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Generalized independent families and dense sets of Box-Product spaces

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Abstract. A generalization of independent families on a set S is introduced, based on which various topologies on S can be defined. In fact, the set S with any such topology is homeomorphic to a dense subset of the corresponding box product space (Theorem 2.2). From these results, a general version of the Hewitt-Marczewski-Pondiczery theorem for box product spaces can be established. For any uncountable regular cardinal θ, the existence of maximal generalized independent families with some simple conditions, and hence the existence of irresolvable dense subsets of θ-box product spaces of discrete spaces of small sizes, implies the consistency of the existence of measurable cardinal (Theorem 4.5).

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1. Introduction

Following notation in [5], a (θ, κ)-*independent family* on S is a subfamily $\mathcal{I} \subseteq \mathcal{P}(S)$ such that for any two disjoint subfamilies $\mathcal{I}_0, \mathcal{I}_1 \subseteq \mathcal{I}$ with $|\mathcal{I}_0 \cup \mathcal{I}_1| < \theta$, the set $\bigcap\{A : A \in \mathcal{I}_0\} \cap \bigcap\{S \setminus A : A \in \mathcal{I}_1\}$ has cardinality κ. Given a space $\langle X, \mathcal{T}\rangle$, it is *irresolvable* ([9]) if X does not have two disjoint dense subsets. Following [3], let $S(\langle X, \mathcal{T}\rangle)$ be the smallest cardinal κ such that every family of pairwise disjoint nonempty open sets has size strictly less than κ. Please refer to [10] about cardinals and ideals, and [6] and [3] for topological terminologies.

The Hewitt-Marczewski-Pondiczery theorem and Hausdorff's theorem (i.e., there are uniformly independent families of size 2^κ on any set S of size κ. See [8], [6] for more details) are equivalent, since each separated (θ, κ)-independent family of size $2^{|S|}$ on a set S induces a Tychonoff topology on S which is homeomorphic to a dense subset of $\{0,1\}^{2^{|S|}}$. Such kind of topologies induced by independent families appeared in a different form in van Douwen's paper [4] and then the paper [5] by F.W. Eckertson.

On the other hand, Kunen [11] established the equiconsistency between the existence of maximal σ-independent family and the existence of measurable cardinals. Later Kunen, Szymanski and Tall in [12] (see also [14]) studied the properties of the ideal of nowhere dense subsets of a λ-Baire irresolvable space, and also gave a method to construct a λ-Baire open-hereditarily irresolvable (the term "strongly irresolvable" was used. We follow the notation in [4]) topology from a λ-complete ideal with a lifting. In [14], it was shown that a λ-complete ideal on λ with certain conditions has a lifting.

In this paper, we study a generalization of independent families and its relation to box product spaces. We provide a generalization of the equivalence between Hausdorff's theorem and the Hewitt-Marczewski-Pondiczery theorem to generalized independent families and dense subsets of box product spaces (Theorem 3.2) (See also [7]). We show, in Section 2, that various topologies can be defined on a set S by any generalized independent family on S, and any such topology is homeomorphic to a dense subset of the corresponding θ-box product spaces. This general equivalence enables us to obtain similar work (Section 4) like that in [11] and [12] by substituting Baire irresolvable dense subsets with irresolvable dense subsets of box product spaces.

2. Generalized independent families and induced topologies

An independent family can be viewed as a family of partitions on some set S, in which each partition consists of two subsets. In general, we can consider the following generalized version.

Definition 2.1. *Let $\mathcal{I}= \{\{I_\alpha^\beta : \beta < \lambda_\alpha\} : \alpha < \tau\}$ be a family of partitions on an infinite set S with each $\lambda_\alpha \geq 2$, and let $\kappa, \lambda, \theta \geq \omega$ be three cardinals.*

- *If for any $J \in [\tau]^{<\theta}$ and any $f \in \Pi_{\alpha\in J}\lambda_\alpha$ the intersection $\cap\{I_\alpha^{f(\alpha)} : \alpha \in J\}$ has size at least κ, then $\mathcal{I}$ is called a "(θ,κ)-generalized independent family" on S, and a "(θ,κ,λ)-generalized independent family" when $\lambda_\alpha = \lambda$ for all $\alpha < \tau$.*
- *$\mathcal{I}$ is called "separated" if for any $\{x, y\} \in [S]^2$, there exists an $\alpha < \tau$ and $\beta < \lambda_\alpha$ such that $x \in I_\alpha^\beta$ and $y \notin I_\alpha^\beta$.*

A $(\theta,\kappa,2)$-generalized independent family is a (θ,κ)-independent family, and a σ-independent family defined in [11] is an $(\omega_1, 1)$-independent family.

Let $\mathcal{I}= \{\{I_\alpha^\beta : \beta < \lambda_\alpha\} : \alpha < \tau\}$ be a (θ,κ)-generalized independent family on some infinite set S, and let $\{\langle X_\alpha, \mathcal{T}_\alpha\rangle : \alpha < \tau\}$ be a family of topological spaces such that $|X_\alpha| = \lambda_\alpha$ for each $\alpha < \tau$. For each $\alpha < \tau$, index the α-th partition of $\mathcal{I}$ by $\{I_\alpha^x : x \in X_\alpha\}$, and for each nonempty open subset $U \in \mathcal{T}_\alpha$, define $B_\alpha^U = \bigcup\{I_\alpha^x : x \in U\}$. Set $\mathcal{B}_\alpha := \{B_\alpha^U : \varnothing \neq U \in \mathcal{T}_\alpha\}$. The family $\mathcal{B}_\alpha$ is a sub-base for a topology on S. We denote it by S_{X_α}, and we use $\mathcal{I}_{\{X_\alpha\}}$ to denote the topology generated by $\{S_{X_\alpha} : \alpha < \tau\}$. When each $\langle X_\alpha, \mathcal{T}_\alpha\rangle$ is discrete, the topology $\mathcal{I}_{\{X_\alpha\}}$ is called "*the simple topology*" induced by $\mathcal{I}$.

It is clear that $\langle S, \mathcal{I}_{\{X_\alpha\}}\rangle$ is a P_θ-space whenever θ is regular. The space is Hausdorff if $\mathcal{I}$ is separated and each $\langle X_\alpha, \mathcal{T}_\alpha\rangle$ is Hausdorff, and zero-dimensional if in addition each $\langle X_\alpha, \mathcal{T}_\alpha\rangle$ is zero-dimensional. In the rest of this section, we only consider Hausdorff spaces and separated families.

Theorem 2.2. *Let $\mathcal{I}$ and $\{\langle X_\alpha, \mathcal{T}_\alpha\rangle : \alpha < \tau\}$ be as above. Any space $\langle S, \mathcal{I}_{\{X_\alpha\}}\rangle$ is homeomorphic to a dense subset of $\square^\tau_\theta\langle X_\alpha, \mathcal{T}_\alpha\rangle$*

Proof. For each $\alpha < \tau$, define $f_\alpha : \langle S, \mathcal{I}_{\{X_\alpha\}}\rangle \to \langle X_\alpha, \mathcal{T}_\alpha\rangle$ such that $f_\alpha(I^x_\alpha) = x$. By our definition of $\mathcal{I}_{\{X_\alpha\}}$, we know that f_α is a continuous map. Since $\mathcal{I}$ is separated, the family $\{f_\alpha : \alpha < \tau\}$ separates points in $\langle S, \mathcal{I}_{\{X_\alpha\}}\rangle$.

Consider the map $f = \Delta_{\alpha<\tau} f_\alpha : \langle S, \mathcal{I}_{\{X_\alpha\}}\rangle \to \square^\tau_\theta\langle X_\alpha, \mathcal{T}_\alpha\rangle$ such that $f(s) = \{f_\alpha(s)\}_{\alpha<\tau}$ for all $s \in S$. Certainly f is a one-one map, and f separates points. We need to show the following: (1) f is continuous; (2) the range of f is dense in $\square^\tau_\theta\langle X_\alpha, \mathcal{T}_\alpha\rangle$; (3) f separates points and closed sets.

To see that f is continuous, it is enough to show that for any set $A \in [\tau]^{<\theta}$ and any family $\{\varnothing \neq U_\alpha \in \mathcal{T}_\alpha : \alpha \in A\}$ of nonempty open sets, the pre-image of the corresponding open set of $\square_{\alpha\in A} U_\alpha$ is open. By the definition of f, a point $s \in S$ is in the pre-image of that open set if and only if $f_\alpha(s) \in U_\alpha$ for all $\alpha \in A$, and $f_\alpha(s) \in U_\alpha$ if and only if there exists some $x \in U_\alpha$ such that $s \in I^x_\alpha \subseteq B^{U_\alpha}_\alpha$. Hence $s \in \bigcap\{B^{U_\alpha}_\alpha : \alpha \in A\} \in \mathcal{I}_{\{X_\alpha\}}$. Therefore the pre-image of $\square_{\alpha\in A} U_\alpha$ is $\bigcap\{B^{U_\alpha}_\alpha : \alpha \in A\}$, which is open in $\mathcal{I}_{\{X_\alpha\}}$.

For (2), we need to show that there exists a point $s \in S$ such that $f(s)$ is in the corresponding open set of $\square_{\alpha\in A} U_\alpha$. Using the same argument as above, it is enough to show that $\bigcap\{B^{U_\alpha}_\alpha : \alpha \in A\}$ is nonempty. Since $\mathcal{I}$ is a θ-generalized independent family, it is clear that $\bigcap\{B^{U_\alpha}_\alpha : \alpha \in A\} \neq \varnothing$. Hence the range of f is dense in $\square^\tau_\theta\langle X_\alpha, \mathcal{T}_\alpha\rangle$.

It remains to show that f separates points and closed sets. Let s be a point and let F be a closed subset in $\langle S, \mathcal{I}_{\{X_\alpha\}}\rangle$ such that $s \notin F$. Since $\mathcal{B}$ is a base for $\mathcal{I}_{\{X_\alpha\}}$, for some set $A \in [\tau]^{<\theta}$ and some family $\{\varnothing \neq U_\alpha \in \mathcal{T}_\alpha : \alpha \in A\}$, we have $s \in \bigcap\{B^{U_\alpha}_\alpha : \alpha \in A\} \subseteq F^c$. Obviously $f(s)$ is in the corresponding open set of $\square_{\alpha\in A} U_\alpha$. We show that the corresponding open set of $\square_{\alpha\in A} U_\alpha$ is disjoint from $f(F)$. Since the projection into any $|A| < \theta$ many products is open and continuous, it suffices to prove that $\square_{\alpha\in A} U_\alpha \cap \Delta_{\alpha\in A} f_\alpha(F) = \varnothing$. But this can be proved by the same argument used before: if $f_\alpha(t) \in U_\alpha$ for some $t \in F$, then $t \in I^x_\alpha$ for some $x \in U_\alpha$ and hence $t \in B^{U_\alpha}_\alpha$, which implies that $t \in \bigcap\{B^{U_\alpha}_\alpha : \alpha \in A\} \subseteq F^c$, contradicting our early assumption.

Since $\{f\}$ is continuous, separates points, and separates points and closed sets, it is a homeomorphism onto its range. It maps $\langle S, \mathcal{I}_{\{X_\alpha\}}\rangle$ onto a dense subset of $\square^\tau_\theta\langle X_\alpha, \mathcal{T}_\alpha\rangle$. □

The following corollary is clear.

Corollary 2.3. *Let $\mathcal{I} = \{\{I^\beta_\alpha : \beta < \lambda_\alpha\} : \alpha < \tau\}$ be a (θ, κ)-generalized independent family on S, and let $\{\langle X_\alpha, \mathcal{T}_\alpha\rangle : \alpha < \tau\}$ be a family of topological spaces such that $d(\langle X_\alpha, \mathcal{T}_\alpha\rangle) \leq \lambda_\alpha$ for all $\alpha < \tau$. Then $d(\square^\tau_\theta\langle X_\alpha, \mathcal{T}_\alpha\rangle) \leq |S|$.*

The converse of Theorem 2.2 can be established for box product spaces of discrete spaces.

Theorem 2.4. *For any dense subset D in $\Box_\theta^\tau D(\lambda_\alpha)$, there exists a $(\theta,1)$-generalized independent family $\mathcal{I}$ on D. The set D is irresolvable if and only if $\mathcal{I}$ is a maximal $(\theta,1)$-generalized independent family.*

3. The Hewitt-Marczewski-Pondiczery theorem for box product spaces

Definition 3.1. *Let κ,θ,λ be two cardinals with κ,θ infinite. Let S be an infinite set of size κ. The cardinal $i(\kappa,\theta,\lambda)$ is the smallest cardinal τ such that there are no $(\theta,1,\lambda)$-generalized independent families on S of size τ.*

The following generalizes the Hewitt-Marczewski- Pondiczery theorem.

Theorem 3.2. *Let S be a set and let θ,τ,λ be three cardinals with θ infinite. Then the following are equivalent.*

- $\tau < i(|S|,\theta,\lambda)$.
- $d(\Box_\theta^\tau\langle X_\alpha,\mathcal{T}_\alpha\rangle) \le |S|$ *holds for any family of topological spaces* $\{\langle X_\alpha,\mathcal{T}_\alpha\rangle : \alpha<\tau\}$ *with each* $d(X_\alpha)\le\lambda$.

Proof. (1)→(2). By Corollary 2.3. (2)→ (1). Let D be a dense subset of $\Box_\theta^\tau D(\lambda)$ such that $|D|=|S|$. For each $\alpha<\tau$ and $\beta<\lambda$, let $I_\alpha^\beta = D\cap\{\{x_\zeta\}_{\zeta<\tau} \in \Box_\theta^\tau D(\lambda) : x_\alpha=\beta\}$. Then the family $\mathcal{I}= \{\{I_\alpha^\beta : \beta<\lambda\} : \alpha<\tau\}$ is a $(\theta,1,\lambda)$-independent family on D. Hence there is a $(\theta,1,\lambda)$-independent family of size τ on S, which implies $\tau < i(|S|,\theta,\lambda)$. □

Comfort and Negrepontis in [2] showed that $|S|^{<\theta}=|S|$ is equivalent to the statement that there exists a subfamily of S^S of size $2^{|S|}$ that is of θ-large oscillation, which implies the existence of a $(\theta,1,|S|)$-independent family of size $2^{|S|}$ on S. On the other hand, assuming there exists a $(\theta,1,|S|)$-independent family $\mathcal{I}$ of size $2^{|S|}$ on S, for each $\beta<2^{|S|}$ let $f_\beta : S\to S$ be such that $f_\beta(I_\beta^s)=s$ for each $s\in S$. Then the family $\{f_\beta : \beta<2^{|S|}\}$ is a family of θ-large oscillation. Hence, we have the following theorem.

Theorem 3.3. $i(|S|,\theta,|S|)=(2^{|S|})^+$ *if and only if* $|S|^{<\theta}=|S|$.

We show in the following theorem that, in general, the cardinal $i(|S|,\theta,|S|)$ is regular.

Theorem 3.4. *Let θ, λ be two infinite cardinals such that $\theta\le\lambda$. Then $i(\lambda,\theta,\lambda)$ is regular.*

Proof. Let $\tau < i(\lambda,\theta,\lambda)$ and let $\{\tau_\alpha : \alpha<\tau\}$ be cardinals such that $\tau_\alpha < i(\lambda,\theta,\lambda)$. Let also $\mu=\sup\{\tau_\alpha : \alpha<\tau\}$. By Theorem 3.2, for each $\alpha<\tau$, the box product $\Box_\theta^{\tau_\alpha}\lambda$ has density λ. By Theorem 3.2 again, the space $\Box_\theta^\mu\lambda = \Box_\theta^\tau(\Box_\theta^{\tau_\alpha}\lambda)$ has density λ. Hence $\mu < i(\lambda,\theta,\lambda)$ according to Theorem 3.2. □

It is clear that for any infinite set S and two cardinals $\lambda_1\le\lambda_2$, we have $(2^{|S|})^+ \ge i(|S|,\theta,\lambda_1) \ge i(|S|,\theta,\lambda_2)$. When $|S^{<\theta}|=|S|$, we have $i(|S|,\theta,\lambda) = i(|S|,\theta,|S|)=(2^{|S|})^+$ for any $\lambda<|S|$.

4. Maximal generalized independent families

The simple topology induced by a maximal $(\theta, 1)$-generalized independent family is irresolvable. Similarly, the simple topology induced by a maximal $(\theta, 1, \lambda)$-independent family is λ-irresolvable. In this section, we show that for any uncountable regular cardinal θ, the existence of maximal $(\theta, 1)$-generalized independent families with some simple conditions (equivalently, the existence of irresolvable dense subsets of θ-box product spaces with some simple conditions) implies the consistency of the existence of measurable cardinals.

Lemma 4.1. *Suppose $\langle X, \mathcal{T}\rangle$ is an open-hereditarily irresolvable space and $\mathcal{T}$ is a P_θ-topology for some regular cardinal θ. Let $\mathcal{N}$ denote the ideal of nowhere dense subsets, and let λ be the smallest cardinal such that $\mathcal{N}$ is not λ-complete. Then for any $\gamma < \gamma^+ < \lambda$ and $\beta < \theta$, $\mathcal{N}$ is $(\gamma^\beta)^+$-complete.*

Proof. Since the topology is open hereditarily irresolvable, $\mathcal{N}= \{A \subseteq S : A^o = \varnothing\}$. For a contradiction, let us assume that there exists $Y_f \in \mathcal{N}$ for each $f \in \gamma^\beta$ such that the Y_f are disjoint and $\bigcup_f Y_f \supseteq U$ for some nonempty open set U. We claim that there exists some member $Y_g \notin \mathcal{N}$. Inductively define $g : \beta \to \gamma$ and a decreasing chain of non-empty basic open sets $\{U^\zeta : \zeta < \beta\}$ so that

- $U^0 = U$,
- $U^\zeta = \bigcap\{U^\eta : \eta < \zeta\}$,
- $U^{\zeta+1} \subseteq U^\zeta$ and $U^{\zeta+1} \subseteq \bigcup\{Y_f : f(\zeta) = g(\zeta)\}$.

When $\zeta < \theta$ is a limit, the set $\bigcap\{U^\eta : \eta < \zeta\}$ defined in (2) is a non-empty open set, since $\mathcal{T}$ is a P_θ-topology. For (3), we have γ-many disjoint sets $\{N_\alpha = \bigcup\{Y_f : f(\zeta) = \alpha\} : \alpha \in \gamma\}$. The union of these sets contains U and hence U^ζ. Since the topology is open hereditarily irresolvable and $\mathcal{N}$ is γ^+-complete, one of these sets $\{N_\alpha \cap U^\zeta : \alpha < \gamma\}$, say $N_\alpha \cap U^\zeta$, has non-empty interior $U^{\zeta+1}$. Set $g(\zeta + 1) = \alpha$.

We have $\bigcap\{U^\zeta : \zeta < \beta\} \subseteq \bigcap_{\zeta<\beta} \bigcup\{Y_f : f(\zeta) = g(\zeta)\} = Y_g$ contradicting $Y_g \in \mathcal{N}$. □

In [2], Comfort and Negrepontis introduced the notion of strongly θ-inaccessible: a cardinal λ is called *strongly θ-inaccessible* if $\beta^\gamma < \lambda$ whenever $\beta < \lambda$ and $\gamma < \theta$. Given a cardinal θ, denote by θ_{in} the smallest cardinal λ such that λ is strongly θ-inaccessible.

Lemma 4.2. *Let everything be as in Lemma 4.1. Then*

- *λ is regular;*
- *if λ is a successor cardinal, say $\lambda = \lambda'^+$, then λ' is strongly θ-inaccessible.*
- *if λ is a limit cardinal, then λ is strongly θ-inaccessible*

Proof. (1) is trivial. If $\lambda = \lambda'^+$, then for any $\gamma < \lambda'$ and $\beta < \theta$, $(\gamma^\beta)^+ \leq \lambda'$ (Lemma 4.1), and hence $(\gamma^\beta) < \lambda'$. This gives (2). (3) is trivial. □

Let everything be as in Lemma 4.1. Let us assume further that $S(\langle X, \mathcal{T}\rangle) \leq \lambda$ with λ defined in Lemma 4.1. Then it is easy to see that the ideal $\mathcal{N}$ is

λ-saturated. Under these assumption, there exists a λ-saturated (hence λ^+-saturated) λ-complete ideal over λ (using the proof of Lemma 27.1 in [10]). Lemma 35.10 and Theorem 86 in [10] show that λ is a measurable cardinal in some model of ZFC.

In the following we show that for any uncountable regular cardinal θ, if there exists a maximal $(\theta, 1)$-generalized independent family with some conditions, then the induced simple topology satisfies above conditions.

Theorem 4.3. *Let θ be a regular cardinal. Suppose there exists a maximal $(\theta, 1)$-generalized independent family $\mathcal{I} = \{\{I_\alpha^\beta : \beta < \lambda_\alpha\} : \alpha < \tau\}$ on a set S with each $\lambda_\alpha < \theta_{in}$. Let $\mathcal{N}$ be the ideal of nowhere dense set of the simple topology induced by $\mathcal{I}$ and let λ be the smallest cardinal such that $\mathcal{I}$ is not λ-complete. Then*

- *there is a nonempty open set U of the simple topology such that U with the subspace topology satisfies all conditions in Lemma 4.1 and the ideal $\mathcal{I}_U$ of nowhere dense set of U is λ-saturated; and*
- $2^{<\theta} = \theta$

Proof. (i) Let $\langle S, \mathcal{T}\rangle$ be the simple topology induced by $\mathcal{I}$. Since $\mathcal{I}$ is a maximal $(\theta, 1)$-generalized independent family, it is irresolvable. Using a standard argument ([9]), there is a nonempty basic open set U the subspace topology on which is hereditarily irresolvable.

Let $\mathcal{N}_U$ be the set of all nowhere dense subsets in $\langle U, \mathcal{T}\rangle$, i.e, the set U with the subspace topology inherited from $\mathcal{T}$. By Lemma 4.2, if λ is a limit cardinal, then λ is strongly θ-inaccessible. If λ is a successor cardinal, say $\lambda = \lambda'^+$, then λ' is strongly θ-inaccessible.

Using Theorem 2.3 in [2], we have that $S(\langle S, \mathcal{T}\rangle)$, and hence $S(\langle U, \mathcal{T}\rangle)$, is $\leq \lambda$ if λ is a limit cardinal, and $< \lambda$ otherwise. Hence $\mathcal{N}_U$ is λ-saturated.

(ii) The proof here uses a similar argument as that of Lemma 1.4 in [11]. For each $\theta' < \theta$ we produce a map from θ onto $2^{\theta'}$.

Index θ as $A \times B$ with $A = \{a_\eta : \eta < \theta\}$ and $B = \{b_\zeta : \zeta < \theta'\}$. Consider the family $\{I_\alpha^0 : \alpha < \theta\} = \{I_{a_\eta b_\zeta}^0 : \eta < \theta, \zeta < \theta'\}$. For each $x \in X$, define $\phi_x : \theta \to 2^{\theta'}$ so that $\phi_x(\eta)(\zeta) = 1$ if and only if $x \in I_{a_\eta b_\zeta}^0$. For each $f \in 2^{\theta'}$, let R_f be $\{x \in X : f \notin range(\phi_x)\} = \{x \in X : f \neq \phi_x(\eta) \textit{ for all } \eta < \theta\}$. We show that $\bigcap_{f \in 2^{\theta'}} (X \setminus R_f) \neq \varnothing$ by proving $R_f \in \mathcal{N}$ and applying Lemma 4.1, which shows that for some x, ϕ_x is onto.

Suppose that R_f contains $U = \bigcap\{U_\alpha^{\sigma(\alpha)} : \alpha \in A\}$, a non-empty basic open set, for some set $A \in [\tau]^{<\theta}$ and some $\sigma \in \Pi_{\alpha \in A}\lambda_\alpha$. Then there is an $\eta < \theta$ such that $A \cap \cup\{(a_\eta, b_\zeta) : \zeta < \theta'\} = \varnothing$. Now consider the open set $U' = U \cap \{I_{a_\eta b_\zeta}^0 : \zeta < \theta' \textit{ and } f(\zeta) = 1\} \cap \{S \setminus I_{a_\eta b_\zeta}^0 : \zeta < \theta' \textit{ and } f(\zeta) = 0\}$. It is clear that $U' \neq \varnothing$ and $U' \subseteq \{s \in S : \phi_s(\eta) = f\} \cap U \subseteq (S \setminus R_f) \cap U$, a contradiction. ∎

The following theorem is a direct corollary of Theorem 4.3.

Corollary 4.4. *For any uncountable regular cardinal* θ*, the existence of a maximal* $(\theta, 1)$*-generalized independent family* $\mathcal{I} = \{\{I_\alpha^\beta : \beta < \lambda_\alpha\} : \alpha < \tau\}$ *on a set* S *with each* $\lambda_\alpha < \theta_{in}$ *implies the consistency of the existence of measurable cardinals, and* $2^{<\theta} = \theta$*.*

The corresponding conclusion is about the existence of irresolvable dense subsets in a θ-box product space.

Theorem 4.5. *Let* θ *be an uncountable regular cardinal, and let* $\{\lambda_\alpha \geq 2 : \alpha < \tau\}$ *be a family of cardinals with each* $\lambda_\alpha < \theta_{in}$*.*

If there exists an irresolvable dense subset S *of the* θ*-box product space* $\square_\theta^\tau D(\lambda_\alpha)$*, then*

- *It is consistent that there exists a measurable cardinal; and*
- $2^{<\theta} = \theta$*.*

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