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Density topology and pointwise convergence

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Dedicated to Professor S. Naimpally on the occasion of his 70th birthday.

Abstract. We shall show that the space of all approximately continuous functions with the topology of pointwise convergence is not homeomorphic to its category analogue.

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1. Density topology and pointwise convergence.

Let S be a σ-algebra of Lebesgue measurable subsets of the real line $\mathbb{R}$, $\mathcal{L} \subset S$ – a σ-ideal of null sets, $\mathcal{B}$ – a σ-algebra of subsets of $\mathbb{R}$ posessing a property of Baire and $\mathcal{I} \subset \mathcal{B}$ – a σ-ideal of sets of the first category. The sets of the first and the second category are considered only with respect to the natural topology.

Recall that a point $x_0 \in \mathbb{R}$ is a density point of the set $A \in S$ if and only if

$$\lim_{h \to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1,$$

where λ stands for the Lebesgue measure on S.

Let $\Phi(A)$ be a set of all density points of $A \in S$. If we denote $A \sim B$ in the case when $A \triangle B \in \mathcal{L}$ then we have (compare [6], Th. 22.2):

Theorem 1.1. (1) $\Phi(A) \sim A$ *for each* $A \in S$ *(Lebesgue density theorem),*
(2) *if* $A, B \in S$ *and* $A \sim B$*, then* $\Phi(A) = \Phi(B)$*,*
(3) $\Phi(\varnothing) = \varnothing$*,* $\Phi(\mathbb{R}) = \mathbb{R}$*,*
(4) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ *for each* $A, B \in S$*.*

Observe that from 1 it follows immediately that $\Phi(A) \in S$ for each $A \in S$. The function $\Phi : S \to S$ is usually called a lower density operator.

Theorem 1.2. ([6], Th. 22.5) A family $\mathcal{T}_d = \{A \in S : A \subset \Phi(A)\} = \{\Phi(E) \setminus P : E \in S \text{ and } P \in \mathcal{L}\}$ is a topology on the real line stronger than the natural topology.

The topology $\mathcal{T}_d$ is usually called the density topology. For further properties of $\mathcal{T}_d$ see, for example, [4] or [9].

A real function of a real variable is called approximately continuous if it is continuous when the domain is equipped with the density topology and the range – with the natural topology. Since $(\mathbb{R}, \mathcal{T}_d)$ is a Tikhonov (completely regular) topological space ([4]), the density topology is the coarsest topology for the class of all approximately continuous functions.

Observe that the following conditions are equivalent (see [8]) for a set $A \in S$:

1) 0 is a density point of A,
2) $\lim_{n\to\infty} \frac{\lambda(A\cap(-\frac{1}{n},\frac{1}{n}))}{\frac{2}{n}} = 1$,
3) $\lim_{n\to\infty} \lambda((n \cdot A) \cap (-1,1)) = 2$ (where $n \cdot A = \{nx : x \in A\}$,
4) $\{\chi_{n\cdot A\cap(-1,1)}\}_{n\in N}$ converges to $\chi_{(-1,1)}$ in measure,
5) for each increasing sequence $\{n_m\}_{m\in N}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in N}$ such that

$$\lim_{p\to\infty} \chi_{(n_{m_p}\cdot A)\cap(-1,1)} = \chi_{(-1,1)} \quad \text{almost everywhere.}$$

The equivalence of 1)-4) is immediate, while the equivalence of 4) and 5) follows from a well known theorem of Riesz.

The above observation was a starting point to study a category analogue of a density point, density topology and approximate continuity.

Definition 1.3. ([8]) We say that 0 is an $\mathcal{I}$-density point of a set $A \in \mathcal{B}$ if and only if for each increasing sequence $\{n_m\}_{m\in N}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in N}$ such that

$$\lim_{p\to\infty} \chi_{(n_{m_p}\cdot A)\cap(-1,1)} = \chi_{(-1,1)}$$

except on a set of the first category (in abbr. $\mathcal{I}$-a.e.). We say that x_0 is an $\mathcal{I}$-density point of $A \in \mathcal{B}$ if and only if 0 is an $\mathcal{I}$-density point of a set $A - x_0 = \{x - x_0 : x \in A\}$.

Let $\Phi_{\mathcal{I}}(A)$ be a set of all $\mathcal{I}$-density points of $A \in \mathcal{B}$. If we denote now $A \sim B$ in the case when $A \triangle B \in \mathcal{I}$, then we have

Theorem 1.4. ([8])

(1) $\Psi_{\mathcal{I}}(A) \sim A$ *for each* $A \in \mathcal{B}$,
(2) *if* $A, B \in \mathcal{B}$ *and* $A \sim B$, *then* $\Phi_{\mathcal{I}}(A) = \Phi_{\mathcal{I}}(B)$,
(3) $\Phi_{\mathcal{I}}(\varnothing) = \varnothing$, $\Phi_{\mathcal{I}}(\mathbb{R}) = \mathbb{R}$,
(4) $\Phi_{\mathcal{I}}(A \cap B) = \Phi_{\mathcal{I}}(A) \cap \Phi_{\mathcal{I}}(B)$ *for each* $A, B \in \mathcal{B}$.

Theorem 1.5. ([8]) *A family* $\mathcal{T}_{\mathcal{I}} = \{A \in \mathcal{B} : A \subset \Phi_{\mathcal{I}}(A)\} = \{\Phi_{\mathcal{I}}(E) \setminus P : E \in \mathcal{B} \text{ and } P \in \mathcal{I}\}$ *is a topology on the real line stronger than the natural topology.*

The topology $\mathcal{T}_{\mathcal{I}}$ is called the $\mathcal{I}$-density topology. For further properties of $\mathcal{T}_{\mathcal{I}}$ see, for example, [8] or [3]. A real function of a real variable is called $\mathcal{I}$-approximately continuous if it is continuous when the domain is equipped with the $\mathcal{I}$-density topology and the range – with the natural topology. Unfortunately, $(\mathbb{R}, \mathcal{T}_{\mathcal{I}})$ is a not a Tikhonov topological space ([8]). However, the coarsest topology for $\mathcal{I}$-approximately continuous functions, which must be completely regular, is studied in details in [5] and [7]. We shall trace a description of such a topology (called the deep $\mathcal{I}$-density topology) after [3].

Definition 1.6. A point $x_0 \in \mathbb{R}$ is called a deep $\mathcal{I}$-density point of $A \in \mathcal{B}$ if there exists a closed (in the natural topology) set $F \subset A \cup \{x_0\}$ such that x_0 is an $\mathcal{I}$-density point of F.

Let $\Phi_{\mathcal{ID}}(A)$ be a set of all deep $\mathcal{I}$-density points of $A \in \mathcal{B}$.

Theorem 1.7. *A family $\mathcal{T}_{\mathcal{ID}} = \{A \in \mathcal{B} : A \subset \Phi_{\mathcal{ID}}(A)\}$ is a topology stronger than the natural topology and weaker than the $\mathcal{I}$-density topology. Moreover $\mathcal{T}_{\mathcal{ID}}$ is a completely regular topology.*

From the above theorem it follows immediately that the class of $\mathcal{I}$-approximately continuous functions is equal to the class of deeply $\mathcal{I}$-approximately continuous real functions of a real variable (i.e. the class of functions which are continuous when the domain is equipped with the deep $\mathcal{I}$-density topology and the range with the natural topology).

Since both spaces $(\mathbb{R}, \mathcal{T}_d)$ and $(\mathbb{R}, \mathcal{T}_{\mathcal{ID}})$ are Tikhonov topological spaces, it is reasonable to consider the spaces $C_p(\mathbb{R}_d)$ and $C_p(\mathbb{R}_{\mathcal{I}})$ of all approximately continuous and all $\mathcal{I}$-approximately continuous functions with the topology of pointwise convergence. The question: are $C_p(\mathbb{R}_d)$ and $C_p(\mathbb{R}_{\mathcal{I}})$ homeomorphic seems to be interesting. In this note we shall try to find an answer.

First of all, observe that the problem is not trivial by virtue of the following theorem:

Theorem 1.8. *The spaces $(\mathbb{R}, \mathcal{T}_d)$ and $(\mathbb{R}, \mathcal{T}_{\mathcal{ID}})$ are not homeomorphic.*

Proof. Suppose that $h : \mathbb{R} \underset{onto}{\to} \mathbb{R}$ is a homeomorphism between $(\mathbb{R}, \mathcal{T}_d)$ and $(\mathbb{R}, \mathcal{T}_{\mathcal{ID}})$. If $E \subset \mathbb{R}$ is $\mathcal{T}_d$-connected set, then $h(E)$ is $\mathcal{T}_{\mathcal{ID}}$-connected set. From [4] it follows that the family of all $\mathcal{T}_d$-connected sets coincides with the family of all sets connected in the natural topology (i.e. with the family of all intervals — open, half-open, closed, bounded or unbounded). The same holds for the topology $\mathcal{T}_{\mathcal{I}}$ (see [8]). Since $\mathcal{T}_{\mathcal{DI}}$ is between $\mathcal{T}_{\mathcal{I}}$ and the natural topology, it has the same family of connected sets. So for h we see that the image of an arbitrary interval is an interval. From this it is easy to conclude that h is a strictly monotone and continuous (in the sense that both the domain and the range are equipped with the natural topology) function, in fact h is a homeomorphism from $(\mathbb{R}, \text{nat})$ to $(\mathbb{R}, \text{nat})$. Let $E \in \mathcal{T}_d$ be a set which is nowhere dense in the natural topology (for example $E = C \cap \Phi(C)$, where C is a nowhere dense Cantor set of positive measure). Then $h(E)$ is also nowhere dense in the natural topology. But from the definition of deep $\mathcal{I}$-density point and deep $\mathcal{I}$-density topology it follows immediately that each set in $\mathcal{T}_{\mathcal{ID}}$ is of the second category in fact, it must contain a nondegenerate closed interval (see the proof of Th. 2 below). So $h(E) \notin \mathcal{T}_{\mathcal{ID}}$ – a contradiction. □

Theorem 1.9. *The spaces $C_p(\mathbb{R}_d)$ and $C_p(\mathbb{R}_{\mathcal{I}})$ are not homeomorphic.*

Proof. Suppose that $C_p(\mathbb{R}_d)$ and $C_p(\mathbb{R}_{\mathcal{I}})$ are homeomorphic. Then $d(\mathbb{R}, \mathcal{T}_d) = d(\mathbb{R}, \mathcal{T}_{\mathcal{ID}})$ ([1], p. 26), where d denotes the density of the topological space, i.e. the smallest cardinal number of dense subsets of this space. But from the definition of deep $\mathcal{I}$-density topology it follows that each $\mathcal{T}_{\mathcal{ID}}$-open set includes a (nondegenerate) closed interval, because $\mathcal{T}_{\mathcal{ID}}$-open set includes a closed set (in the natural topology) of the second category. Hence the set $E \subset \mathbb{R}$ is dense in the topology $\mathcal{T}_{\mathcal{ID}}$ if and only if it is dense in the natural topology and $d(\mathbb{R}, \mathcal{T}_{\mathcal{DI}}) = \aleph_0$. Simultaneously if the set E is dense in the topology $\mathcal{T}_d$, then $\lambda^*(E) > 0$ (in fact, $\lambda^*(E \cap (a,b)) = b - a$ for each interval (a,b)). So $d(\mathbb{R}, \mathcal{T}_d) > \aleph_0$ – a contradiction. Observe that the exact value of $d(\mathbb{R}, \mathcal{T}_d)$ depends essentially of the system of axioms (see [2]). □

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