APPLIED GENERAL TOPOLOGY

A topological approach to Best Approximation Theory

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ABSTRACT. The main goal of this paper is to put some light in several arguments that have been used through the time in many contexts of Best Approximation Theory to produce proximinality results. In all these works, the main idea was to prove that the sets we are considering have certain properties which are very near to the compactness in the usual sense. In the paper we introduce a concept (the wrapping) that allow us to unify all these results in a whole theory, where certain ideas from Topology are essential. Moreover, we do not only cover many of the known classical results but also prove some new results. Hence we prove that exists a strong interaction between General Topology and Best Approximation Theory.

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1. INTRODUCTION

One of the central problems in Best Approximation Theory can be roughly formulated in the following way. Let X be a set, A a nonempty subset of X and $x \in X$. If there exists a real valued function d on $X \times X$ that provides a notion of gap between points in X, we want to know about the existence of points $a \in A$ such that

$$d(x,a) = \inf_{b \in A} d(x,b).$$

The points $a \in A$ satisfying the previous relation are called the best approximations to x from A. The subset A is said to be proximinal if for all $x \in X$ there exists a best approximation to x from A.

The proximinality of a subset A can be assured if there exists a topology for X large enough to be the functions $d(x, \cdot) : A \longrightarrow \mathbb{R}$ sequentially lower semi-continuous and small enough in order to be the domain A countably compact. The main tools used in the known proofs of the results concerning the existence of best approximations can be clearly distinguished in the following cases.

(a): If X is a normed space and the gap function is the metric derived from the norm, the Functional Analysis arguments will give a common thread of compactness. As a sample of the most notorious results of this kind we give the following list.
- Nonempty closed subsets of finite dimensional subspaces of normed linear spaces are proximinal.
- Nonempty weakly closed subsets of a reflexive Banach space (in particular, any nonempty closed convex subset) are proximinal.
- Nonempty weakly star closed subsets of the dual space of a normed space are proximinal.

(b): If X is not normed, or X is normed but the gap function d is not a metric. In this situation the set X is usually a function space and the techniques used in the proofs will depend on each particular case. The arguments will derive, in general, from Measure Theory and they will provide the convergence of certain sequences of functions. As an example we mention that the subset of non decreasing functions in the Orlicz space $\mathcal{L}_\phi(0, 1)$ is proximinal in the sense that for each function $f \in \mathcal{L}_\phi(0, 1)$ there is a non decreasing function $g \in \mathcal{L}_\phi(0, 1)$ that minimizes the gap function

$$d(f, \cdot) = \int_0^1 \phi(|f - \cdot|)d\mu$$

over the subset of non decreasing functions in $\mathcal{L}_\phi(0, 1)$.

Another interesting situation is when X is quasi-metric space (see for example [21], where a characterization of best approximation result is proved in this context).

The main purpose of this paper is to give a common thread of compactness in the proofs of proximinality. To achieve our aim we introduce and extend some well known facts of Best Approximation Theory to general topological spaces with some structure. Our description covers and extends some classical situations in normed spaces (see [6]) and in metric spaces. Moreover some recent developments in topological vector spaces are covered through our method.

The present paper is organized as follows. In Section 2 we state the main definitions. We use the following idea: let (X, ρ) be a metric space, $x \in X$ and A a nonempty subset of X. The set of best approximations to x from A is defined by

$$P(x, A) = \{a \in A : \rho(x, a) = \rho(x, A)\},$$

where $\rho(x, A) = \inf\limits_{a \in A} \rho(x, a)$. If $B(x, r)$ denotes the closed ball of center x and radius r, we have

$$P(x, A) = A \cap B(x, \rho(x, A)) = A \cap \left(\bigcap_{r > \rho(x, A)} B(x, r) \right).$$

This description gives the idea to extend some notions of Best Approximation Theory to more general spaces. In order to estimate the gap between points in some space we will introduce the concept of *wrapping* as an increasing family of closed sets with nonempty interior whose union is the whole space. Section 3 is devoted to the application of the framework introduced in the previous section to Metric Spaces, Topological Vector Spaces and Function Spaces. We cover some of the classical situations and some recent descriptions due to different authors. In Section 4 we show that our description gives a common thread of compactness for the proofs of proximinality in some of the most interesting and well known examples. Finally, we also characterize proximinality in terms of countable compactness. The situation described here in order to introduce the notion of wrapping has been applied by other authors not only in Approximation Theory but also in Selection theory (in Michael's approach, see [15]) and in Fixed Point Theory (see [12]).

2. Definitions and Preliminaries Results

Let (X, τ) be a Hausdorff topological space and let us denote by $\mathcal{C}(X)$ the family of closed subsets of X. A *wrapping* for X is a function $\xi : X \times [0, \infty) \longrightarrow \mathcal{C}(X)$ such that, for each $x \in X$, satisfies:

(i) $\xi(x, 0) = \{x\}$,
(ii) for all $r > 0$ there is an open set $\theta(x, r)$ with $x \in \theta(x, r) \subset \xi(x, r)$,
(iii) for all $r, s \geq 0$, if $r \leq s$ then $\xi(x, r) \subseteq \xi(x, s)$,
(iv) for all $s \geq 0$, $\bigcup\limits_{r > s} \xi(x, r) = X$.

The set $\xi(x, r)$ is called the ξ-*ball* of center x and radius r.

If ξ is a function from $X \times [0, \infty)$ into the power set $\mathcal{P}(X)$ that fulfills the axioms (iii) and (iv) of the above definition, then it is called a *pre–wrapping* for X.

Fixed a wrapping ξ for X, A a nonempty subset of X and $x \in X$, we define:

- the radius set of the ξ-balls of center x with nonempty intersection with A

(2.1) $$F_\xi(x, A) = \{r \geq 0 : A \cap \xi(x, r) \neq \varnothing\},$$

- the ξ-*distance* of x to A

(2.2) $$d_\xi(x, A) = \inf F_\xi(x, A),$$

- the set of best ξ-*approximations* to x from A

$$(2.3) \qquad P_\xi(x, A) = A \bigcap \left(\bigcap_{r \in F_\xi(x,A)} \xi(x, r) \right).$$

The subset A will be called ξ-*proximinal* if $P_\xi(x, A)$ is nonempty for all $x \in X$, and it will be called ξ-*Chebyshev* if there is a function $p_\xi(\cdot, A) : X \longrightarrow A$ such that $P_\xi(x, A) = \{p_\xi(x, A)\}$ for all $x \in X$ (i.e. if $P_\xi(x, A)$ is a singleton for all $x \in X$).

By means of the previous definitions we have the following statements:

(i) $P_\xi(\cdot, A)$ is a function from X to $\mathcal{P}(A)$,

(ii) $x \in A$ if and only if $P_\xi(x, A) = \{x\}$,

(iii) the set $F_\xi(x, A)$ is either the interval $(d_\xi(x, A), \infty)$ or $[d_\xi(x, A), \infty)$. Moreover, in the last case we have $P_\xi(x, A) = A \bigcap \xi(x, d_\xi(x, A)) \neq \varnothing$,

(iv) $d_\xi(x, A) = \inf_{a \in A} d_\xi(x, \{a\}) = \inf_{a \in A} \inf\{r \geq 0 : a \in \xi(x, r)\}$,

(v) $\{y \in X : d_\xi(x, \{y\}) \leq s\} = \bigcap_{r > s} \xi(x, r)$, for each $x \in X$,

(vi) $d_\xi(\cdot, A)$ is a function from X to $[0, \infty)$ and $d_\xi(\cdot, A)|_{\overline{A}} = 0$.

Associated to the concept of wrapping, we would like to introduce the following properties: Given ξ a wrapping for X, we say that it satisfies the

Intersection property: If for all $x \in X$ and $s \geq 0$, $\xi(x, s) = \bigcap_{r > s} \xi(x, r)$.

Triangular property: If for all $x \in X$, $r \geq 0$, $y \in \xi(x, r)$ and $s \geq 0$, $\xi(y, s) \subset \xi(x, r + s)$.

A-Adherence property: If for all $x \in X$, if $A \bigcap \left(\bigcap_{r > 0} \xi(x, r) \right) \neq \varnothing$ then $x \in A$.

If ξ satisfies the intersection property it is clear that

$$(2.4) \qquad P_\xi(x, A) = A \cap \xi(x, d_\xi(x, A)) = \{a \in A : d_\xi(x, \{a\}) = d_\xi(x, A)\},$$

and so $P_\xi(x, A) \neq \varnothing$ if and only if $F_\xi(x, A) = [d_\xi(x, A), \infty)$.

In case that ξ satisfies the triangular property, and $x, y, z \in X$, we have that for each $\varepsilon > 0$,

$$y \in \xi(x, d_\xi(x, \{y\}) + \frac{\varepsilon}{2}) \text{ and } z \in \xi(y, d_\xi(y, \{z\}) + \frac{\varepsilon}{2}),$$

so $z \in \xi(x, d_\xi(x, \{y\}) + d_\xi(y, \{z\}) + \varepsilon)$ for all $\varepsilon > 0$. Therefore

$$(2.5) \qquad d_\xi(x, \{z\}) \leq d_\xi(x, \{y\}) + d_\xi(y, \{z\})$$

and hence the function $\rho : X \times X \longrightarrow [0, \infty)$ defined by

$$\rho(x, y) = d_\xi(x, \{y\}) + d_\xi(y, \{x\})$$

is a pseudometric in X. Reciprocally, if (2.5) is fulfilled and ξ satisfies the intersection property, it also satisfies the triangular property.

Without difficulty it can be shown that $\{x\} = \bigcap_{r>0} \xi(x, r)$ for all $x \in X$, if and only if for all $A \subset X$ the wrapping ξ has the A-adherence property.

We can characterize the closed subsets in A by means of the proximinality and adherence properties as follows

Proposition 2.1. *Let (X, τ) be a Hausdorff regular space. A nonempty subset A of X is closed if and only if there exists a wrapping ξ for X with the A-adherence property such that A is ξ-proximinal.*

Note 1. *A Hausdorff space is regular if for each point x and each closed subset A, if $x \notin A$, then there are disjoint open sets U and V such that $x \in U$ and $A \subset V$.*

Proof Let A be a closed subset of X. If $A = X$ we can choose the wrapping defined by $\xi(x, 0) = \{x\}$ and $\xi(x, r) = X$ for $r > 0$. If A is a proper subset of X and $x \in X \setminus A$ there are disjoint open sets U_x, V_x such that $x \in U_x$ and $A \subset V_x$; we define the wrapping ξ by:

$$\xi(x, r) = \begin{cases} \{x\} & \text{if } r = 0, \\ X \setminus V_x & \text{if } 0 < r < 1, \\ X & \text{if } r \geq 1, \end{cases}$$

if $x \notin A$, and $\xi(x, 0) = \{x\}$ and $\xi(x, r) = X$ for $r > 0$ in other case. Then, in case that $x \in A$ we have $P_\xi(x, A) = \{x\}$ and if $x \notin A$ it follows that $P_\xi(x, A) = A$. Consequently A is ξ-proximinal, and it is straightforward to verify that ξ has the A-adherence property.

To show the converse suppose that A is not closed and let $x \in \overline{A} \setminus A$. For every wrapping ξ, since the open sets $\theta(x, r)$ have nonvoid intersection with A, we have that $P_\xi(x, A) = A \bigcap \left(\bigcap_{r>0} \xi(x, r) \right)$. So if A is ξ-proximinal, the wrapping cannot have the A-adherence property. $\square$

Note 2. *The previous result is in the same line as Theorem 6 of* [11].

Let ρ be a metric in a set X. For each $x \in X$, the function $\rho_x : X \longrightarrow [0, \infty)$ defined by $\rho_x(y) = \rho(y, x)$ gives the distance to x. By definition, the metric topology for X is the smallest one containing the sets

$$S(x, r) := \{y \in X : \rho(y, x) < r\} = \rho_x^{-1}([0, r)),$$

for all $x \in X$ and $r \geq 0$. Hence the metric topology for X is the smallest one that makes the functions ρ_x continuous at x. If $A \subset X$, the uniform continuity of the function $\rho(\cdot, A)$ follows from the symmetry and the triangle inequality of the metric ρ. With a similar pattern we can state the following result.

Proposition 2.2. *Let ξ be a wrapping for X and $\varnothing \neq A \subset X$. If ξ satisfies the triangular property and $d_\xi(\cdot, \{x\})$ is continuous at $x \in X$, then $d_\xi(\cdot, A)$ is continuous at x.*

Proof Let $\{x_i\}_{i \in I}$ be a net in X such that $x_i \longrightarrow x$. Given $\varepsilon > 0$, there is a $i_1 \in I$ such that $x_i \in \xi(x, \frac{\varepsilon}{2})$ for $i \geq i_1$. For each $i \in I$, let $a_i \in A \cap \xi(x_i, d_\xi(x_i, A) + \frac{\varepsilon}{2})$. Since ξ satisfies the triangular property, then $a_i \in \xi(x, d_\xi(x_i, A) + \varepsilon)$ for $i \geq i_1$, and hence

$$(2.6) \qquad\qquad d_\xi(x, A) - d_\xi(x_i, A) \leq \varepsilon.$$

To prove the other inequality, first observe that $x \in \xi(x_i, d_\xi(x_i, \{x\}) + \frac{\varepsilon}{4})$ for all $i \in I$ and let $a_\varepsilon \in A \cap \xi(x, d_\xi(x, A) + \frac{\varepsilon}{4})$. Since $a_\varepsilon \in \xi(x_i, d_\xi(x_i, \{x\}) + d_\xi(x, A) + \frac{\varepsilon}{2})$, then $d_\xi(x_i, A) - d_\xi(x, A) \leq d_\xi(x_i, \{x\}) + \frac{\varepsilon}{2}$. Finally, notice that there exists $i_2 \in I$ such that

$$(2.7) \qquad\qquad d_\xi(x_i, A) - d_\xi(x, A) \leq \varepsilon,$$

for all $i \geq i_2$, since $d_\xi(\cdot, \{x\})$ is continuous at x.

If i_0 is an upper bound of $\{i_1, i_2\}$, it follows from (2.6) and (2.7) that

$$|d_\xi(x_i, A) - d_\xi(x, A)| \leq \varepsilon,$$

for all $i \geq i_0$. $\square$

3. Metric Spaces, Topological Vector Spaces and Function Spaces.

3.1. Metric Spaces. Let (X, ρ) be a metric space, and let $\varnothing \neq A \subset X$. In the sequel, the closed ball of center x and radius r will be denoted by $B(x, r)$. The function $\xi : X \times [0, \infty) \longrightarrow \mathcal{C}(X)$ defined by

$$\xi(x, r) = B(x, r)$$

is the "natural" wrapping for X, and it clearly satisfies the intersection and the triangular properties. We recall that

$$P(x, A) = \{a \in A : \rho(x, a) = \rho(x, A)\} = A \cap B(x, \rho(x, A))$$

where $\rho(x, A)$ is the distance of x to A. We can generalize this wrapping by merely introducing a real function f with suitable properties and considering the balls $B(x, f(r))$. In some sense, the best approximation problem with this new kind of wrapping will be show to be equivalent to the standard problem in the space (X, ρ_f), where ρ_f is a metric derived from ρ and f.

More precisely, a function $f : [0, \infty) \longrightarrow [0, \infty)$ is said to be *of type A* (abbr. TA) if

   (i) $f(r) = 0$ if and only if $r = 0$,
   (ii) f is right continuous $(\lim_{r \downarrow s} f(r) = f(s))$,
  (iii) f is superadditive $(f(r) + f(s) \leq f(r + s))$.

Let f be a TA function and $0 \leq r < s$. We have $f(r) < f(r)+f(s-r) \leq f(s)$, thus f is strictly increasing. Moreover, $nf(1) \leq f(n)$ for each nonnegative integer n, so that $\lim_{r \to \infty} f(r) = \infty$.

Let us now give some natural examples of TA functions: The first one is given by $f(r) = r + [r]$ (where $[r]$ denotes the greatest integer smaller or equal to r) is a TA function. On the other hand, if $f : [0, \infty) \longrightarrow [0, \infty)$ is a convex function that vanishes only at 0, then it is an strictly increasing continuous TA function: if $p \in [0, \infty)$, then

(3.8) $f(\lambda p) \leq \lambda f(p) + (1 - \lambda)f(0) = \lambda f(p),$

for all $\lambda \in [0, 1]$. Therefore, if $0 < r \leq s$ and $\lambda \in (0, 1]$, we have

$$\frac{1}{\lambda} f(\lambda(r + s)) \leq f(r + s).$$

Taking $\lambda = \dfrac{s}{r + s}$ and using (3.8), we get

$$f(r) + f(s) \leq \frac{r}{s} f(s) + f(s) \leq f(r + s).$$

Let (X, ρ) be a metric space, and let f be a TA function. By the properties of f, it is clear that the function ξ_f, defined on $X \times [0, \infty)$ by

$$\xi_f(x, r) = B(x, f(r))$$

is a wrapping for X, that satisfies the intersection and the triangular properties. Moreover, we can state the following result.

Proposition 3.1. *Let (X, ρ) be a metric space, A a nonempty subset of X and f a TA function. For every $x \in X$, we have*

$P_{\xi_f}(x, A) = A \cap B(x, f(d_{\xi_f}(x, A)))$ *where* $d_{\xi_f}(x, A) = \inf\{r \geq 0 : f(r) \geq \rho(x, A)\}$.

Proof The first relation follows from the intersection property of the wrapping ξ_f. Clearly,

$$\{r \geq 0 : f(r) > \rho(x, A)\} \subset \{r \geq 0 : A \cap B(x, f(r)) \neq \varnothing\} \subset \{r \geq 0 : f(r) \geq \rho(x, A)\},$$

so that we have an inverted chain of inequalities of infimums of such sets. Now f is right continuous and strictly increasing, so that $\inf\{r \geq 0 : f(r) > \rho(x, A)\} = \inf\{r \geq 0 : f(r) \geq \rho(x, A)\}$ and the second statement of the proposition holds.$\square$

Let f be a TA function, and let the *right inverse* of f be defined by

$$f_{ex}^{-1}(x) = \inf\{r \geq 0 : f(r) \geq x\} = \inf f^{-1}([x, \infty)),$$

for $x \geq 0$. We can prove now the following result.

Proposition 3.2. *Let f_{ex}^{-1} be the right inverse of a TA function f. Then f_{ex}^{-1} is a function from $[0, \infty)$ onto $[0, \infty)$ which is non decreasing, continuous, subadditive ($f_{ex}^{-1}(x + y) \leq f_{ex}^{-1}(x) + f_{ex}^{-1}(y)$) and vanishes only at 0.*

Proof Since $\lim_{r\to\infty} f(r) = \infty$, then $f^{-1}([x,\infty))$ is nonempty (and bounded below) for each $x \geq 0$. Under the hypothesis on f (strictly increasing and right continuous), for each $x \geq 0$ there exists $s \geq 0$ such that $f^{-1}([x,\infty)) = [s,\infty)$, hence $f_{ex}^{-1}(x) = \min f^{-1}([x,\infty))$.

Let $r \geq 0$ and $x = f(r)$. It is clear that $f_{ex}^{-1}(x) = r = f^{-1}(x)$, hence f_{ex}^{-1} is a surjective extension of the inverse function $f^{-1} : f([0,\infty)) \longrightarrow [0,\infty)$. Also, if $0 \leq x \leq y$, then $[y,\infty) \subset [x,\infty)$ and

$$f_{ex}^{-1}(x) = \min f^{-1}([x,\infty)) \leq \min f^{-1}([y,\infty)) = f_{ex}^{-1}(y),$$

so that f_{ex}^{-1} is non decreasing.

Let $x > 0$ and let $a = \lim_{y\downarrow x} f_{ex}^{-1}(y)$ and $b = \lim_{y\uparrow x} f_{ex}^{-1}(y)$. By the surjectivity of f_{ex}^{-1}, it follows that $a = b$. In the same way we get that $0 = f_{ex}^{-1}(0) = \lim_{y\downarrow 0} f_{ex}^{-1}(y)$. Hence f_{ex}^{-1} is continuous in $[0,\infty)$. The condition $f_{ex}^{-1}(x) = 0$ if and only if $x = 0$, is a direct consequence of the right continuity of f.

Finally, let $x, y > 0$. There exist $r, s > 0$ such that

$$f(t) < x \leq f(r) \text{ for } t \in [0,r), \text{ and } f(t) < y \leq f(s) \text{ for } t \in [0,s),$$

so that $x + y \leq f(r) + f(s) \leq f(r+s)$, and $f_{ex}^{-1}(x+y) \leq f_{ex}^{-1}(f(r+s)) = r + s = f_{ex}^{-1}(x) + f_{ex}^{-1}(y)$. Hence f_{ex}^{-1} is a subadditive function.□

Let ρ be a metric for X and let f be a TA function. One can easily verify that the composition

$$\rho_f = f_{ex}^{-1} \circ \rho$$

is a metric in X. In fact, ρ and ρ_f are equivalent metrics. Moreover, if B_σ denotes the closed ball in the metric space (X, σ), we have that

(3.9) $$B_\rho(x, f(r)) = B_{\rho_f}(x, r),$$

which follows from

$$B_\rho(x, f(r)) = \{y \in X : \rho(y,x) \leq f(r)\} \subset \{y \in X : f_{ex}^{-1}(\rho(y,x)) \leq r\} = B_{\rho_f}(x,r),$$

and from

$$B_{\rho_f}(x,r) = \{y \in X : f_{ex}^{-1}(\rho(y,x)) \leq r\} \subset \{y \in X : \rho(y,x) \leq f(r)\} = B_\rho(x, f(r)),$$

(the last inclusion is due to $\rho(x,y) \leq f(f_{ex}^{-1}(\rho(x,y)))$).

We can also establish the following relation

$$
\begin{aligned}
d_{\xi_f}(x, A) &= \inf\{r \geq 0 : f(r) \geq \rho(x, A)\} = f_{ex}^{-1}(\rho(x, A)) \\
&= f_{ex}^{-1}(\inf_{a\in A} \rho(x,a)) = \inf_{a\in A} f_{ex}^{-1}(\rho(x,a)) = \inf_{a\in A} \rho_f(x,a) \\
&= \rho_f(x, A).
\end{aligned}
$$

(3.10)

In view of (3.9) and (3.10), Proposition 3.1 takes the form

Proposition 3.3. *Let (X, ρ) be a metric space, A a nonempty subset of X and f a TA function. For every $x \in X$, we have*

$$P_{\xi_f}(x, A) = A \cap B_{\rho_f}(x, \rho_f(x, A)).$$

Hence the best approximation problem in the metric space (X, ρ) with the wrapping $\xi_f(x, r) = B_\rho(x, f(r))$, is equivalent to the approximation problem in (X, ρ_f) with the usual wrapping $\xi(x, r) = B_{\rho_f}(x, r)$.

3.2. **Topological Vector Spaces.** In [17] the following problem has been considered. Let X be a separated locally convex space and let $f : X \longrightarrow \mathbb{R}$ be a continuous convex function satisfying $f(0) = 0$. If A is a nonempty closed subset of X, the authors define the number

$$f_A(x) = \inf\{f(x - a) : a \in A\},$$

and the so-called *f–projection* set

$$P_{f,A}(x) = \{a \in A : f(x - a) = f_A(x)\}.$$

They define properties of A related to the set valued mapping $P_{f,A}(\cdot)$ and explore several relationships between these properties and the continuity of this mapping. Taking into account that for each $r > 0$ the sub-level set

$$C_r = \{x \in X : f(x) \le r\}$$

is a closed convex absorbing set containing 0 in its interior, is immediate to verify that the function ξ defined by

$$\xi(x, r) = \begin{cases} \{x\} & \text{if } r = 0, \\ x - C_r & \text{if } r > 0, \end{cases}$$

is a wrapping for X that satisfies the intersection property. Of course we have $d_\xi(x, A) = f_A(x)$ and $P_\xi(x, A) = P_{f,A}(x)$, hence this minimization problem can be described and studied with the tools we have introduced in this paper.

Also the following problem can be found in [2]. Let C be a closed bounded convex subset of a Banach space X which has the origin as an interior point and let f_C denote the Minkowski functional with respect to C. Given a nonempty closed bounded subset $A \subset X$ and a point $x \in X$, we consider the minimization problem which consists in proving the existence of a point $a_0 \in A$ such that

$$f_C(a_0 - x) = \inf\{f_C(a - x) : a \in A\}.$$

In this subsection we are going to introduce a wrapping in a topological vector space in order to cover and extend the problem above.

Let (X, τ) be a (Hausdorff) topological vector space, and let C be a convex neighborhood of 0. For all $x \in X$, it is easy to check that

(i) for each $r > 0$, the set $x + rC$ is an convex neighborhood of x contained in the closed convex set $x + r\overline{C}$,

(ii) if $0 \le r \le s$, then $x + r\overline{C} \subset x + s\overline{C}$,

(iii) for each $y \in X$, there exists $t \geq 0$ such that y belongs to $x + t\overline{C}$, and so

$$\bigcup_{r \geq 0} (x + r\overline{C}) = X.$$

In the light of the above properties, we can state the following result (we remind the reader that a halfline in X is a set of the form $\{x + ty : t \geq 0\}$, where $x, y \in X$ and $y \neq 0$).

Proposition 3.4. *Let (X, τ) be a Hausdorff topological vector space with a convex neighborhood of the origin, C, whose closure does not contain halflines. The function $\xi_C : X \times [0, \infty) \longrightarrow \mathcal{C}(X)$ defined by*

$$\xi_C(x, r) = x + r\overline{C}$$

is a wrapping for X that satisfies the intersection and the triangular properties.

Proof We will prove the following two statements:

(1): For each $x \in X$, $r \geq 0$, and for each $y \in x + r\overline{C}$ and $s \geq 0$,

$$y + s\overline{C} \subset x + (r + s)\overline{C}.$$

(2): For each $x \in X$ and $s \geq 0$,

$$x + s\overline{C} = \bigcap_{r > s} (x + r\overline{C}).$$

As the first step for the proof of (1) we shall prove that for all $r, s \geq 0$,

(3.11) $$r\overline{C} + s\overline{C} = (r + s)\overline{C}.$$

Let $x, y \in \overline{C}$ and let r, s be positive. By convexity of $\overline{C}$, it follows that

$$\frac{1}{r + s}(rx + sy) = \frac{r}{r + s}x + \frac{s}{r + s}y \in \overline{C},$$

which implies that $r\overline{C} + s\overline{C} \subset (r + s)\overline{C}$. The reverse inclusion is trivial.

Now, let $r, s > 0$ and let $y \in x + r\overline{C}$. If z belongs to $y + s\overline{C}$, then, using (3.11), we have $z \in x + r\overline{C} + s\overline{C} = x + (r + s)\overline{C}$. This concludes the proof of (1).

The proof of (2) goes as follows. Since trivially

$$x + s\overline{C} \subset \bigcap_{r > s} (x + r\overline{C}),$$

we have to prove the reverse inclusion. First consider $s > 0$, and let $y \in \bigcap_{r > s} (x + r\overline{C})$. If $\{r_n\}$ is a sequence of numbers greater than s and converging to s, we have that

$$\overline{C} \ni \frac{1}{r_n}(y - x) \to \frac{1}{s}(y - x),$$

and hence $y \in x + s\overline{C}$. This implies

$$\bigcap_{r > s} (x + r\overline{C}) \subset x + s\overline{C}$$

for each $s > 0$. Finally, let $y \in \bigcap_{r>0} (x + r\overline{C})$. By

$$\frac{1}{r}(y - x) \in \overline{C}$$

for each $r > 0$, if $y \neq x$, then the halfline $\{t(y - x) : t \geq 0\}$ is contained in $\overline{C}$, which is absurd. So

$$\bigcap_{r>0}(x + r\overline{C}) \subset \{x\},$$

and this concludes the proof.$\square$

Observe that an open set which does not contain halflines needs not to be bounded. Consider, for example, the set

$$C = \{\{x_n\} \in l_1 : |x_n| < n \text{ for each } n\},$$

where l_1 denotes the space of sequences $\{x_n\}_{n=1}^{\infty}$ of real numbers which are absolutely summable. It is clear that C is a convex set containing 0. Moreover, let $\mathbf{x} = \{x_n\} \in C$ and define

$$\varepsilon_{\mathbf{x}} = \min_{n}\{n - |x_n|\}.$$

If $\mathbf{y} \in \{\mathbf{z} \in l_1 : \|\mathbf{z} - \mathbf{x}\|_1 < \varepsilon_{\mathbf{x}}\}$, then, for each n,

$$|y_n| - |x_n| \leq |y_n - x_n| \leq \|\mathbf{y} - \mathbf{x}\|_1 < \varepsilon_{\mathbf{x}} \leq n - |x_n|,$$

and therefore C is open. It is straightforward to verify that the convex set

$$\overline{C} = \{\{x_n\} \in l_1 : |x_n| \leq n \text{ for each } n\}$$

does not contain halflines and, however, it is unbounded (if $\mathbf{x}_n = \{(n-1)\delta_{mn}\}_{m=1}^{\infty}$, then $\mathbf{x}_n \in C$ and $\sup_{n} \|\mathbf{x}_n\|_1 = \infty$).

Let C be a convex subset of a Hausdorff Topological Vector Space, with 0 as an interior point and which contains no halflines. The Minkowski functional of C, defined by

$$f_C(x) = \inf\{r > 0 : \frac{x}{r} \in C\},$$

is a non-negative, positive homogeneous, convex, continuous and subadditive function, that vanishes only at 0. Moreover, the convex sets

$$C_1 = \{x \in X : f_C(x) < 1\} \text{ and } C_2 = \{x \in X : f_C(x) \leq 1\}$$

(open and closed, respectively) satisfy

$$C_1 \subset C \subset \overline{C} \subset C_2.$$

The Minkowski functional provides the following characterization of the set of best ξ_C-approximations.

Proposition 3.5. *Let (X, τ) be a Hausdorff topological vector space with a convex neighborhood of the origin, C, whose closure does not contain halflines, and let A be a nonempty subset of X. For each $x \in X$, we have*

$$(3.12) \qquad \begin{aligned} P_{\xi_C}(x, A) &= A \cap \left(x + d_{\xi_C}(x, A)(\overline{C} \setminus C)\right) \\ &= \{a \in A : f_{\overline{C}}(a - x) = d_{\xi_C}(x, A)\}, \end{aligned}$$

where $d_{\xi_C}(x, A) = \inf\limits_{a \in A} f_{\overline{C}}(a - x)$.

Proof It follows from (2.4) that $P_{\xi_C}(x, A) = A \cap \left(x + d_{\xi_C}(x, A)\overline{C}\right)$.

We first consider the case $d_{\xi_C}(x, A) > 0$. For $a \in P_{\xi_C}(x, A)$, we will show that $a \notin x + d_{\xi_C}(x, A)C$. Suppose this is not so. If $\{t_n\}$ is a sequence of positive numbers such that $t_n \to 0$, then $(1 + t_n)a - t_n x \to a$, and hence there exists $t_m > 0$ and $c \in C$ such that $(1 + t_m)a - t_m x = x + d_{\xi_C}(x, A)c$, so that

$$a = x + \frac{d_{\xi_C}(x, A)}{1 + t_m}c \in x + \frac{d_{\xi_C}(x, A)}{1 + t_m}\overline{C}.$$

Hence we have $d_{\xi_C}(x, A) \leq \dfrac{d_{\xi_C}(x, A)}{1 + t_m}$, an absurd. This shows that

$$A \cap \left(x + d_{\xi_C}(x, A)\overline{C}\right) \subset A \cap \left(x + d_{\xi_C}(x, A)(\overline{C} \setminus C)\right).$$

The reverse inclusion and the case $d_{\xi_C}(x, A) = 0$ are obvious. Thus we have proved that

$$P_{\xi_C}(x, A) = A \cap \left(x + d_{\xi_C}(x, A)(\overline{C} \setminus C)\right).$$

The proof ends by noting that

$$d_{\xi_C}(x, \{a\}) = \inf\{r \geq 0 : a \in x + r\overline{C}\} = \inf\left\{r \geq 0 : \frac{a - x}{r} \in \overline{C}\right\} = f_{\overline{C}}(a - x).$$

$\square$

We have also the following characterization of the ξ_C-distance of x to A.

Proposition 3.6. *If*

$$\varnothing \neq A \cap \left(x + r\overline{C}\right) \subset x + r(\overline{C} \setminus C)$$

for some $r \geq 0$, then $r = d_{\xi_C}(x, A)$ and consequently $P_{\xi_C}(x, A)$ is nonempty.

Proof It suffices to consider $r > 0$. First observe that $d = d_{\xi_C}(x, A) \leq r$. Supposing $d < r$, we get

$$\varnothing \neq A \cap \left(x + \frac{d + r}{2}\overline{C}\right) \subset A \cap \left(x + r\overline{C}\right) \subset x + r(\overline{C} \setminus C).$$

Thus there exist $c_1 \in \overline{C}$ and $c_2 \in \overline{C} \setminus C$, such that

$$\frac{d + r}{2}c_1 = rc_2.$$

Let $t = \frac{d+r}{2r} \in (0,1)$. Since $c_1 + \frac{1-t}{-t}C$ is a neighborhood of c_1, there exists $c_3 \in C \cap \left(c_1 + \frac{1-t}{-t}C \right)$. Therefore

$$c_2 = tc_1 = tc_3 + t(c_1 - c_3) \in tC + (1-t)C \subset C.$$

This contradicts that $c_2 \notin C$, and we conclude $d = r$. $\square$

The set of best ξ_C-approximations to x from A inherits the invariance properties of the space X, and it is easy to
verify that for $s > 0$, $t \geq 0$ and $y \in X$, we have

$$P_{\xi_{(sC)}}(tx - y, tA - y) = tP_{\xi_C}(x, A) - y.$$

A convex subset U of a topological vector space will be called *strictly convex* provided that the relations $x, y \in \overline{U} \setminus int(U)$ and $x \neq y$ imply $\frac{x+y}{2} \in int(U)$. Note that if U is a convex neighborhood of the origin, then $int(\overline{U}) = U$ and $\overline{U}$ is strictly convex if U is so. It is possible to establish uniqueness as a consequence of strict convexity of the set that generates the wrapping.(For normed linear spaces see Lemma 3.2. of [19])

Proposition 3.7. *Let (X, τ) be a Hausdorff topological vector space with a convex neighborhood of the origin, C, whose closure does not contain halflines, and let A be a nonempty, convex and ξ_C-proximinal subset of X. If C is strictly convex, then A is ξ_C-Chebyshev. Moreover, if for each convex and ξ_C-proximinal subset A of X we have that A is ξ_C-Chebyshev, then C is strictly convex.*

Proof First consider that C is strictly convex. Let $a_1, a_2 \in P_{\xi_C}(x, A)$. In case $a_1 \neq a_2$, we have $d = d_{\xi_C}(x, A) > 0$, and so $\frac{a_i - x}{d} \in \overline{C} \setminus C$. By the assumption on C, we have

$$\frac{\frac{a_1 + a_2}{2} - x}{d} \in C.$$

Since A is convex, this means that $\frac{a_1 + a_2}{2} \in A \cap (x + dC)$, which contradicts (3.12). This proves the first claim.

Now suppose that C is not strictly convex. Then there exist distinct points $a_1, a_2 \in \overline{C} \setminus C$ such that $\frac{a_1 + a_2}{2} \in \overline{C} \setminus C$. Let the mapping g from $[0,1]$ into X be defined by

$$g(t) = (1-t)a_1 + ta_2.$$

Since g is continuous and $[0,1]$ is compact, the convex set $A = g([0,1])$ is compact. From (3.12) and the fact that $f_{\overline{C}}(\cdot - x)$ is continuous, it follows that A is ξ_C-proximinal. Moreover, since $f_{\overline{C}}(a) = 1 = d_{\xi_C}(0, A)$ for each $a \in A$ (this is not difficult to check), then $A = P_{\xi_C}(0, A)$. Hence A is not ξ_C-Chebyshev. This proves the second claim.$\square$

For $a \in A$, S_a will denote the set of all points in X having a as a best ξ_C-approximation, i.e.

$$S_a = \{x \in X : a \in P_{\xi_C}(x, A)\} = \{x \in X : f_{\overline{C}}(a - x) = d_{\xi_C}(x, A)\}.$$

Clearly S_a is nonempty ($a \in S_a$). Let $\{x_d\}_{d \in D}$ be a net in S_a which converges to x. Then $f_{\overline{C}}(a - x_d) = d_{\xi_C}(x_d, A)$. Since $f_{\overline{C}}(a - \cdot)$ and $d_{\xi_C}(\cdot, A)$ are continuous, we have

$$f_{\overline{C}}(a - x) = d_{\xi_C}(x, A),$$

hence $x \in S_a$ and S_a is closed.

It is interesting to note that if we assume that A is convex we can prove a geometrical condition of the sets S_a. This condition has been established, for normed linear spaces, in [19].

Proposition 3.8. *Let (X, τ) be a Hausdorff topological vector space with a convex neighborhood of the origin, C, whose closure does not contain halflines, and let A be a nonempty and convex subset of X. Then for each $a \in A$, the set S_a is a cone with vertex a.*

Proof If $x \in S_a$, and we denote $d = d_{\xi_C}(x, A)$, then there exists $c_1 \in \overline{C} \setminus C$ such that $a = x + dc_1$. We shall prove that for each $t \geq 0$

(3.13) $\varnothing \neq A \cap \left((1 - t)a + tx + td\overline{C}\right) \subset (1 - t)a + tx + td(\overline{C} \setminus C)$.

Suppose this is not so. Then for some $b \in A$ and some $c_2 \in C$, we would have

$$b = (1-t)a + tx + tdc_2 = (1-t)x + (1-t)dc_1 + tx + tdc_2 = x + d((1-t)c_1 + tc_2).$$

In case that $t \in (0, 1)$, by the fact that $(1 - t)c_1 + tc_2$ belongs to C, we have $b \in x + dC$, an absurd. If $t \in (1, \infty)$, using that A is convex, we get $\frac{1}{t}b + (1 - \frac{1}{t})a = \frac{1}{t}x + d((\frac{1}{t} - 1)c_1 + c_2) + (1 - \frac{1}{t})x + (1 - \frac{1}{t})dc_1 = x + dc_2 \in A$, again a contradiction.

From (3.13) and proposition 3.6 , we deduce

$$d_{\xi_C}((1 - t)a + tx, A) = td_{\xi_C}(x, A),$$

for all non-negative t. Therefore

$$f_{\overline{C}}(a - ((1 - t)a + tx)) = tf_{\overline{C}}(a - x) = td_{\xi_C}(x, A) = d_{\xi_C}((1 - t)a + tx, A).$$

Whence $(1 - t)a + tx$ belongs to S_a. $\square$

Without the hypothesis of convexity on A, we can establish the following corollaries of the above result.

Corollary 3.9. *Let (X, τ) be a Hausdorff topological vector space with a convex neighborhood of the origin, C, whose closure does not contain halflines, and let $\varnothing \neq A \subset X$. If $a \in A$ and $x \in S_a$, then $(1 - t)a + tx \in S_a$ for each $t \in [0, 1]$.*

Corollary 3.10. *Let (X, τ) be a Hausdorff topological vector space with a convex neighborhood of the origin, C, whose closure does not contain halflines, and let A be a nonempty and ξ_C-Chebyshev subset of X. Then*

$$p_{\xi_C}((1 - t)p_{\xi_C}(x, A) + tx, A) = p_{\xi_C}(x, A),$$

for each $x \in X$ and $t \in [0, 1]$.

The previous two statements are well known in the context of normed linear spaces (see [24]).

3.3. **Function Spaces.** We recall that a convex function ϕ from $[0, \infty)$ into $[0, \infty)$ is said to be Δ_2-convex at 0 if there exists $K > 0$ such that

$$(3.14) \qquad \phi(2x) \leq K \, \phi(x),$$

for each $x \geq 0$.

If $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a Δ_2-convex function at 0, with $\phi(0) = 0$ and $\phi \not\equiv 0$ then it follows rather easily that

    (a) For all $x > 0$, $\phi(x) > 0$,
    (b) ϕ is superadditive ,
    (c) ϕ is strictly increasing,
    (d) There exists $M \geq 1$ such that $\phi(x+y) \leq M(\phi(x)+\phi(y))$ for all $x, y \geq 0$.

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, $\phi : [0, \infty) \longrightarrow [0, \infty)$ a Δ_2-convex function at 0 and let $\mathcal{L}_\phi(\Omega) := \mathcal{L}_\phi(\Omega, \mathcal{A}, \mu)$ denote the class of the μ-equivalent measurable functions, $f : \Omega \longrightarrow \overline{\mathbb{R}}$, such that

$$\int_\Omega \phi(|f|)d\mu < \infty,$$

where we assume that $f = g$ if $\mu\{\omega \in \Omega : f(\omega) \neq g(\omega)\} = 0$. It is well known that $\mathcal{L}_\phi(\Omega)$ is a real linear space.

For $r \geq 0$, $f \in \mathcal{L}_\phi(\Omega)$, we define the sets of functions,

$$\theta(f, r) = \left\{ g \in \mathcal{L}_\phi(\Omega) : \int_\Omega \phi(|f - g|)d\mu < r \right\}$$

and

$$\xi(f, r) = \left\{ g \in \mathcal{L}_\phi(\Omega) : \int_\Omega \phi(|f - g|)d\mu \leq r \right\}.$$

In order to define a topology in $\mathcal{L}_\phi(\Omega)$ we consider the family τ_ϕ of subsets of $\mathcal{L}_\phi(\Omega)$,

$$(3.15) \qquad \tau_\phi = \{O \subset \mathcal{L}_\phi(\Omega) : \forall f \in O, \ \exists r > 0 \text{ such that } \theta(f, r) \subset O\}$$

Proposition 3.11. *The following conditions hold true*
    (a) *$(\mathcal{L}_\phi(\Omega), \tau_\phi)$ is a Hausdorff topological space.*
    (b) *$\{\theta(f, r) : f \in \mathcal{L}_\phi(\Omega), \ r > 0\}$ is a basis for τ_ϕ.*
    (c) *For all $f \in \mathcal{L}_\phi(\Omega)$ and $r \geq 0$, $\xi(f, r)$ is τ_ϕ–closed.*

Proof
(a) By definition (3.15), $\varnothing$ and $\mathcal{L}_\phi(\Omega)$ belong to τ_ϕ. Moreover, if $\{O_\lambda\}_{\lambda \in \Lambda}$ is an arbitrary family of sets in τ_ϕ, is immediate that $\cup_{\lambda \in \Lambda} O_\lambda \in \tau_\phi$. Finally, let $O_i, i = 1, 2, \cdots, n$, be a finite family of sets in τ_ϕ. For all $f \in \cap_{i=1}^n O_i$, there are $r_i > 0$ such that $\theta(f, r_i) \subset O_i, i = 1, 2, \cdots, n$. Taking $r_0 = \min_{1 \leq i \leq n} r_i$, we have

$$\theta(f, r_0) = \bigcap_{i=1}^n \theta(f, r_i) \subset \bigcap_{i=1}^n O_i,$$

and therefore $\cap_{i=1}^n O_i \in \tau_\phi$. This proves that τ_ϕ is a topology in $\mathcal{L}_\phi(\Omega)$.

Now let $f, g \in \mathcal{L}_\phi(\Omega)$, $f \neq g$ and consider

$$\alpha = \int_\Omega \phi(|f - g|)d\mu > 0.$$

In order to prove that $(\mathcal{L}_\phi(\Omega), \tau_\phi)$ is a Hausdorff topological space, we will show that

(3.16) $$\theta\left(f, \alpha/(2M)\right) \cap \theta\left(g, \alpha/(2M)\right) = \varnothing.$$

If (3.16) is false, then there exists $h \in \mathcal{L}_\phi(\Omega)$, such that $h \in \theta\left(f, \alpha/(2M)\right) \cap \theta\left(f, \alpha/(2M)\right)$. But this is not possible since

$$
\begin{aligned}
\int_\Omega \phi(|f - g|)d\mu &\leq \int_\Omega \phi(|f - h| + |g - h|)d\mu \\
&\leq M\left(\int_\Omega \phi(|f - h|)d\mu + \int_\Omega \phi(|g - h|)d\mu\right) < \alpha.
\end{aligned}
$$

(b) By definition of τ_ϕ it suffices to show that $\theta(f, r)$ belongs to τ_ϕ. We will prove that for all $g \in \theta(f, r)$ there exists $s > 0$ such that $\theta(g, s) \subset \theta(f, r)$. Suppose the contrary. Then there exists a sequence $\{g_n\}$ in $\mathcal{L}_\phi(\Omega)$, such that

$$\int_\Omega \phi(|g - g_n)|)d\mu < \frac{1}{2^n} \quad \text{and} \quad \int_\Omega \phi(|f - g_n|)d\mu \geq r.$$

By the Jensen inequality

$$\phi\left(\frac{1}{\mu(\Omega)}\int_\Omega |g - g_n|d\mu\right) \leq \frac{1}{\mu(\Omega)}\int_\Omega \phi(|g - g_n|)d\mu,$$

we obtain

$$\int_\Omega |g - g_n|d\mu \leq \mu(\Omega)\phi^{-1}\left(\frac{1}{2^n\mu(\Omega)}\right),$$

and therefore $\|g - g_n\|_{\mathcal{L}_1(\Omega)} \to 0$. Then there exists a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \to g$, μ-almost everywhere in Ω. Since ϕ is continuous, then $\phi(|f - g_{n_k}|) \to \phi(|f - g|)$, μ-a.e. On the other hand,

$$\phi(|f - g_{n_k}|) \leq M\left(\phi(|f - g|) + \phi(|g - g_{n_k}|)\right) \leq M\left(\phi(|f - g|) + h\right),$$

where $h = \sum_{k=1}^\infty \phi(|g - g_{n_k}|)$. Since, $\phi(|g - g_{n_k}|) \geq 0$ on Ω, then

$$\int_\Omega h d\mu = \sum_{k=1}^\infty \int_\Omega \phi(|g - g_{n_k}|)d\mu < \sum_{k=1}^\infty \frac{1}{2^{n_k}} < \infty$$

and therefore $M\left(\phi(|f - g|) + h\right) \in \mathcal{L}_1(\Omega)$. Finally, applying the Lebesgue Dominated Convergence Theorem, we get

$$\int_\Omega \phi(|f - g|)d\mu = \lim_{k \to \infty}\int_\Omega \phi(|f - g_{n_k}|)d\mu \geq r,$$

and we obtain a contradiction.

(c) If we consider the complement of $\xi(f, r)$, the proof follows the same pattern of (b). $\square$

Then we easily deduce the following

Corollary 3.12. *The family* $\{\xi(f,r) : f \in \mathcal{L}_\phi(\Omega), \ r \geq 0\}$ *is a wrapping for* $(\mathcal{L}_\phi(\Omega), \tau_\phi)$.

4. ξ-Proximinality

In this section we prove a simple yet general proximinality result. It is general because it includes, as special cases, some of the most interesting and well known examples of proximinality. Is simple because the proof requires nothing but the definition of countable compactness.

We recall that a Hausdorff topological space is called countably compact if every countable open covering has a finite subcovering. Countable compactness admits a Heine-Borel type argument: a Hausdorff topological space is countably compact if and only if every family of closed subsets having the finite intersection property also has the countable intersection property. As a consequence, each descending sequence of nonempty closed subsets has nonempty intersection. Finally, let us remember that a space is countably compact if and only if every sequence of points of the space has an accumulation point.

The following result generalizes a result by Singer ([24], p. 383).

Theorem 4.1. *Let* ξ *be a pre–wrapping for* (X, τ) *and let* A *be a nonempty subset of* X. *If* τ' *is a Hausdorff topology in* X *such that for each* $x \in X$ *there exists a non increasing sequence* $\{\varepsilon_n\}$ *of positive numbers tending to* 0 *such that*

(i) $A \cap \xi(x, d_\xi(x, A) + \varepsilon_1)$ *is* τ'-*countably compact,*

(ii) $A \cap \xi(x, d_\xi(x, A) + \varepsilon_n)$ *is* τ'-*closed for* $n > 1$,

then A *is* ξ-*proximinal.*

Proof If for each $x \in X$ we define, for $n \geq 1$,

$$A_n = A \cap \xi(x, d_\xi(x, A) + \varepsilon_n),$$

then $\{A_{n+1}\}$ is a non increasing sequence of nonempty and τ'-closed subsets of A_1. Since A_1 is τ'-countably compact,

$$(4.17) \qquad \varnothing \neq \bigcap_{n \geq 1} A_{n+1} = A \cap \bigcap \{\xi(x, r) : r > d_\xi(x, A)\}.$$

If we assume $F_\xi(x, A) = (d_\xi(x, A), \infty)$, then by 4.17 we have that $P_\xi(x, A)$ is nonempty. The same conclusion follows trivially in case $F_\xi(x, A) = [d_\xi(x, A), \infty)$. Thus A is ξ-proximinal. $\square$

It may be surprising that some of the most famous existence results in normed linear spaces and also results in function spaces that are not normed can be obtained as consequences of the result above. The following list of examples is intended to be a representative sampling of this fact.

A: Nonempty closed subsets of finite dimensional subspaces of normed linear spaces are proximinal.

We consider the "natural" wrapping for the normed linear space, i. e. $\xi(x, r) = B(x, r)$, where $B(x, r)$ stands for the closed ball of center x and radius r. If A is a closed subset of a finite dimensional subspace, then the sets $A \cap B(x, r)$ are bounded closed subsets contained in a finite dimensional space, hence they are compact.

This example is emblematic because it gives an affirmative answer, with an elegant formulation, to the problem that gave rise to Best Approximation Theory, namely, the possibility on finding, within the algebraic polynomials of degree least or equal to a fixed n, the nearest one (in the sense of uniform norm) to some continuous fixed function.

This result was proved by F. Riesz [20] in 1918, although the corresponding result in the setting of polynomial approximation was proved by Chebyshev in 1859 (see [3], [4] and [25]).

Note 3. *For the following example, we recall first that $\sigma(X, X^*)$ stands for the weak topology on X, i. e., the topology defined by the family of seminorms $\{p_{x^*} : x^* \in X^*\}$, where $p_{x^*}(x) = |x^*(x)|$ for each $x \in X$. On the other hand, $\sigma(X^*, X)$ stands for the weak star topology on X^*, i. e., the topology defined by the family of seminorms $\{p_x : x \in X\}$, where $p_x(x^*) = |x^*(x)|$ for each $x^* \in X^*$.*

    B: Every nonempty $\sigma(X, X^*)$–closed subset of a reflexive Banach space X is proximinal.

We consider again the natural wrapping for X, i. e. $\xi(x, r)$ denotes the closed ball of center x and radius r. If X is reflexive (i.e., the natural embedding of X into its double dual, X^{**}, is surjective), then the unit ball $B(0, 1)$ is $\sigma(X, X^*)$–compact (in fact, this is a characterization of reflexivity, see [9]). The functions f_x and g_α (with $x \in X$ and $\alpha \neq 0$), defined by

$$f_x(y) = x + y \qquad \text{and} \qquad g_\alpha(y) = \alpha y \qquad \text{for all } y \in X$$

are homeomorphisms in $(X, \sigma(X, X^*))$. Then, as the continuous image of a compact set is compact, we have that

$$B(x, r) = x + rB(0, 1) = f_x(g_r(B(0, 1)))$$

is $\sigma(X, X^*)$-compact for all $x \in X$ and all $r > 0$. So if A is a nonempty and $\sigma(X, X^*)$–closed subset of a reflexive space, then the sets

$$A \cap B(x, r)$$

are weakly compact. Thus, A is ξ-proximinal.

Mazur proved (see [14]) that the norm closure of a convex subset equals its weak closure. Then the nonempty closed convex subsets of a reflexive Banach space are proximinal. This appears firstly in a paper of Day (see [5]) in 1941. In fact, he gives this result for Banach spaces with a weakly compact unit ball. Day used the results of Milman and Alaoglu and Birkhoff which gave the same result for uniformly convex spaces. By other way, Smulian (see [23]) gave a characterization of the reflexivity of Banach spaces in terms that every descending sequence of nonempty closed, bounded and convex subsets

had nonvoid intersection. The previous example can be deduced from this characterization.

C: Nonempty and $\sigma(X^*, X)$-closed subsets of the dual space X^* of a normed space X are proximinal.

For $x^* \in X^*$ and $r \geq 0$, the closed balls (in the usual norm of the dual spaces) of center x^* and radius r are used to define the wrapping. More precisely,

$$\xi(x^*, r) = B^*(x^*, r) = \{y^* \in X^* : \|y^* - x^*\| \leq r\}$$

where $\|x^*\| = \sup\{|x^*(x)| : x \in B(0, 1)\}$.

The Alaoglu theorem states that $B^*(0^*, 1)$ is $\sigma(X^*, X)$-compact. The balls $B^*(x^*, r)$ are then weakly star compact. If A is a nonempty and $\sigma(X^*, X)$-closed subset of X^*, then the sets

$$A \cap B^*(x^*, r)$$

are $\sigma(X^*, X)$-compact. Thus A is ξ-proximinal.

The previous example, for the case of linear subspaces, appears firstly in a paper of Hirschfeld ([10], 1958) with a wrong proof. Later, in 1960, it appeared in a paper of Phelps (see [18])

D: The subset of non decreasing functions in $\mathcal{L}_\infty(0, 1)$ is proximinal.

Let μ be the Lebesgue measure on the interval $(0, 1)$. Recall that an extended real valued Lebesgue measurable function f on $(0, 1)$ is said to be essentially bounded if there exists some real number $a \geq 0$ such that $\mu(\{x \in (0, 1) : |f(x)| > a\}) = 0$. If f is essentially bounded then the essential supremum of f is defined by

$$\|f\|_\infty = \inf\{a \geq 0 : \mu(\{x \in (0, 1) : |f(x)| > a\}) = 0\}.$$

Let $\mathcal{L}_\infty(0, 1)$ denote the set of all essentially bounded Lebesgue measurable functions on $(0, 1)$, two functions being identified if they differ only on a set of measure zero, and let $A \subset \mathcal{L}_\infty(0, 1)$ be the subset of non decreasing functions from $(0, 1)$ into $\overline{\mathbb{R}}$. Under pointwise linear operations, $(\mathcal{L}_\infty(0, 1), \|\cdot\|_\infty)$ is a real Banach space, and each equivalence class in $\mathcal{L}_\infty(0, 1)$ contains a bounded function.

It is clear that the function ξ defined by

$$\xi(f, r) = B(f, r) = \{g \in \mathcal{L}_\infty(0, 1) : \|g - f\|_\infty \leq r\}$$

where $f \in \mathcal{L}_\infty(0, 1)$ and $r \geq 0$, is a wrapping for $(\mathcal{L}_\infty(0, 1), \|\cdot\|_\infty)$.

Using a bounded function as a representative of each equivalence class $f \in \mathcal{L}_\infty(0, 1)$ we have, for $r > 0$,

$$(4.18) \qquad\qquad B(0, r) \subset \prod_{x \in (0,1)} [-r, r].$$

With the aid of Tychonoff theorem we can state that $\prod_{x \in (0,1)} [-r, r]$ is compact in the cartesian product topology and without effort we can prove that the nonempty set $A \cap B(f, \|f\|_\infty)$ is closed in the product topology. The remainder

of the proof is therefore devoted to showing that the set $A \cap B(f, \|f\|_\infty)$ is compact in the product topology. But, taking into account the previous comments, this a direct consequence of (4.18) and the fact that $B(f, \|f\|_\infty) \subset B(0, 2\|f\|_\infty)$.

> **E:** The subset of non decreasing functions in $\mathcal{L}_\phi(0,1)$ is ϕ-proximinal, i.e. for each function $f \in \mathcal{L}_\phi(0,1)$ there is a non decreasing function $g \in \mathcal{L}_\phi(0,1)$ such that
>
> $$\int_0^1 \phi(|f-g|)d\mu \leq \int_0^1 \phi(|f-h|)d\mu$$
>
> for each non decreasing function $h \in \mathcal{L}_\phi(0,1)$.

Let us consider the wrapping ξ described in Subsection 3.3. Then, for a fixed $f \in \mathcal{L}_\phi(0,1)$ and for $r \geq 0$, we have $\xi(f,r) = \left\{ g \in \mathcal{L}_\phi(0,1) : \int_0^1 \phi(|f-g|)d\mu \leq r \right\}$. We shall denote, for shortness, $d = d_\xi(f,A)$ and $A_\varepsilon = A \cap \xi(f, d+\varepsilon)$, where ε is any positive number. We will show that A is ξ-proximinal by proving that for all $\varepsilon > 0$, every sequence in A_ε has an accumulation point in the topology τ_ϕ (hence A_ε is τ_ϕ-countably compact), and that A_ε is τ_ϕ-closed.

First consider a sequence $\{g_n\}$ in A_ε. Then using the Jensen inequality we get

$$\int_0^1 |f-g_n|d\mu \leq \phi^{-1}\left(\int_0^1 \phi(|f-g_n|)d\mu \right),$$

and therefore $\|g_n\|_{L_1} \leq K \|f\|_{L_1} + \phi^{-1}(d+\varepsilon)$. By the previous inequality it is straightforward to verify that the functions g_n are uniformly bounded in each closed subinterval $[a,b] \subset (0,1)$. Applying the Helly theorem we get a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \to g$, μ-almost everywhere in $(0,1)$. Since ϕ is continuous, then $\phi(|f-g_{n_k}|) \to \phi(|f-g|)$, μ-a.e. The function g is non decreasing and by Fatou Lemma we have

$$(4.19) \qquad \int_0^1 \phi(|f-g|)d\mu \leq d+\varepsilon.$$

On the other hand, by (4.19) and the fact that $\phi(|g|) \leq M\left(\phi(|f-g|) + \phi(|f|)\right)$, we can assure that $g \in \mathcal{L}_\phi(0,1)$ and therefore $g \in A_\varepsilon$.

Since $|g - g_{n_k}| \leq |f-g| + |f-g_{n_k}|$, then

$$\phi(|g-g_{n_k}|) \leq M\left(\phi(|f-g|) + \phi(|f-g_{n_k}|)\right) \in \mathcal{L}_1(0,1).$$

Applying the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{k\to\infty} \int_0^1 \phi(|g-g_{n_k}|)d\mu = 0.$$

Thus g is an accumulation point (in the τ_ϕ topology) of $\{g_n\}$.

The sets A_ε are τ_ϕ-closed since they are τ_ϕ-countably compact and τ_ϕ is first countable.

To close this section, we characterize, for proper closed subsets of regular spaces, the proximinality property in terms of countable compactness. This result is in the same spirit as Theorem 5 of [11]

Proposition 4.2. *Let (X, τ) be a Hausdorff regular space. A proper and closed subset A of X is countably compact if and only if it is ξ-proximinal for any wrapping ξ.*

Proof Let us consider $x \in X \setminus A$. The regularity of X implies that there are open sets O_x, O_A such that $x \in O_x$, $A \subset O_A$ and $O_x \cap O_A = \varnothing$. Hence x is an interior point of the closed set $U_A = X \setminus O_A$. Now suppose that A is not countably compact and let $\{F_n\}$ be a sequence of nonempty and relative closed subsets of A such that $F_{n+1} \subset F_n$ and $\cap_{n=1}^{\infty} F_n = \varnothing$. Since A is closed, the relative closed subsets F_n are closed.

For $0 < r \leq 1$, let n_r denotes the integer such that

$$\frac{1}{2^{n_r}} < r \leq \frac{1}{2^{n_r - 1}}.$$

We define the wrapping ξ in X by

$$\xi(x, r) = \begin{cases} \{x\} & \text{if } r = 0, \\ U_A \cup F_{n_r} & \text{if } 0 < r \leq 1, \\ X & \text{if } r > 1, \end{cases}$$

and, for $y \neq x$, $\xi(y, 0) = \{y\}$ and $\xi(y, r) = X$ for $r > 0$. Thus, we have

$$P_\xi(x, A) = A \cap \left(\bigcap_{n=1}^{\infty} (U_A \cup F_n) \right) = \varnothing.$$

This implies that A is not proximinal with respect to the wrapping described above. $\square$

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