-group as in Step 5, a contradiction. Let us prove that $T \trianglelefteq G$. If α is a generator of T, we know that $|\alpha^G|$ is π'-number. Then $(|G : K|, |\alpha^G|) = 1$, so we have $G = K\mathbf{C}_G(\alpha)$ and $\alpha^G = \alpha^K \subseteq K \cap N$. Therefore, $\alpha^G \subseteq T$ as wanted.

Since the class size of every p'-element of T is a p'-number then, by Lemma 2, $T = \mathbf{O}_p(T) \times \mathbf{O}_{p'}(T)$. However, by definition of T, we have $\mathbf{O}_p(T) = 1$, or equivalently $M = \mathbf{O}_{p'}(T)$. Now, notice that if $s \in \pi$ and $s \neq p$, then the class size of every element of T is an s'-number so, it is well know that T has a Sylow s-subgroup central and we can write $T = \mathbf{O}_{\pi}(T) \times \mathbf{O}_{\pi'}(T)$. On the other hand, $|N : T| = |N : K \cap N||K \cap N : T|$ where $|N : K \cap N| = |KN : K|$ is a π-number and $|K \cap N : T|$ is a power of $p \in \pi$. Therefore $\mathbf{O}_{\pi'}(T) = \mathbf{O}_{\pi'}(N)$ and $\mathbf{O}_{\pi'}(N)$ is a Hall π'-subgroup of N. We have proved that the class size of every p'-element of N is a π'-number, so by Lemma 1, we have $N = \mathbf{O}_{\pi'}(N) \times \mathbf{O}_{\pi}(N)$. We apply induction to $\mathbf{O}_{\pi}(N) < N$ and the proof is finished. □

We give an example showing that the converse of Theorem A is not true.

Example 1. We take the Special Linear group $H = \mathrm{SL}(2, 5)$ which is a group of order 120 that acts Frobeniusly on $K = \mathbb{Z}_{11} \times \mathbb{Z}_{11}$. Let $P \in \mathrm{Syl}_5(H)$ and we consider $\mathbf{N}_H(P)$. Then, we define $N = KP$, which is trivially a normal subgroup of $G = K\mathbf{N}_H(P)$. We have that the set of the G-conjugacy class sizes of N is $\{1, 20, 242\}$. Consequently, there are not two non-central G-classes of N such that any non-central G-class of N has size coprime with one of both.

Let us look at several examples illustrating Theorem A.

Example 2. We take the following groups from the library *SmallGroups* of GAP $G_1 = \mathrm{Id}(324, 8)$ and $G_2 = \mathrm{Id}(168, 44)$ that have the normal subgroups exposed now. The abelian 3-subgroup $P = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, respectively. It follows that the set of conjugacy class sizes of P is $\{1, 2, 3, 3\}$ and the set of conjugacy class sizes of A is $\{1, 7\}$. We construct $N = P \times A$ and $G = G_1 \times G_2$. We have that N is a normal subgroup of G and the set of G-conjugacy class sizes of N is $\{1, 2, 3, 7, 14, 21\}$ so $d(\Gamma_G(N)) = 3$ and N satisfies that it is the direct product of a 3-group and $A \leq \mathbf{Z}(N)$. Note that in this example it follows that $\mathbf{O}_{\pi'}(N) = 1$ and $\pi = \{2, 3\}$.

Example 3. In order to illustrate the quasi-Frobenius case it is enough to consider any group G and a normal subgroup $N = G$ such that $\Gamma(N)$ has two connected components. Thus, by applying Theorem of [3] we know that N is a quasi-Frobenius group with abelian kernel and complement.

Acknowledgements

The results in this paper are part of the third author's Ph.D. thesis at the University Jaume I of Castellon. The research of the first and second authors is supported by the Valencian Government, Proyecto PROMETEOII/2015/011. The first author is also partially supported by Universitat Jaume I, grant P11B-2012-05.

References

[1] A. Beltrán and M.J. Felipe, Prime powers as conjugacy class lengths of π-elements. Bull. Austral. Math. Soc. (69) (2004), 317-325.

[2] A. Beltrán, M.J. Felipe and C. Melchor, Graphs associated to conjugacy classes of normal subgroups in finite groups. J.Algebra, (443) (2015), 335-348.

[3] E.A. Bertram, M. Herzog and A. Mann, On a graph related to conjugacy classes of groups. Bull. London Math. Soc., **22** (6) (1990), 569-575.

[4] A.R. Camina, Arithmetical conditions on the conjugacy class numbers of a finite group. J.London Math. Soc. (2) 5 (1972), 127-132.

[5] S. Dolfi, Arithmetical conditions of the length of the conjugacy classes in finite groups. **174**, (3) (1995), 753-771.

[6] C. Casolo, S. Dolfi, The diameter of a conjugacy class graph of finite groups. Bull. London Math. Soc. 28 (1996), 141-148.

[7] L.S. Kazarin, On groups with isolated conjugacy classes, Izv. Vyssh. Uchebn. Zaved. Mat. (7) (1981), 40-45.