Monatshefte für Mathematik manuscript No.
(will be inserted by the editor)

$\mathfrak{Z}$-permutable subgroups of finite groups

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Abstract Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a finite group G, that is, a set composed of a Sylow p-subgroup of G for each p dividing the order of G. A subgroup H of G is called $\mathfrak{Z}$-permutable if H permutes with all members of $\mathfrak{Z}$. The main goal of this paper is to study the embedding of the $\mathfrak{Z}$-permutable subgroups and the influence of $\mathfrak{Z}$-permutability on the group structure.

Keywords finite group, p-soluble group, p-supersoluble group, $\mathfrak{Z}$-permutable subgroup, subnormal subgroup

Mathematics Subject Classification (2000) 20D10 · 20D20 · 20D35 · 20D40

1 Introduction and statements of results

All groups in this paper will be finite.

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A subgroup H of a group G is said to *permute* with a subgroup K of G if HK is a subgroup of G. H is said to be *S-permutable* in G if H permutes with all Sylow subgroups of G. This property extends normality and permutability and was introduced by Kegel in [11]. In this paper, he proved some interesting properties which turn out to be useful in establishing results concerning the group structure. In particular, it is proved there that every S-permutable subgroup must be subnormal ([11, Satz 1]).

On the other hand, we say that a set $\mathfrak{Z}$ of Sylow subgroups of a group G is a *a complete set of Sylow subgroups of* G if $\mathfrak{Z}$ contains exactly one Sylow subgroup for each prime dividing the order of G; $\mathfrak{Z}$ is called a *Sylow basis of* G if the Sylow subgroups in $\mathfrak{Z}$ are pairwise permutable. Sylow basis were introduced and studied by Hall in [7]. The results of this paper show that the existence and conjugacy of Sylow bases is a characteristic property of soluble groups.

In [1], Asaad and Heliel introduced and studied the notion of a $\mathfrak{Z}$-permutable subgroup, where $\mathfrak{Z}$ is a complete set of Sylow subgroups of a group G. A subgroup of G is called $\mathfrak{Z}$-*permutable* if it permutes with every member of a complete set $\mathfrak{Z}$ of Sylow subgroups of G. It is clear that S-permutability implies $\mathfrak{Z}$-permutability but the converse does not hold in general. In fact, $\mathfrak{Z}$-permutable subgroups are not subnormal in general, and subnormal $\mathfrak{Z}$-permutable subgroups are not S-permutable either as the following example shows:

Example 1 Let $E = \langle x, y \rangle$ be an extraspecial group of order 27 and exponent 3. Let a be an automorphism of order 2 of G given by $x^a = x^{-1}$, $y^a = y^{-1}$. Let $G = E \rtimes \langle a \rangle$ be the corresponding semidirect product. Then $\mathfrak{Z} = \{E, \langle a \rangle\}$ is a complete set of Sylow subgroups of G. The subgroup $H = \langle x \rangle$ is $\mathfrak{Z}$-permutable, but it does not permute with the Sylow 2-subgroup $\langle ay \rangle$. Therefore, H is not S-permutable. However, H is a subnormal subgroup of G.

Throughout the first part of our paper, proving important properties of S-permutable type of the subnormal $\mathfrak{Z}$-permutable subgroups has been our main focus.

The embedding of S-permutable subgroups was studied by Kegel [11, Satz 1] and Deskins [4, Theorem 1] (see also [3, Theorem 1.2.14]). They proved that if A is an S-permutable subgroup of G, then $\langle A^G \rangle / \mathrm{Core}_G(A)$ is nilpotent. Our first main result shows how a subnormal $\mathfrak{Z}$-permutable subgroup is embedded in the group.

Theorem 1 *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G. Let A be a subnormal $\mathfrak{Z}$-permutable subgroup of G. Then $\langle A^G \rangle / \mathrm{Core}_G(A)$ is soluble. If, in addition, $\mathfrak{Z}$ is a Sylow basis of G, then $\langle A^G \rangle / \mathrm{Core}_G(A)$ is nilpotent.*

The alternating group of degree 6 is a non-subnormal $\mathfrak{Z}$-permutable subgroup of the alternating group of degree 7 which is not soluble. Moreover, every core-free maximal subgroup of a soluble primitive group is $\mathfrak{Z}$-permutable. Therefore subnormality is necessary in the above theorem.

A classical result of Kegel ([11, Satz 2], see also [3, Theorem 1.2.19]) shows that the set of all S-permutable subgroups of a group G is a sublattice of the subgroup lattice of G. Kegel's result also holds for subnormal $\mathfrak{Z}$-permutable subgroups. It is consequence of the following theorem.

Theorem 2 *Let p be a prime and U and V subgroups of a group G. If U and V permute with a Sylow p-subgroup G_p of G and U is subnormal in G, then $U \cap V$ permutes with G_p.*

The hypothesis of the subnormality of U is necessary in the above theorem, even for soluble groups, as an example of Doerk [5, Beispiel 1] shows.

Corollary 1 *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G. Then the set of all subnormal $\mathfrak{Z}$-permutable subgroups of a group G is a sublattice of the lattice of all subgroups of G.*

If $\mathfrak{Z}$ is a Sylow basis of a group G, the set of all $\mathfrak{Z}$-permutable subgroups is a sublattice of the subgroup lattice of G ([6, Chapter I, Theorem 4.29]). However, we do not know whether the set of all $\mathfrak{Z}$-permutable subgroups (not necessarily subnormal) of a group G is a sublattice of the lattice of all subgroups of G.

There are several papers in the literature where global information about a group is obtained by assuming that some distinguished subgroups are $\mathfrak{Z}$-permutable ([1, 8, 9, 12, 13, 14, 15, 17]). The second part of the paper concerns situations in this spirit, but we require only that some p-subgroups, for a given prime p, have the required property.

In order to state our results in this direction, we recall that a group is said to be *p-supersoluble* if it is p-soluble and every p-chief factor has order p, where *p is a prime that we hold fixed.*

In the sequel, $\mathfrak{Z} = \{G_q \mid q \in \pi(G)\}$ *will denote a complete set of Sylow subgroups of a group G, where G_q is a Sylow q-subgroup of G.*

Asaad and Heliel [1, Theorem 3.1] showed that if all maximal subgroups of the Sylow subgroups in $\mathfrak{Z}$ are $\mathfrak{Z}$-permutable, then G is supersoluble. A local approach to this theorem is the following.

Theorem 3 *Let G be a group. Assume that all maximal subgroups of $G_p \in \mathfrak{Z}$ are $\mathfrak{Z}$-permutable. Then either G_p is cyclic or G is p-supersoluble.*

If p is the smallest prime dividing the order of G, and the Sylow p-subgroups are cyclic, then G is p-nilpotent by [10, IV, Satz 2.8]. Furthermore, if G is p-supersoluble, then G every p-chief factor is central and so G is p-nilpotent. Therefore we have:

Corollary 2 ([14, Theorem 3.1]) *If p is the smallest prime dividing the order of a group G and the maximal subgroups of $G_p \in \mathfrak{Z}$ are $\mathfrak{Z}$-permutable, then G is p-nilpotent.*

Corollary 3 ([1, Theorem 3.1]) *Assume that G is a group whose maximal subgroups of the Sylow subgroups in $\mathfrak{Z}$ are $\mathfrak{Z}$-permutable. Then G is supersoluble.*

The next natural step in our analysis to consider the structural impact of the $\mathfrak{Z}$-permutability of the second maximal subgroups of the Sylow p-subgroup in $\mathfrak{Z}$.

Suppose that every 2-maximal subgroup of $G_p \in \mathfrak{Z}$ is $\mathfrak{Z}$-permutable and that G_p does not have cyclic maximal subgroups. Then every maximal subgroup of G_p is $\mathfrak{Z}$-permutable and G_p is not cyclic. By Theorem 3, G is p-supersoluble. Therefore we have:

Corollary 4 *Let G be a group. Suppose that all 2-maximal subgroups of $G_p \in \mathfrak{Z}$ are $\mathfrak{Z}$-permutable. Either G_p has a cyclic maximal subgroup or G is p-supersoluble.*

In [14, Theorem 3.3], the authors proved the following result:

Theorem 4 ([14, Theorem 3.3]) *Assume that p is the smallest prime dividing the order of a group G. Suppose that all 2-maximal subgroups of $G_p \in \mathfrak{Z}$ are $\mathfrak{Z}$-permutable. If G is A_4-free, then G is p-nilpotent.*

Our goal in the sequel is to present an improvement of this theorem.

According to Corollary 4, if all 2-maximal subgroups of $G_p \in \mathfrak{Z}$ are $\mathfrak{Z}$-permutable, then either G_p has a cyclic maximal subgroup or G is p-supersoluble. Futhermore, a p-supersoluble group G is p-nilpotent provided that p is the smallest prime dividing the order of G. Hence we only must consider the case when G_p has a cyclic maximal subgroup. We prove:

Theorem 5 *Assume that p is the smallest prime dividing the order of a group G. Suppose that all 2-maximal subgroups of $G_p \in \mathfrak{Z}$ are $\mathfrak{Z}$-permutable. If G_p has a cyclic maximal subgroup, then G is p-soluble.*

A key fact for the proof of Theorem 5 is that G cannot be non-abelian simple. This was established in Step 3 of the proof of [14, Theorem 3.3].

Theorem 6 *Assume that p is the smallest prime dividing the order of a group G. If every 2-maximal subgroup of $G_p \in \mathfrak{Z}$ is $\mathfrak{Z}$-permutable, then either G is p-nilpotent or G has an epimorphic image isomorphic to Σ_4.*

In [12, Theorem 3.3], the authors proved that if p is the smallest prime dividing the order of a group of G and every cyclic subgroup of G_p with order p or order 4 (if $p = 2$) is $\mathfrak{Z}$-permutable in G, then G is p-nilpotent.

Our last results concern the $\mathfrak{Z}$-permutability of the minimal subgroups of the Sylow p-subgroup in $\mathfrak{Z}$ and include the above result as a particular case.

Theorem 7 *Let G be a p-soluble group such that every cyclic subgroup of G_p with order p or order 4 (if $p = 2$) is $\mathfrak{Z}$-permutable in G. Then G is p-supersoluble.*

Theorem 8 *Let G be a group such that every cyclic subgroup of G_p with order p or order 4 (if $p = 2$) is $\mathfrak{Z}$-permutable in G. Either G_p has order p or G is p-soluble.*

Corollary 5 *Let G be a group such that every cyclic subgroup of G_p with order p or order 4 (if $p = 2$) is $\mathfrak{Z}$-permutable in G. Then either G_p has order p or G is p-supersoluble.*

Corollary 6 ([12, Theorem 3.3]) *If p is the smallest prime dividing the order of G and every cyclic subgroup of G_p with order p or order 4 (if $p = 2$) is $\mathfrak{Z}$-permutable in G, then G is p-nilpotent.*

2 Preliminaries

Suppose that G is a group and N a normal subgroup of G. Following [1], we write $\mathfrak{Z}N = \{G_qN : G_q \in \mathfrak{Z}\}$, $\mathfrak{Z}N/N = \{G_qN/N : G_q \in \mathfrak{Z}\}$, and $\mathfrak{Z} \cap X = \{G_q \cap X : G_q \in \mathfrak{Z}\}$ for all subgroups X of G.

Lemma 1 ([1, Lemma 2.1]) *Let G be a group and N a normal subgroup of G.*

1. *$\mathfrak{Z} \cap N$ and $\mathfrak{Z}N/N$ are complete sets of Sylow subgroups of N and G/N, respectively.*
2. *If U is a $\mathfrak{Z}$-permutable subgroup of G, then UN/N is $\mathfrak{Z}N/N$-permutable. If U is contained in N, then U is $\mathfrak{Z} \cap N$-permutable.*

The following well-known fact, which follows from the repeated application of [10, Kapitel I, Hilfssatz 7.7a)], will be used in this paper without further notice.

Lemma 2 *Let S be a subnormal subgroup of a group G and let Q be a Sylow q-subgroup of G, where q is a prime. Then $Q \cap S$ is a Sylow q-subgroup of S.*

The following result, due to Vdovin, turns out to be crucial in the proofs of some of our results.

Theorem 9 *If, for every prime $q \neq p$, G possesses a Hall $\{p,q\}$-subgroup, then G is p-soluble.*

The above theorem is a consequence of the following lemma, whose proof requires a bit of notation.

Let q be a natural number, and r an odd prime such that $\gcd(q,r) = 1$. Let $e(q,r)$ denote the multiplicative order of q modulo r, that is, the least natural number t with $q^t \equiv 1 \pmod{r}$. For an odd q, we set $e(q,2) = 1$ if $q \equiv 1 \pmod 4$ and $e(q,2) = 2$ otherwise.

Lemma 3 *Let r be a prime. Then, for every simple group S with $r \in \pi(S)$, there exists $s \in \pi(S)$ such that S does not possess a Hall $\{r,s\}$-subgroup.*

Proof Suppose, by contradiction, that there exists a finite simple group S and a prime $r \in \pi(S)$ such that, for every $s \in \pi(S)$, S possesses a Hall $\{r,s\}$-subgroup H. Burnside's p^aq^b-theorem implies that H is soluble. We proceed case by case.

Assume first that S is an alternating group A_n of degree $n \geq 5$. If $r \neq 2$, then [16, Table 2] implies that for every odd $s \in \pi(G) \setminus \{p\}$, we have that S does not possess a Hall $\{r,s\}$-subgroup. If $r = 2$, then [16, Table 2] implies that S does not have a Hall $\{2,5\}$-subgroup.

Now assume that S is sporadic. Then the claim follows from [16, Tables 3 and 4].

Finally, assume that S is a finite simple group of Lie type over a field of characteristic p and order q. If $r = p$, then [16, Theorem 8.3] implies that every Hall π-subgroup of S with $r \in \pi$ is contained in a Borel subgroup B or is parabolic. Since B is a proper subgroup of S, there exists $s \in \pi(|S : B|)$, and so B cannot contain a Hall $\{r,s\}$-subgroup of S. Therefore a Hall $\{r,s\}$-subgroup H of S is parabolic. Theorems 8.5, 8.6, and 8.7 and Table 6 from [16] imply that in this case $\{r,s\} = \{2,3\}$

and $S \in \{\mathrm{SL}_3(2), \mathrm{SL}_3(3), \mathrm{SL}_4(2), \mathrm{SL}_5(2)\}$. In all these cases, there is no Hall $\{r,s\}$-subgroup for $s \in \pi((q^n-1)/(q-1))$, which is always contained in $\pi' \cap \pi(G)$ if G has a proper Hall π-subgroup with $|\pi| \geq 2$.

Assume that $r \neq p$ is that r is odd. If S is an exceptional group of Lie type and S is neither a Suzuki nor a Ree group, then [16, Table 7] implies that $S \in E_{\{r,s\}}$ if and only if $e(q,r) = e(q,s)$. Now by the decomposition of $|S|$ as a product of polynomials in q, there exists an odd $s \in \pi(S)$ with $e(q,r) \neq e(q,s)$, and so S does not have a Hall $\{r,s\}$-subgroup. If S is either a Suzuki or a Ree group, then the claim follows immediately from [16, Table 8]. If S is a classical group of Lie type, then [16, Table 7] implies that S has a Hall $\{r,s\}$-subgroup only if either $e(q,r) = e(q,s)$ or $e(q,s) = b(r)$, where $b(r) \in \{1,r\}$ if $G = \mathrm{PSL}_n(q)$, $b(r) \in \{2,2r\}$ if $G = \mathrm{PSU}_n(q)$, $b(r) = 2e(q,r)$ if $e(q,r)$ is odd and $G = {}^2D_n(q)$, $b(r) = e(q,r)/2$ if $e(q,r)$ is even, 4 does not divide $e(q,r)$ and $G = {}^2D_n(q)$. In particular, if $e(q,s) \neq e(q,r)$, then $e(q,s)$ can take at most two values. If the rank of S is at least 2, then $|\{e(q,s) \mid s \in \pi(S) \setminus \{p\}\}| \geq 3$, and so there exists s such that $e(q,r) \notin \{e(q,r), b(r)\}$ and therefore $S \notin E_{\{r,s\}}$. If the rank of S is less than 2, then $S \cong \mathrm{PSL}_2(q)$. If $S = \mathrm{PSL}_2(q)$ and $e(q,r) = 1$, then there exists $s \in \pi(S)$ with $e(q,s) = 2$ and [16, Tables 7 and 10] imply that $S \notin E_{\{r,s\}}$. If $S \equiv \mathrm{PSL}_2(q)$ and $e(q,r) = 2$, then $S \notin E_{\{r,p\}}$.

Finally, assume that $r = 2$ and $r \neq p$. If $S \neq \mathrm{SL}_3(3)$, then $S \notin E_{\{2,p\}}$ by the arguments presented for the case $r = p$. If $S = \mathrm{SL}_3(3)$, then $S \notin E_{\{2,13\}}$.

Corollary 7 *Let G be a group and $p \in \pi(G)$. Assume that*

1. *all maximal subgroups of $G_p \in \mathfrak{Z}$ are $\mathfrak{Z}$-permutable and G_p is not cyclic, or*
2. *all 2-maximal subgroups of $G_p \in \mathfrak{Z}$ are $\mathfrak{Z}$-permutable and G_p has no cyclic maximal subgroups.*

Then G is p-soluble.

Proof Assume that 1 holds. Then G_p possesses two maximal subgroups M_1 and M_2, both $\mathfrak{Z}$-permutable. Then $M_1M_2 = G_p$ is $\mathfrak{Z}$-permutable. This implies that G_pG_q is a Hall $\{p,q\}$-subgroup of G for each $q \neq p$. By Theorem 9, G is p-soluble.

Assume that 2 holds. Let M_1 be a maximal subgroup of G_p. Since M_1 is not cyclic, M_1 possesses two maximal subgroups M_{11} and M_{12}. Since both of them are $\mathfrak{Z}$-permutable, $M_1 = M_{11}M_{12}$ is also $\mathfrak{Z}$-permutable. Hence 1 holds and G is p-soluble.

3 Proofs of the main results

Proof (of Theorem 1) We prove that $A/\mathrm{Core}_G(A)$ is soluble by induction on the order of G. Since $A/\mathrm{Core}_G(A)$ is $(\mathfrak{Z}\mathrm{Core}_G(A)/\mathrm{Core}_G(A))$-permutable in $G/\mathrm{Core}_G(A)$ by Lemma 1, we can assume that $\mathrm{Core}_G(A) = 1$. Let r be a prime dividing $|G|$ and let R be the Sylow r-subgroup of G in $\mathfrak{Z}$. Consider $X = AR$. Let q be a prime different from r and let G_q be the Sylow q-subgroup of G in $\mathfrak{Z}$. Since A is subnormal in G, $G_q \cap A$ is a Sylow q-subgroup of A. Moreover, A is $(\mathfrak{Z} \cap X)$-permutable, because $\mathfrak{Z} \cap X = \{R\} \cup \{G_q \cap A \mid q \neq r\}$. Moreover A is subnormal in X. Assume that X is a proper subgroup of G. By induction, the soluble residual $A^{\mathfrak{S}}$ of A is contained in $\mathrm{Core}_X(A) = \mathrm{Core}_R(A)$. Consequently, $A^{\mathfrak{S}} = (A^{\mathfrak{S}})^{\mathfrak{S}} \leq \mathrm{Core}_R(A)^{\mathfrak{S}} \leq A^{\mathfrak{S}}$. It follows that $A^{\mathfrak{S}} = \mathrm{Core}_R(A)^{\mathfrak{S}}$ is a normal subgroup of X. In particular, $R \leq \mathrm{N}_G(A^{\mathfrak{S}})$. Suppose that for every Sylow subgroup R of G in $\mathfrak{Z}$, AR is a proper subgroup of G. It follows that $R \leq \mathrm{N}_G(A^{\mathfrak{S}})$ for each $R \in \mathfrak{Z}$. Hence $A^{\mathfrak{S}}$ is a normal subgroup of G. Thus $A^{\mathfrak{S}} \leq \mathrm{Core}_G(A) = 1$. Consequently A is soluble, as wanted.

Therefore there exists a prime r and a Sylow r-subgroup R of G in $\mathfrak{Z}$ such that $G = AR$. Let q be a prime different from r and let Q be a Sylow q-subgroup of G. The subnormality of A implies that $Q \cap A$ is a Sylow q-subgroup of A. By order considerations, $Q \cap A = Q$ and so Q is a Sylow q-subgroup of A. It follows that $\mathrm{O}^r(G) \leq \mathrm{Core}_G(A) = 1$. In particular, G is a r-group and so G is soluble. Hence, A is soluble.

We conclude that $\langle A^G \rangle / \mathrm{Core}_G(A)$ is soluble.

Suppose that $\mathfrak{Z}$ is a Sylow basis of G. We shall show that $A^G/\mathrm{Core}_G(A)$ is nilpotent. Without loss of generality we may assume that $\mathrm{Core}_G(A) = 1$. Let $B = \bigcap_q \mathrm{O}^q(A)$ be the nilpotent residual of A. Let r be a prime dividing $|G|$ and $g \in G$. Then $g = xy$, where x is an element of G_r and y is a r'-element of $Z = \prod_{q \neq r} G_q$. It follows that $B_r = B \cap G_r = \mathrm{O}^r(A) \cap G_r$ is a Sylow r-subgroup of B. Applying [3, Lemma 1.1.11], we have that $\mathrm{O}^r(A) = \mathrm{O}^r(AG_r)$, which is a normal subgroup of AG_r, and that $\mathrm{O}^{r'}(A) = \mathrm{O}^{r'}(AZ)$, which is a normal subgroup of AZ. In particular, G_r normalises $\mathrm{O}^r(A)$ and Z normalises $\mathrm{O}^{r'}(A)$. Moreover, B_r is contained in $\mathrm{O}^{r'}(A)^y = \mathrm{O}^{r'}(A)$. Since G_r normalises B_r, it follows that $B_r^g = B_r^y$ is contained in $\mathrm{O}^{r'}(A)$ and so it is a subgroup of A. Consequently, the normal closure $\langle B_r^G \rangle$ is contained in A and then $\langle B_r^G \rangle \leq \mathrm{Core}_G(A) = 1$. Hence $B = 1$ and A is nilpotent.

Therefore, $\langle A^G \rangle / \mathrm{Core}_G(A)$ is nilpotent.

Proof (of Theorem 2) We argue by induction on $|G|$. Assume that VG_p is a proper subgroup of G. Then $U \cap VG_p$ is a subnormal subgroup of VG_p. Since $UG_p \cap VG_p = (U \cap VG_p)G_p$ is a subgroup of G, $U \cap VG_p$ permutes with the Sylow p-subgroup G_p of VG_p. The induction hypothesis implies that G_p permutes with $(U \cap VG_p) \cap V = U \cap V$. Therefore we may assume that $G = VG_p$. An analogous argument with the subnormal subgroup U of UG_p and $V \cap UG_p$ shows that $G = UG_p$. Let $q \neq p$ be a prime dividing $|G|$ and let G_q be a Sylow q-subgroup of G contained in V. Then $G_q \cap U$ is a Sylow q-subgroup of U since U is subnormal in G. Hence G_q is contained in U by order considerations. This means that $U \cap V$ contains a Sylow q-subgroup of G for all primes $q \neq p$. Therefore $|G : U \cap V|$ is a power of p. Applying [6, Chapter A, Lemma 1.6 (b)], we conclude that $G = G_p(U \cap V)$ and G_p permutes with $U \cap V$, as required.

Proof (of Theorem 3) Suppose that every maximal subgroup of $G_p \in \mathfrak{Z}$ is $\mathfrak{Z}$-permutable and that G_p is not cyclic. By Corollary 7, G is p-soluble. Assume that G is not p-supersoluble and consider G of least possible order. Let N be a minimal normal subgroup of G. Let M/N be a maximal subgroup of PN/N. Then $M/N = M_1N/N$ for some maximal subgroup M_1 of P. Since M_1 is $\mathfrak{Z}$-permutable, it follows that M/N is $\mathfrak{Z}N/N$ permutable by Lemma 1. Then G_pN/N has all maximal subgroups $\mathfrak{Z}N/N$-permutable. Assume that N is a p'-group. Since $G_pN/N \cong G_p$, we have that G_pN/N is not cyclic and so G/N is p-supersoluble by the choice of G. This implies that G itself is p-supersoluble, against the hypothesis. Hence N is a p-group. Suppose that N

is contained in $\Phi(G_p)$, the Frattini subgroup of G_p. Then G_p/N is not cyclic. Hence G/N is p-supersoluble by the choice of G. Moreover, N is also contained in $\Phi(G)$. Since the class of all p-supersoluble groups is a saturated formation, it follows that G is p-supersoluble, contrary to assumption.

Consequently, N is not contained in $\Phi(G_p)$. Let M_1 be a maximal subgroup of G_p such that $NM_1 = G_p$. Let $q \in \pi(G) \setminus \{p\}$ and let G_q be the Sylow q-subgroup of G in $\mathfrak{Z}$. Thus $1 = G_q \cap G_p = G_q \cap NM_1 = (G_q \cap N)(G_q \cap M_1)$. By [6, Chapter A, Lemma 1.2], $(N \cap M_1)G_q = NG_q \cap M_1 G_q$ is a subgroup of G. Futhermore, $(N \cap M_1)G_q \cap N = (N \cap M_1)(G_q \cap N) = N \cap M_1$ is a normal subgroup of $(N \cap M_1)G_q$. Hence G_q normalises $N \cap M_1$. On the other hand, $N \cap M_1$ is a normal subgroup of M_1 and, since N is abelian, is centralised by N. Therefore $N \cap M_1$ is normalised by $NM_1 = G_p$. Consequently, $N \cap M_1$ is a normal subgroup of G properly contained in N. Hence $N \cap M_1 = 1$. But $|G_p : M_1| = |NM_1 : M_1| = |N : N \cap M_1| = p$, hence N has order p. If M_1 were not cyclic, then G/N would be p-supersoluble by the minimal choice of G. Thus, G would be p-supersoluble, against supposition. Therefore, M_1 is cyclic and G/N has cyclic Sylow p-subgroups. This implies that every p-chief factor of G/N is cyclic and G/N is p-supersoluble. Thus, G is p-supersoluble. This final contradiction completes the proof.

Proof (of Theorem 5) We prove that G is p-soluble by induction on the order of G. Applying Step 3 of the proof of [14, Theorem 3.3], G cannot be non-abelian simple. Let M be a maximal normal subgroup of G. Assume that G_p is contained in M. By Lemma 1, $\mathfrak{Z} \cap M$ is a complete set of Sylow subgroups of M and every 2-maximal subgroup of $G_p = M_p \in \mathfrak{Z} \cap M$ is $(\mathfrak{Z} \cap M)$-permutable. By induction, M is p-soluble. Futhermore, G/M is a p'-group. Thus G is p-soluble. Therefore we may assume that p divides $|G/M|$. Then $M_p = M \cap G_p$ is a proper subgroup of G_p. Let S be a maximal subgroup of G_p containing M_p. Suppose that S is cyclic. Then G_pM/M has a cyclic maximal subgroup. By Lemma 1, $\mathfrak{Z}M/M$ is a complete set of Sylow subgroups of G/M and every 2-maximal subgroup of G_pM/M is $\mathfrak{Z}M/M$-permutable. Therefore, by induction, G/M is p-soluble. Futhermore, since M_p is cyclic, we have that M is p-nilpotent by [10, Kapitel IV, Satz 2.8]. Therefore, M is p-soluble and so is G. Hence we may assume that S is not cyclic. Then S has two different maximal subgroups which are $\mathfrak{Z}$-permutable. Thus S is $\mathfrak{Z}$-permutable. Let $q \in \pi(G) \setminus \{p\}$ and let G_q be the Sylow q-subgroup of G in $\mathfrak{Z}$. It follows that $M_q = M \cap G_q$ is a Sylow q-subgroup of M. Now, G_q permutes with S and M. Applying Theorem 2, G_q permutes with $M \cap S = M \cap G_p = M_p$. Hence, $M_pM_q = M_p(M \cap G_q) = M \cap M_pG_q$ is a Hall $\{p,q\}$-subgroup of G. By Theorem 9, M is p-soluble. Consequently, G is p-soluble, as wanted.

Proof (of Theorem 6) Suppose that G is soluble and every 2-maximal subgroup of $G_p \in \mathfrak{Z}$ is $\mathfrak{Z}$-permutable. Assume, arguing by contradiction, that neither G is a p-nilpotent group nor G has an epimorphic image isomorphic to Σ_4. By Corollary 4 and Theorem 5, G is p-soluble.

Let N be a minimal normal subgroup of G. The quotient group G/N inherits the hypothesis of the theorem. Therefore G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, it follows that $N = \mathrm{Soc}(G)$ is a minimal normal subgroup of G which is complemented in G by a core-free maximal

p-nilpotent subgroup of G, M say. Moreover, $C_G(N) = N$ and N is a p-group. Hence $N \leq G_p$, the Sylow p-subgroup in $\mathfrak{Z}$. Then $G_p = N(G_p \cap M)$ and there exists a maximal subgroup M_1 of G_p containing $M_p = G_p \cap M$ such that $NM_1 = G_p$. Assume that M_p is a maximal subgroup of G_p. Then $|N| = |G_p : M_p| = p$ and G is p-supersoluble. This implies that G is p-supersoluble, which contradicts our assumption. Therefore M_p is not a maximal subgroup of G and so M_p is contained in a 2-maximal subgroup S of $G_p = NS$. Let $p \neq q \in \pi(G)$. Thus $1 = G_q \cap G_p = G_q \cap NS = (G_q \cap N)(G_q \cap S)$. By [6, Chapter A, Lemma 1.2], $(N \cap S)G_q = NG_q \cap SG_q$ is a subgroup of G. In particular, $(N \cap S)G_q \cap N = (N \cap S)(G_q \cap N) = N \cap S$ is a normal subgroup of $(N \cap S)G_q$ and G_q normalises $N \cap S$. On the other hand, $N \cap S$ is a normal subgroup of G_p. Consequently, $N \cap S$ is a normal subgroup of G properly contained in N. Hence $N \cap S = 1$ and so $|N| = p^2$. Since $C_G(N) = N$, it follows that q divides $p + 1$. Hence $p = 2$ and G/N is isomorphic to Σ_3. Consequently, G is isomorphic to Σ_4. This contradiction proves the theorem.

Our hypothesis in the next two theorems is that subgroups of G_p with order p or 4 (if $p = 2$) are $\mathfrak{Z}$-permutable. Let us collect together the arguments common to these two results.

Every subgroup of $G_p O_{p'}(G)/O_{p'}(G)$ of order p or 4 (if $p = 2$) is of the form $TO_{p'}(G)/O_{p'}(G)$ for some subgroup T of G_p with order p or 4 (if $p = 2$). Then, by Lemma 1, every subgroup of $G_p O_{p'}(G)/O_{p'}(G)$ is $\mathfrak{Z}O_{p'}(G)/O_{p'}(G)$-permutable, Hence, arguing by induction or minimal counterexample, we assume that $O_{p'}(G) = 1$. Hence $F(G)$, the Fitting subgroup of G, is a p-group.

Assume that $1 \neq F(G)$ and let z be an element of $Z\big(F(G)\big)$ of order p and let y be an element of order p of $F(G)$. Then $\langle z, y\rangle$ is an elementary abelian subgroup of G_p and G_q normalises $\langle w\rangle$ for each $w \in \langle z, y\rangle$ and each $q \neq p$ because $\langle w\rangle = \langle w\rangle G_q \cap F(G)$ is a normal subgroup of $\langle w\rangle G_q$. Hence p'-elements of G induce power automorphims in the abelian socle S of G. Applying [3, Lemma 2.1.3], all the G-chief factors of G below S are cyclic and G-isomorphic.

If N is a central minimal normal subgroup of G, then $\Omega_1\big(O_p(G)\big)$ is centralised by all p'-elements of G. Futhermore, if $p = 2$, every subgroup Z of order 4 of $F(G)$ is normalised by every $2'$-element of G. Since the automorphism group of Z is of order 2, it follows that $O^2(G)$ centralises every subgroup with order 2 and order 4 of $F(G)$. In this case, we can apply [10, IV, Satz 5.12], to conclude that $O^p(G)$ centralises $F(G)$.

If G is p-soluble, then $C_G(F(G)) \leq F(G)$ by [10, VI, Hilfssatz 6.5]. Consequently, G is a p-group.

Proof (of Theorem 7) Assume that all subgroups of $G_p \in \mathfrak{Z}$ with order p and order 4 (if $p = 2$), with G a p-soluble, non-p-supersoluble group of the smallest possible order, are $\mathfrak{Z}$-permutable.

By the above arguments, p is odd, $O_{p'}(G) = 1$. Since G is p-soluble, it follows that S, the abelian socle of G, is just $\mathrm{Soc}(G)$ and every minimal normal subgroup of G is not central in G and has order p. Let N be one of them. Then $C_G(N)$ is a proper normal subgroup of G. Let M be a maximal normal subgroup of G containing $C_G(N)$. Since N has order p, $G/C_G(N)$ is a cyclic group of order dividing $p - 1$. In particular, $|G : M|$ is a p'-group. Since $O_{p'}(M) \leq O_{p'}(G) = 1$, it follows that $O_{p',p}(M) = O_p(M)$. The minimal choice of G implies that M is a p-supersoluble group. Hence $M/\mathrm{O}_p(M)$ is an abelian group of exponent dividing $p-1$. Therefore $\mathrm{O}_p(M)$ is a Sylow p-subgroup of G. In particular, $\mathrm{O}_p(M) = G_p$ is a normal subgroup of G.

Since G is not p-supersoluble, then G contains a minimal non-p-supersoluble subgroup H. Hence H is one of the groups of [2, Theorem 9]. We will follow the notation of this paper.

Assume that $|H|$ is divisible only by two primes, p and q. Then the Sylow p-subgroup H_p of H is contained in G_p. The Sylow q-subgroup H_q of H is contained in a conjugate G_q^x of G_q, with $x \in G$. By taking $H^{x^{-1}}$ if necessary, we can assume that H_q is contained in G_q. Let x be an element of order p of H. Then $\langle x\rangle G_q$ is a subgroup of G. Now $\langle x\rangle G_q \cap G_p = \langle x\rangle(G_q \cap G_p) = \langle x\rangle$ is a normal subgroup of $\langle x\rangle G_q$. In particular, H_q normalises $\langle x\rangle$. This rules out the groups of types 2, 4, 6, 8, and 10. Moreover, every element of order a power of q acts in the same way on all elements of order p. This rules out the groups of type 3, 5, 7, and 9, since there are elements x of H_p such that H_q does not normalise $\langle x\rangle$. Suppose that H is a group of type 1. If $s = 1$, we consider the generator c of C, of order p, which is not normalised by the Sylow q-subgroup H_q. If $s > 1$, then if c is a generator of C, $c^{p^{s-1}}$ has order p and is centralised by H_q, but H_q does not centralise the elements of order p of M. This contradicts the fact that H_q induces the same automorphism on all cyclic subgroups of order p.

Assume now that H has order divisible by three primes, p, q, and r. Then H is one of the groups of types 11 or 12. As above, we can assume that the Sylow q-subgroup H_q is contained in G_q. As before, G_q normalises $\langle x\rangle$ for each $x \in G_p$, in particular, H_q normalises $\langle x\rangle$ for each $x \in H_p$. If G is a group of type 12, then $H_p = P$ is an extraspecial group of order p^3 and exponent p and the elements of $H_q = M$ act on the cyclic subgroups of P in the same way. This is impossible since M does not centralise P. Assume that G is a group of type 11. There exists $z \in G$ such that the Sylow r-subgroup H_r of H is contained in G_r^z. Let c be the generator of C. Given $y \in G_r$, there exists an integer $t(y)$ such that if x is an element of order p of G_p, $x^y = x^{t(y)}$ for each $y \in G_r$. In particular, given an element x of $H_p^{z^{-1}}$, $x^c = x^{t(c)}$. Since c acts in the same way on all elements of G_p, for every element x of H_p, $x^c = x^{t(c)}$. But this implies that MC acts as a group of power automorphisms on P, in particular, MC acts as an abelian group on P. This implies that H is p-supersoluble, a contradiction.

Proof (of Theorem 8) Let G be a group in which every cyclic subgroup with order p or order 4 (if $p = 2$) of G_p is $\mathfrak{Z}$-permutable. Assume that the order of G_p is greater than p. We prove that G is p-soluble by induction on the order of G. Applying the above arguments, we may assume that $\mathrm{O}_{p'}(G) = 1$ and every abelian minimal normal subgroup of G is of order p.

Let M be a maximal normal subgroup of G. Then, by Lemma 1, M satisfies the hypotheses of the theorem. Therefore either M_p is of order p or M is p-soluble. If M is p-soluble, then M is p-supersoluble by Theorem 7. Since $\mathrm{O}_{p'}(M) \le \mathrm{O}_{p'}(G) = 1$, it follows that $M_p = G_p \cap M$ is a normal Sylow p-subgroup of M by [3, Lemma 2.1.6].

Let A be a maximal normal subgroup of G such that $A \neq M$. Then $G = AM$. Applying [3, Theorem 1.1.19], there exist Sylow p-subgroups A_p and M_p of A and M, respectively, such that A_pM_p is a Sylow p-subgroup of G. If $|A_p| = |M_p| = p$, then

$|G_p| = p^2$ and, by Theorem 3, G is p-supersoluble. Suppose that the order of M_p is greater than p. Then M_p is normal in G. If A were p-supersoluble, then A_p would be also normal in G and so would be G_p. Then G is p-soluble. Assume that $|A_p| = p$. Then M_p is not contained in A and so $G = AM_p$. This means that G/A is of order p and $|G_p| = p^2$. By Theorem 3, G is p-supersoluble. Therefore G is p-soluble.

Therefore we may assume that M is the unique maximal normal subgroup of G. Assume that MG_p is a proper subgroup of G. Then $\mathfrak{Z} \cap MG_p$ is a complete set of Sylow subgroups of MG_p and every subgroup of G_p with order p or order 4 (if $p = 2$) is $(\mathfrak{Z} \cap MG_p)$-permutable. By induction, MG_p is p-soluble and M_p is a normal subgroup of G. Since $\mathrm{O}_{p'}(G) = 1$, it follows that $M_p \neq 1$ and $\mathrm{F}(G) = \mathrm{F}(M)$ is a non-trivial p-group. Suppose that $p = 2$. Since M is 2-soluble, we conclude that M is a 2-group and so G is 2-soluble. Assume that p is odd and every abelian minimal normal subgroup of G is non-central. Let N be one of them. Then $\mathrm{C}_G(N)$ is contained in M and so G/M is a p'-group. Consequently, M_p is a normal Sylow p-subgroup of G and G is p-soluble.

We may therefore assume that $G = MG_p$ and $|G : M| = p$. If M were not p-soluble, then M_p must be of order p and so the order of the Sylow p-subgroups of G would be p^2. By Theorem 3, G is p-supersoluble.

Consequently, in all cases, G is p-soluble and the induction argument is complete.

Proof (of Corollary 5) Assume that G is a group in which every cyclic subgroup with order p or order 4 (if $p = 2$) of G_p is $\mathfrak{Z}$-permutable. If the order of G_p is greater than p, then G is p-soluble by Theorem 8. If the order of G_p is p, then G has Hall $\{p,q\}$-subgroups for all $q \in \pi(G)$. By Theorem 9, G is p-soluble. In both cases, we have that G is p-soluble. Applying Theorem 7, G is p-supersoluble.

Acknowledgements

This Project was funded by the Deanship of Scientific Research (DSR), at King Abdulaziz University, Jeddah, under grant no. (1/31/RG). The authors, therefore, acknowledge with thanks DSR technical and financial support. The second and the third authors have been supported by the research project MTM2010-19938-C03-01 from the *Ministerio de Ciencia e Innovación*, Spain, and the first author has also been supported by the National Natural Science Foundation of China (Grant No. 11271085).

We also thank E. Vdovin for providing us with a proof of Theorem 9.

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