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# Two transfinite chains of separation conditions between $T_1$ and $T_2$

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Abstract. Two new families of separation conditions have arisen in the study of the impact that the algebraic properties of topological algebras have on the topologies that may occur on their underlying spaces. We describe the relative strengths of these families of separation conditions for general spaces.

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## 1. Introduction

The separation conditions, or axioms, $T_0$, $T_1$ and $T_2$ are very well known, as is the fact that the implications

$$T_2 \Longrightarrow T_1 \Longrightarrow T_0$$

hold for any topological space. Another important separation condition, called *sobriety*, is known to be stronger than $T_0$, weaker than $T_2$ and independent of $T_1$.

John Coleman [1], motivated by the study of topologies occurring in some topological algebras , defined new separation conditions called $j$-step Hausdorffness for each $j \geq 1$ ($\mathrm{H}_j$ for short). The relative strengths of the $T_i$ conditions and the $\mathrm{H}_j$ conditions are indicated by

$$T_2 \Longleftrightarrow \mathrm{H}_1 \Longrightarrow \mathrm{H}_2 \Longrightarrow \mathrm{H}_3 \Longrightarrow \mathrm{H}_4 \cdots \Longrightarrow T_1 \Longrightarrow T_0$$

where none of the unidirectional arrows are reversible.

In [2], Keith Kearnes and the present author, while extending and clarifying some of the results of [1], introduced symmetrized versions of Coleman's $\mathrm{H}_j$ conditions, which were labeled $\mathrm{sH}_j$.

In the cited papers, these conditions have been defined for any natural number $j$, and their occurrence in the underlying spaces of topological algebras with some prescribed algebraic properties was the object of study.

Here, we consider these separation conditions for topological spaces in general, and describe their strengths, relative to each other and to the well known conditions of sobriety and the $T_i$ axioms. Also, following Paolo Lipparini's suggestion, we allow the index $j$ to range over all ordinals, rather than just the natural numbers.

## 2. Preliminaries

A topological space $X$ is $T_0$ if whenever $a$ and $b$ are distinct points of $X$ there is a closed subset of $X$ containing one of the points that does not contain the other. $X$ is $T_1$ if for each $a \in X$ the singleton set $\{a\}$ is closed. $X$ is $T_2$, or *Hausdorff*, if for each $a \in X$ the intersection of the closures of the neighborhoods of $a$ is $\{a\}$.

This nonstandard definition of $T_2$ suggests the following generalization:

**Definition 2.1.** *Let $A$ be a topological space. For each $a \in A$ and each ordinal $\alpha$ we define a subset $\Delta^A_\alpha(a)$ of $A$ — also denoted by $\Delta_\alpha(a)$ if there is no cause for ambiguity — recursively by*

$$\Delta_0(a) = A$$
$$\Delta_{\beta+1}(a) = \{\, b \mid \forall \textit{ open } U, V \textit{ with } a \in U, b \in V, \quad U \cap V \cap \Delta_\beta(a) \neq \varnothing \,\}$$
$$\Delta_\gamma(a) = \bigcap_{\beta<\gamma} \Delta_\beta(a) \quad \textit{(if } \gamma \textit{ is a limit ordinal)}$$

*We say that a point $a \in A$ is* $\alpha$-step Hausdorff *if $\Delta_\alpha(a) = \{a\}$. We say that a space is* $\alpha$-step Hausdorff, *or* $\mathrm{H}_\alpha$, *if each of its points is $\alpha$-step Hausdorff.*

This definition implies that $\Delta_1(a)$ is the intersection of the closures of the neighborhoods of $a$. Thus $\Delta_1(a)$ is a closed subspace of $A$ containing $a$. Each $\Delta_{\beta+1}(a)$ is the intersection of the closures of neighborhoods of $a$ in the subspace $\Delta_\beta(a)$ under the relative topology (i. e., $\Delta_{\beta+1}(a)$ coincides with $\Delta_1^{\Delta_\beta(a)}(a)$). In particular, $\Delta_\alpha(a)$ is closed in $A$ for all $a$ and $\alpha$. Clearly, a space is $\mathrm{H}_1$ if and only if it is Hausdorff since both properties say exactly that $\Delta_1(a) = \{a\}$ for all $a \in A$.

Since each $\Delta_\alpha(a)$ is closed, and since $\mathrm{H}_\alpha$ asserts that $\Delta_\alpha(a) = \{\, a \,\}$ for all $a \in A$, it follows that $\mathrm{H}_\alpha \Longrightarrow T_1$.

**Definition 2.2.** *For each ordinal $\alpha$, we let the symbol $\Delta_\alpha$ also denote the binary relation defined by*

$$a\, \Delta_\alpha\, b \Longleftrightarrow a \in \Delta_\alpha(b)$$

Two elements $a$, $b$ of a topological space are sometimes called *unseparable* if they cannot be separated by open sets: thus $a$ and $b$ are unseparable if and only if $a\, \Delta_1\, b$. We will say $a$ is *$\alpha$-unseparable* from $b$ if $a\, \Delta_\alpha\, b$. One should note, though, that the relation $\Delta_\alpha$ need not be symmetric, except of course when $k = 0$ (since $\Delta_0 = A \times A$) and when $k = 1$ (since $\Delta_1$ is the closure of the diagonal of $A \times A$).

We will henceforth adopt the following equivalent definition of $\alpha$-step Hausdorffness.

**Definition 2.3.** *Let $A$ be a topological space. For each ordinal $\alpha$, we will say that $A$ is* $\alpha$-step Hausdorff, *or* $\mathrm{H}_\alpha$, *if the following condition holds for all $a, b \in A$:*

$$a \,\Delta_\alpha\, b \implies a = b \tag{$\mathrm{H}_\alpha$}$$

*In other words,* $\mathrm{H}_\alpha$ *is the assertion that* $\Delta_\alpha$ *is the equality relation.*

A new family of separation conditions, related to the $\mathrm{H}_\alpha$'s and labeled $\mathrm{sH}_\alpha$, is defined as follows:

**Definition 2.4** ([2])**.** *Let $A$ be a topological space, $\alpha$ an ordinal. $A$ is said to be* $\alpha$-step Hausdorff up to symmetry, *or* $\mathrm{sH}_\alpha$, *if the following condition holds for all $a, b \in A$:*

$$a \,\Delta_\alpha\, b \;\wedge\; b \,\Delta_\alpha\, a \Longrightarrow a = b \tag{$\mathrm{sH}_\alpha$}$$

*Thus* $\mathrm{sH}_\alpha$ *asserts that* $\Delta_\alpha$ *is antisymmetric.*

The following Lemma was present in [2], although only for finite ordinals $\alpha$.

**Lemma 2.5.** *For each ordinal $\alpha$, every* $\mathrm{sH}_\alpha$ *space is $T_1$.*

*Proof.* Just note that

(i) ($\mathrm{sH}_\alpha \Longrightarrow T_0$): If $a \neq b$ either $\Delta_\alpha(a)$ is a closed set containing $a$ and not $b$ or $\Delta_\alpha(b)$ is a closed set containing $b$ and not $a$.

(ii) ($T_0 \wedge \neg T_1 \Longrightarrow \neg\,\mathrm{sH}_\alpha$): A $T_0$ space $X$ that fails to be $T_1$ has a subspace $\{ a, b \}$ with induced topology $\{ \varnothing, \{ a \}, \{ a, b \} \}$. For these $a$ and $b$ we have $a \,\Delta_\alpha\, b$ and $b \,\Delta_\alpha\, a$ for all $\alpha$, thus $X$ fails to satisfy $\mathrm{sH}_\alpha$ for any $\alpha$.

$\square$

Since each $\mathrm{sH}_\alpha$ condition is formally weaker than the corresponding $\mathrm{H}_\alpha$, the relative strengths of these conditions may be described by the following diagram.

$$\begin{array}{ccccccccccccc}
T_2 & \Leftrightarrow & \mathrm{H}_1 & \Rightarrow & \mathrm{H}_2 & \Rightarrow & \dots & \Rightarrow & \mathrm{H}_\alpha & \Rightarrow & \dots & & & & \\
 & & \Updownarrow & & \Downarrow & & & & \Downarrow & & & & & & \\
 & & \mathrm{sH}_1 & \Rightarrow & \mathrm{sH}_2 & \Rightarrow & \dots & \Rightarrow & \mathrm{sH}_\alpha & \Rightarrow & \dots & \Rightarrow & T_1 & \Rightarrow & T_0
\end{array}$$

## 3. Every sH space is $T_1$ and sober

In this section we discuss the relation between the separation conditions introduced in the last section and another, well-known, condition called sobriety:

**Definition 3.1.**

(i) *A topological space $X$ is said to be* irreducible *if it contains no two disjoint nonempty open sets. A subset $F$ of a topological space $X$ is said to be* irreducible *if it is irreducible as a subspace (i. e., under the induced topology).*

(ii) *A topological space $X$ is called* sober *if every nonempty closed irreducible subset $F$ of $X$ is the closure of a unique point.*

**Remark 3.2.**

(1) The closure of any point is always (closed and) irreducible: if $F = \mathrm{cl}(a)$, $U$, $V$ are open and $U \cap F \neq \varnothing \neq V \cap F$, then $a \in U \cap V \cap F$.
(2) It follows immediately from the definitions that a $T_1$ space is sober if and only if every nonempty irreducible set is a singleton.

The following examples attest to the well known fact that sobriety is independent of the $T_1$ axiom:

**Example 3.3.**

(1) Let $X$ be an infinite set endowed with the cofinite topology. Then $X$ is $T_1$, but not sober ($X$ itself is closed irreducible).
(2) The *Sierpiński space*, $(\{0,1\}, \{\varnothing, \{0\}, \{0,1\}\})$, is not $T_1$, since $\{0\}$ is not closed. It is sober, for the nonempty closed irreducible sets are $\{1\}$ and $\{0,1\} = \mathrm{cl}(\{0\})$.

We wish to describe the relations between sobriety and the $\mathrm{H}_\alpha$ and $\mathrm{sH}_\alpha$ conditions. We begin with an easy but very useful lemma.

**Lemma 3.4.** *Let $F$ be a nonempty irreducible subset of a topological space $X$, and let $a \in F$. Then $F \subseteq \Delta_\alpha(a)$, for all $\alpha$.*

*Proof.* Let $F$ be nonempty irreducible and let $a \in F$. We argue by transfinite induction to show that $F \subseteq \Delta_\alpha(a)$, for each ordinal $\alpha$. Clearly, $F \subseteq \Delta_0(a) = X$. Let $\alpha > 0$ and suppose $F \subseteq \Delta_\gamma(a)$, for all $\gamma < \alpha$. If $\alpha$ is a limit ordinal, then we clearly have

$$F \subseteq \bigcap_{\gamma<\alpha} \Delta_\gamma(a) = \Delta_\alpha(a)$$

If $\alpha = \beta + 1$ is a successor ordinal, then we have, in particular, $F \subseteq \Delta_\beta(a)$. Consider any $b \in F$, and let $U$ and $V$ be open sets such that $a \in U$, $b \in V$. Clearly,

$$U \cap F \neq \varnothing \neq V \cap F$$

as $a \in U \cap F$ and $b \in V \cap F$; so, by irreducibility of $F$, we have

$$U \cap V \cap F \neq \varnothing$$

and, since $F \subseteq \Delta_\beta(a)$,

$$U \cap V \cap \Delta_\beta(a) \neq \varnothing$$

Thus we see that $b \in \Delta_\alpha(a)$ and, since $b$ was an arbitrary member of $F$, we have $F \subseteq \Delta_\alpha(a)$. □

**Theorem 3.5.** *Every* $\mathrm{sH}_\alpha$ *space (and,* a fortiori*, every* $\mathrm{H}_\alpha$ *space) is $T_1$ and sober.*

*Proof.* Let $X$ be an $\mathrm{sH}_\alpha$ space; then, by the results of Section 2, $X$ is $T_1$. Let $F \subseteq X$ be nonempty and irreducible, and let $a, b \in F$. By Lemma 3.4, we have $a \in F \subseteq \Delta_\alpha(b)$, and $b \in F \subseteq \Delta_\alpha(a)$. By the $\mathrm{sH}_\alpha$ property, it follows that $a = b$. Thus $F$ must be a singleton. Since every nonempty irreducible set is a singleton, $X$ is sober. □

## 4. Not all $T_1$ and sober spaces are sH

In this section, we will provide a counterexample to show that the implication given by Theorem 3.5 cannot be reversed.

**Example 4.1.** Let $X = \mathbb{R} \cup \{ p, q \}$, where $p$ and $q$ are two distinct points not in $\mathbb{R}$. We topologize $X$ by stipulating that the open sets contained in $\mathbb{R}$ are precisely the open sets in the usual Euclidean topology of the real numbers and that sets having either $p$ or $q$ as a member are open if and only if they are cofinite. We will show that $X$ is a $T_1$ and sober topological space that is not $\mathrm{sH}_\alpha$, for any $\alpha$:

$X$ is $T_1$, since the complement of each singleton is clearly open.
It is easy to check that

$$\Delta_1(a) = \{ a, p, q \}, \ \forall a \in \mathbb{R}$$
$$\Delta_1(p) = \Delta_1(q) = X$$

from which it follows that, for any $\alpha > 1$,

$$\Delta_\alpha(a) = \{ a \}, \ \forall a \in \mathbb{R}$$
$$\Delta_\alpha(p) = \Delta_\alpha(q) = X$$

Thus $X$ is not $\mathrm{sH}_\alpha$ for any $\alpha$, as $p \neq q$ but $p \in \Delta_\alpha(q)$ and $q \in \Delta_\alpha(p)$. Now we check that $X$ is sober. First, note that if $a \in F$ for some real number $a$, and $F \subseteq X$ is closed and irreducible, then, by Lemma 3.4, we have $F \subseteq \Delta_2(a) = \{ a \}$, so $F = \{ a \}$ is a singleton. Therefore the only possibility for an irreducible set with more than one element is $F = \{ p, q \}$. But this set is not irreducible: letting $U = X \setminus \{ p \}$, $V = X \setminus \{ q \}$, we have $U$, $V$ open and

$$U \cap F \neq \varnothing \neq V \cap F$$

but

$$U \cap V \cap F = \varnothing$$

Thus every nonempty irreducible set is a singleton, so $X$ is $T_1$ and sober, as claimed.

## 5. All H and all sH conditions are distinct

The purpose of this section is to provide examples of topological spaces that show that the H and sH conditions are all distinct from each other (apart from the noted equivalence $\mathrm{H}_1 \Leftrightarrow \mathrm{sH}_1 \Leftrightarrow T_2$).

In order to make the following arguments clearer, we first introduce some terminology and notation.

**Definition 5.1.** *Let $X$ be a topological space, let $a \in X$, and let $\alpha$ be an ordinal. We will say that $a$ is* strictly $\mathrm{H}_\alpha$ *if $\Delta_\alpha(a) = \{ a \}$, but $\Delta_\beta(a) \neq \{ a \}$, for all $\beta < \alpha$. We will say that $X$ is* strictly $\mathrm{H}_\alpha$ *if $X$ is $\mathrm{H}_\alpha$ and is not $\mathrm{H}_\beta$ for any $\beta < \alpha$.*

Any Hausdorff space with more than one point is a strictly $\mathrm{H}_1$ space, since $\mathrm{H}_0$ only holds in one-point spaces.

To establish the desired results, we need to introduce a few constructions of topological spaces. The following is well known.

**Definition 5.2.** *The* sum *of a family $(X_i, \tau_i)_{i \in I}$ of topological spaces is the space $(X, \tau)$ where $X$ is the disjoint union of the $X_i$, and the union of the $\tau_i$ is a basis for $\tau$.*

A topological space may be strictly $\mathrm{H}_\alpha$ but fail to contain a strictly $\mathrm{H}_\alpha$ point. For instance, a sum of strictly $\mathrm{H}_n$ spaces, for all finite $n$, is a strictly $\mathrm{H}_\omega$ space that has no strictly $\mathrm{H}_\omega$ point. This is inconvenient for our purposes, so we will make use of a slightly modified construction.

Recall that a *pointed topological space* is a pair $(X, *)$ where $X$ is a topological space and $*$ is a point of $X$. The distinguished point $*$ will be referred to as a *base point*. In the sequel, we will often need to work at once with several pointed spaces, sharing the same base point. Two pointed spaces $(X, *)$, $(Y, *)$, with a common base point $*$, will be called *disjoint* if $X \cap Y = \{ * \}$.

**Definition 5.3.** *A pointed topological space $(X, *)$ is* strictly $\mathrm{H}_\alpha$ *if $X$ is $\mathrm{H}_\alpha$ and $*$ is strictly $\mathrm{H}_\alpha$.*

**Definition 5.4.** *Let $((X_i, *))_{i \in I}$ be a family of pointed topological spaces with a common base point, which are pairwise disjoint. The* amalgamated sum *of the family $((X_i, *))_{i \in I}$ is the pointed space $(X, *)$, where*

$$X = \bigcup_{i \in I} X_i$$

*and a subset $U$ of $X$ is open if and only if*

$$U \cap X_i \text{ is } X_i\text{-open, for all } i \in I$$

Note that the amalgamated sum just described can be viewed as a sum in which the base points are all identified: it is the same as the quotient space obtained by factoring the sum of the spaces by the equivalence relation identifying all base points.

**Definition 5.5** ([1, 2])**.** *Let $A$ and $B$ be topological spaces. Let $* \in B$ be such that $\{ * \}$ is closed in $B$. We denote by $A \rightsquigarrow_* B$ the space with underlying set $A \dot{\cup} (B \setminus \{ * \})$ in which a subset $U \subseteq A \rightsquigarrow_* B$ is open if and only if the following three conditions hold:*

a) *$U \cap A$ is $A$-open;*
b) *$U \cap B$ is $B$-open;*
c) *if $U \cap A \neq \varnothing$, then $(U \cap B) \cup \{ * \}$ is $B$-open.*

For each subset $U$ of $A \rightsquigarrow_* B$, we will henceforth let $U_A$, $U_B$ and $U_B^*$ denote $U \cap A$, $U \cap B$ and $(U \cap B) \cup \{*\}$, respectively.

The space $A \rightsquigarrow_* B$ can be understood as the result of replacing the point $*$ of $B$ with a copy of the space $A$. Thus if $U$ is a neighborhood in $A \rightsquigarrow_* B$ of a point $a \in A$, then $U_A$ is a neighborhood of $a$ in the space $A$ and $U_B$ is a punctured neighborhood of the point $*$ in $B$. In order to impose an adequate structure on $A \rightsquigarrow_* B$, we will also make the assumption that the singleton $\{*\}$ is *not* open in $B$ (and thus any two punctured neighborhoods of $*$ have nonempty intersection). In fact, we require a stronger property to hold.

**Definition 5.6.** *Let $(B,*)$ be a pointed topological space, and $\beta > 0$ an ordinal. We will say that $(B,*)$ is* normal strictly $\mathrm{H}_\beta$ *if it is strictly $\mathrm{H}_\beta$ and one the following conditions holds:*

(1) *$\beta$ is a limit ordinal, or*
(2) *$\beta = \gamma + 1$ and $\{*\}$ is* not *open in $\Delta_\gamma^B(*)$.*

The following Lemma describes how, in a space $A \rightsquigarrow_* B$, the $\Delta_\alpha$ relations can be computed from the corresponding relations in $A$ and $B$. The proof of the Lemma is not hard, but is somewhat tedious.

**Lemma 5.7.** *Suppose $A$ is a topological space and $(B,*)$ is a pointed space which is normal strictly $\mathrm{H}_\beta$, for some ordinal $\beta$. Let $X$ denote the space $A \rightsquigarrow_* B$. Then we have the following:*

(i) *For each $b \in B \setminus \{*\}$, and each ordinal $\gamma$,*

$$\Delta_\gamma^X(b) = \begin{cases} \Delta_\gamma^B(b) & \text{if } * \notin \Delta_\gamma^B(b) \\ A \rightsquigarrow_* \Delta_\gamma^B(b) & \text{if } * \in \Delta_\gamma^B(b) \end{cases}$$

(ii) *For each $a \in A$, and each ordinal $\gamma$,*

$$\Delta_\gamma^X(a) = \begin{cases} A \rightsquigarrow_* \Delta_\gamma^B(*) & \text{if } \gamma < \beta \\ \Delta_\delta^A(a) & \text{if } \gamma = \beta + \delta \end{cases}$$

*Sketch of Proof.*

(i) Let $b \in B \setminus \{*\}$ and let $\gamma$ be an ordinal. The desired result follows immediately from the two claims below:

**Claim 5.8.** $\Delta_\gamma^X(b) \cap B = \Delta_\gamma^B(b) \setminus \{*\}$.

**Claim 5.9.**

$$\Delta_\gamma^X(b) \cap A = \begin{cases} A & \text{if } * \in \Delta_\gamma^B(b) \\ \varnothing & \text{if } * \notin \Delta_\gamma^B(b) \end{cases}$$

These two claims may be proved by transfinite induction — the induction step being trivial for limit ordinals and relatively straightforward for successor ordinals.

(ii) First, we prove, by transfinite induction, that the desired result holds for all ordinals $\gamma$ such that $0 \leq \gamma \leq \beta$; in particular, letting $\gamma = \beta$, we get $\Delta_\beta^X(a) = \Delta_0^A(a) = A$.

It is an easy consequence of the definitions that the equality

$$\Delta^X_{\beta+\delta}(a) = \Delta_\delta^{\Delta^X_\beta(a)}(a)$$

always holds, for any space. Thus the rest of the required result readily obtains. □

**Corollary 5.10.** *Let $\alpha$ and $\beta$ be ordinals. Suppose $A$ is a topological space which is strictly* $\mathrm{H}_\alpha$ *and $(B, *)$ is a pointed space which is normal strictly* $\mathrm{H}_\beta$.

*Then $A \rightsquigarrow_* B$ is strictly* $\mathrm{H}_{\beta+\alpha}$. *Furthermore, if, for some point $a \in A$, $(A, a)$ is normal strictly* $\mathrm{H}_\alpha$, *then $(A \rightsquigarrow_* B, a)$ is normal strictly* $\mathrm{H}_{\beta+\alpha}$.

**Theorem 5.11.** *All* $\mathrm{H}_\alpha$ *conditions are distinct.*

*Proof.* We argue by transfinite induction to show that, for each nonzero ordinal $\alpha$, there exists a normal strictly $\mathrm{H}_\alpha$ pointed topological space. Let $(X_1, *)$ denote the space of real numbers, taken with, say, $* = 0$ as the base point: this is clearly a normal strictly $\mathrm{H}_1$ pointed space. Let $\alpha > 1$ and suppose a pointed strictly $\mathrm{H}_\beta$ space $(X_\beta, *)$ has been picked for each ordinal $\beta < \alpha$. If $\alpha = \gamma + 1$ is a successor ordinal, then we let $A = X_1$, $B = X_\gamma$ and $X_\alpha = A \rightsquigarrow_* B$. By Corollary 5.10, $X$ is a strictly $\mathrm{H}_\alpha$ space and we can make it a pointed strictly $\mathrm{H}_\alpha$ space by choosing $* \in A$ as the base point. If $\alpha$ is a limit ordinal, then we let $(X_\alpha, *)$ be the amalgamated sum of the family $((X_\beta, *))_{\beta<\alpha}$. Again, it follows that $(X_\alpha, *)$ is a normal strictly $\mathrm{H}_\alpha$ pointed space. □

The proof of the preceding Theorem provides a recipe for constructing, for each ordinal $\alpha$, a normal strictly $\mathrm{H}_\alpha$ pointed space $(X_\alpha, *)$.

Using the description of the $\Delta$ relations provided by Lemma 5.7, it may be easily shown, by transfinite induction on $\alpha$, that on these spaces $X_\alpha$ all the $\Delta$ relations are symmetric, i. e., for all ordinals $\alpha$, $\gamma$ and all elements $x, y \in X_\alpha$, we have

$$x \in \Delta_\gamma(y) \iff y \in \Delta_\gamma(x) \tag{s}$$

Clearly, for any ordinal $\beta$, a space in which (s) holds will be $\mathrm{sH}_\beta$ if and only if it is $\mathrm{H}_\beta$, so we immediately obtain:

**Theorem 5.12.** *All* $\mathrm{sH}_\alpha$ *conditions are distinct.*

For $\alpha > 1$, each $\mathrm{sH}_\alpha$ condition is distinct from all the H conditions as well:

**Example 5.13** ([2])**.** Let $X = \mathbb{R} \cup \{ p \}$, where $p$ is a point not in $\mathbb{R}$. Topologize $X$ by stipulating that the open subsets of $\mathbb{R}$ are the same as under the Euclidean topology, and the open sets containing $p$ are the cofinite ones. Then it is easily seen that

$$\begin{aligned} &\Delta_1(p) = X \\ &\Delta_1(a) = \{ a, p \} \quad \forall a \in \mathbb{R} \end{aligned}$$

and, for each ordinal $\alpha > 1$,

$$\begin{aligned} \Delta_\alpha(p) &= X \\ \Delta_\alpha(a) &= \{\, a \,\} \quad \forall a \in \mathbb{R} \end{aligned}$$

and thus $X$ is $\mathrm{sH}_2$, but is not $\mathrm{H}_\alpha$, for any $\alpha$.

## 6. Concluding remarks

The following diagram depicts the relative strengths of the separation axioms discussed in this paper. None of the unidirectional arrows may, in general, be reversed.

$$\begin{array}{ccccccccccccccc}
T_2 & \Leftrightarrow & \mathrm{H}_1 & \Rightarrow & \mathrm{H}_2 & \Rightarrow & \dots & \Rightarrow & \mathrm{H}_\alpha & \Rightarrow & \dots & & & & \\
 & & \Updownarrow & & \Downarrow & & & & \Downarrow & & & & & & \\
 & & \mathrm{sH}_1 & \Rightarrow & \mathrm{sH}_2 & \Rightarrow & \dots & \Rightarrow & \mathrm{sH}_\alpha & \Rightarrow & \dots & \Rightarrow & T_1\&\text{Sober} & \Rightarrow & T_1 \\
 & & & & & & & & & & & & \Downarrow & & \Downarrow \\
 & & & & & & & & & & & & \text{Sober} & \Rightarrow & T_0
\end{array}$$

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