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Stone compactification of additive generalized-algebraic lattices

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Abstract. In this paper, the notions of regular, completely regular, compact additive generalized algebraic lattices ([7]) are introduced, and Stone compactification is constructed. The following theorem is also obtained.
Theorem: An additive generalized algebraic lattice has a Stone compactification if and only if it is regular and completely regular.

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1. Introduction

The notions of directed sets, way-below relations, continuous lattices, algebraic lattices were introduced in [12], and applied in the study of domain theory, topological theory, lattice theory, etc..

As a generalization, D. Novak introduced the notions of generalized continuous lattices (M-continuous lattices) and generalized algebraic lattices in [15].

In the study of topological theory and lattice theory, many researchers are interested in the topological representation of a complete lattice. For example: suppose (X, T) is a topological space, all open sets T of a topological space may be viewed as a frame, and a frame may also be viewed as an open sets lattice. For the theory of Frame (Locale), please refer to [13].

On the other hand, suppose (X, C) is a co-topological space. C is the set of all closed subsets. D. Drake, W. J. Thron and S. Papert considered C as a complete lattice $(C, \cup, \cap, \varnothing, X)$([11, 16]). But unfortunately the correspondence between complete lattices and T_0-topological spaces is not one-to-one.

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To solve the problem, Deng also investigated generalized continuous lattices on the basis of [1, 11, 15, 16]. He introduced the notions of maximal systems of subsets, additivity property, homomorphisms, direct sums, lower sublattices in [5, 6, 9, 10]. Finally, the question was settled in [7, 8]. He obtained the equivalence between the category of additive generalized algebraic lattices with lower homomorphisms and the category of T_0-topological spaces with continuous mappings.

This paper is a sequel of [2, 7, 8]. In Section 2, we begin with an overview of generalized continuous lattices, Deng's work, which surveys Preliminaries. In Section 3, we introduce the notions of regular, completely regular and compactness on an additive generalized algebraic lattice, and obtain a Stone compactification.

2. Preliminaries

We introduce some notions for each area, i.e., generalized continuous lattices and additive generalized algebraic lattices.

2.1. Generalized Continuous Lattices.

In [15], D. Novak introduced the notions of generalized way-below relations and systems of subsets.

Let $(P, \leq)$ be a complete lattice, $\prec$ is said to be a generalized way-below relation if (i) $a \prec b \Rightarrow a \leq b$, (ii) $a \leq b \prec c \leq d \Rightarrow a \prec d$.

Obviously, it is a natural generalization of the notion of a way-below relation ([12]).

$M \subseteq 2^P$ is said to be a system of subsets of P, if for $a \in P$, there exists $S \in M$, such that $\downarrow a = \downarrow S$, where $\downarrow a = \{b \mid b \leq a\}$, $\downarrow S = \cup\{\downarrow a \mid a \in S\}$. There are three kinds of common used system of subsets: (i) the system of all finite subsets, (ii) the system of all directed sets and (iii) the system of all subsets.

By means of the notion of systems of subsets, he defined a generalized way-below relation. Suppose M is a system of subsets. For $a, b \in P$, a is said to be way-below b with respect to M, in symbols $a \prec_M b$, if for every $S \in M$ with $b \leq \vee S$, then $a \in \downarrow S$.

Clearly $\prec_M$ is a generalized way-below relation induced by M ([15]). We will denote $\prec_M$ as $\prec$.

$(P, \prec)$ is called a generalized continuous lattice, if for every $a \in P$, we have $a = \vee \Downarrow a$, where $\Downarrow a = \{b \mid b \prec a\}$.

$a \in P$ is called a compact element, if $a \prec a$. Let $K(\prec) = \{a \in P \mid a \prec a\}$. $(P, \prec)$ is called a generalized algebraic lattice, if for every $a \in P$, we have $a = \vee\{\downarrow a \cap K(\prec)\}$. Further study, see [1, 17].

2.2. Additive Generalized Algebraic Lattices.

Suppose $(P, \prec)$ is a generalized continuous lattice, Deng introduced the notion of a maximal system of subsets generated by $\prec$, that is,

$$M(\prec) = \{S \subseteq P \mid \forall a \in P \text{ with } a \prec \vee S, \text{ then } a \in \downarrow S\}.$$

Suppose $(P, \prec)$ is an generalized algebraic lattice, Deng defined a new property: $(P, \prec)$ is additivity, if for $a, b, c \in P$ with $a \prec b \vee c$ implies $a \prec b$ or $a \prec c$ ([7]).

He investigated the connection between additive generalized algebraic lattices and T_0-topological spaces as follows.

From the one direction, suppose $(P, \prec)$ is an generalized algebraic lattice, let $X = K(\prec)$, and $T : P \to 2^X$, $T(a) = \downarrow a \cap K(\prec)$. If $(P, \prec)$ is additivity, then T satisfies: (1) $T(0) = \varnothing$, (2) $T(1) = X$, (3) for $S \in M(\prec) = M(K(\prec))$, $T(\vee S) = \cup T(S)$, (4) for $S \subseteq P$, $T(\wedge S) = \cap T(S)$, (5) $T(a \vee b) = T(a) \cup T(b)$.

Let $C = T(P)$, then (X, C) is a T_0 co-topological space, and $(P, \prec)$ is isomorphic to (X, C), see [7].

From the other direction, we assume (X, C) is a co-topological space, let $Q = \{\{x\}^- \mid x \in X\}$ be the collection of closure of all singletons. Clearly Q is a $\vee-$base for C, i.e., $a \in C$, a is a closed subset, we have $a = \vee \downarrow a$.

$M(Q) = \{S \mid S \subseteq X, \text{ for } a \in Q, \ a \leq \vee S \text{ we have } a \in \downarrow S\}$ is a system of subsets induced by Q, then $(C, \prec_{M(Q)})$ is an additive generalized algebraic lattice with $K(\prec_{M(Q)}) = Q$. In this case, $a \prec_{M(Q)} b$ for $a, b \in C$ if and only if $a \subseteq \{x\}^-$ for some $x \in b$. It is clearly that $\prec_{M(Q)}$ is the specialization order ([12]) which is essentially in topological and domain theory.

Suppose $(P_1, \prec_1)$, $(P_2, \prec_2)$ are two generalized continuous lattices, and $h : P_1 \to P_2$ is said to be a lower homomorphism if it preserves arbitrary joins and the generalized way-below relations. Thus a lower homomorphism h is residuated. Let g be its upper adjoint, we have (g, h) is a Galois connection ([7]).

The lower homomorphism also corresponds to the closed mapping. So he obtained the equivalence between the category of additive generalized algebraic lattices with lower homomorphisms and the category of T_0-topological spaces with continuous mappings in [7].

From the point of view of Deng's work ([7, 8]), an additive generalized algebraic lattice is algebraic abstraction of a topological space. Thus topological theory may be directly constructed on it. The work will benefit the study of the theory of topological algebra and the possible application about additive generalized algebraic lattices. In [2], we constructed Tietze extension theorem. Furthermore, $(C, \prec_{M(Q)})$ is an example of additive generalized algebraic lattice. For another example in commutative ring, see [7].

For other notions and results cited in this paper, please refer to [7, 15].

3. Stone Compactification

In the section, $(P, \prec)$ denotes an additive generalized algebraic lattice. It is T_0, but not T_1 ([7]). $K(\prec)$ is the set of all compact elements of $(P, \prec)$.

Definition 3.1. *For $a \in P$, $a^* = \wedge\{x | a \vee x = 1\}$*

Note 1. *Since* $(P, \prec)$ *is a complete lattice, we have* $a \vee 1 = 1$ *for every* $a \in P$, *so the existence of* a^* *is obvious.*

Proposition 3.2.

(1) $\forall a \in P, \quad a \vee a^* = 1$
(2) $a \prec b \Rightarrow b^* \leq a^*$
(3) $a \wedge b = 0 \Rightarrow a \leq b^*$

Proof. (1) $\forall y \in K(\prec)$, if $y \prec a$, then $y \prec a \vee a^*$. If $y \not\prec a$, then $\forall x \in \{x \mid a \vee x = 1\}$, $y \prec y \leq 1 = a \vee x$. Since $(P, \prec)$ is additive, we have $y \prec x$, which implies $y \prec a^*$. Hence $y \prec a \vee a^*$. Furthermore $(P, \prec)$ is algebraic, $1 = \vee(\downarrow 1 \cap k(\prec))$, we obtain $a \vee a^*$=1.

(2) It is clear.

(3) $\forall y \prec a$, if $y = 0$, certainly $y \prec b^*$. If $y \neq 0$, $a \wedge b = 0$, so $b \wedge y = 0$, which implies $y \not\prec b$. By (1), we have $y \prec 1 = b \vee b^*$. Since $(P, \prec)$ is additive, so $y \prec b^*$. Thus

$$a = \vee \Downarrow a = \vee\{y \mid y \prec a\} \leq b^*$$

□

Note 2. *On* $(P, \prec)$, $\forall a \in P$, $a \wedge a^* = 0$ *is false in general.*

We introduce the notion of regular on $(P, \prec)$.

Definition 3.3. $(P, \prec)$ *is said to be regular, if for* $x \in K(\prec)$, $b \in P$, $x \not\prec b$, *then* $x \wedge b = 0$.

Note 3. *Let* (X, T) *be a point-set topological space, if* $\forall z \in X$, *a closed set* $A \subseteq X$, $z \notin A$ *if and only if* $\{z\}^- \not\subseteq A$, *which equivalent to* $\{z\}^- \not\prec A$ *according to the definition of* $\prec$.

If (X, T) *is regular, then there exist* U, V *two open sets, such that* $z \in U, A \subseteq V$ *and* $U \cap V = \varnothing$. *We obtain* $\{z\}^- \cap A = \varnothing$. *Otherwise if* $\{z\}^- \cap A \neq \varnothing$, *there exists* $y \in \{z\}^- \cap A$, *so* $y \in A \subseteq V$. *By* $y \in \{z\}^-$, *we have* $\{z\} \cap V \neq \varnothing$, *thus* $z \in V$, *a contradiction.*

Definition 3.3 coincides with the above definition when (X, C) *is a co-topological space.*

The notion of compactness is defined as follows.

Definition 3.4. $(P, \prec)$ *is said to be compact if for every* $D \subseteq P$, $\wedge D = 0$ *implies that there exists a finite subset* $D_0 \subseteq D$ *satisfying* $\wedge D_0 = 0$. *That is to say, if* D *has the finite intersection property, then* $\wedge D \neq 0$.

We introduce the notions of a scale, completely regular on $(P, \prec)$.

Definition 3.5. *A family of elements* $\langle c_\alpha \in P \mid \alpha \in [0, 1]$ & α *is a rational number* $\rangle$ *is called a scale of* $(P, \prec)$, *if it satisfies: for* $\alpha < \beta$, *we have* $c_\alpha \prec c_\beta$.

For $a, b \in P$, *if there exists a scale* $\langle c_\alpha \rangle$, *such that* $a \leq c_0$, $c_1 \leq b$. *We denote the relation by* $a \lhd b$.

$(P, \prec)$ *is said to be completely regular, if* $\forall a \in P$, $a = \wedge\{b \mid a \lhd b\}$.

Suppose $(P_\alpha, \prec_\alpha)$ is a family of additive generalized algebraic lattices, $\alpha \in \Lambda$ (a index set), then $(\Pi P_\alpha, \prec_\Pi)$ is the direct product, and $pr_\alpha : \Pi P_\alpha \to P_\alpha$, $\forall a = (a_\alpha) \in \Pi P_\alpha$, $pr_\alpha(a) = a_\alpha$, pr_α is onto upper adjoint. $q_\alpha : (P_\alpha, \prec_\alpha) \to (\Pi P_\alpha, \prec_\Pi)$ is the lower homomorphism of pr_α ([9]).

By the definitions of pr_α and q_α, we know that q_α preserves the generalized way-below relation, and obtain the following proposition.

Proposition 3.6. *Suppose* $(P_\alpha, \prec_\alpha)$ *is regular, completely regular for every* $\alpha \in \Lambda$*, then* $(\Pi P_\alpha, \prec_\Pi)$ *is also regular, completely regular.*

Proof. It is trivial. □

Since every inclusion mapping is a lower homomorphism, it is obvious that every lower sublattice of regular, completely regular $(P, \prec)$ is also regular, completely regular.

Proposition 3.7 (Tychonoff product theorem)**.**
Suppose for every $\alpha \in \Lambda$, $(P_\alpha, \prec_\alpha)$ *is compact, then* $(\Pi P_\alpha, \prec_\Pi)$ *is also compact.*

Proof. It is similar to Bourbaki's proof ([14]).

(1) Let $B \subseteq \Pi P_\alpha$ be the maximal with respect to the finite intersection property ([14])

(2) $pr_\alpha : \Pi P_\alpha \to P_\alpha$ is the onto upper adjoint, then for some $\alpha \in \Lambda$, $\{pr_\alpha(b) \mid b \in B\}$ also has the finite intersection property. Since $(P_\alpha, \prec_\alpha)$ is compact, by Definitions 3.4, $\wedge\{pr_\alpha(b) \mid b \in B\} \neq 0$, so there exists $c \in K(\prec_\alpha)$, $c \neq 0$, $c \prec \wedge\{pr_\alpha(b) \mid b \in B\}$.

(3) q_α is the lower homomorphism of pr_α, so by $c \prec_\alpha pr_\alpha(b)$, we obtain $q_\alpha(c) \prec b$ for every $b \in B$, and $q_\alpha(c) \neq 0$, $q_\alpha(c) \in \Pi P_\alpha$. Thus $\wedge B \neq 0$, which shows that $(\Pi P_\alpha, \prec_\Pi)$ is compact. □

Suppose $I = [0, 1]$, the topology on I induced by $\rho(x, y) = |x - y|$. C_I denotes the family of all closed subsets, thus (I, C_I) is a co-topology on I.

According to Proposition 4.2 ([7]), let $Q = \{\{r\}^- \mid r \in [0, 1]\}$, $M(Q)$ generated by Q. C_I ordered by inclusion relation, forms a complete lattice. The generalized way-below relation $\prec_I$ induced by $M(Q)$, and $M(\prec_I) = M(Q)$. Then $(C_I, \prec_I)$ is an additive generalized algebraic lattice. By Definitions 3.3, 3.4 and 3.5, $(C_I, \prec_I)$ is regular, completely regular and compact. Furthermore, by Propositions 3.6 and 3.7, $(\Pi C_I, \prec_\Pi)$ is also regular, completely regular and compact.

By [7] Lemma 4.5, the system of subsets $M(\prec_I)$ is the collection of classes of closed subsets such that the union of any class is still closed. i.e., $\vee S = \cup S$ for every $S \in M(\prec_I)$, and $\cup S \in C_I$.

By the property of closed sets, for $D \subseteq C_I$, we have $\wedge D = \cap D \in C_I$.

Lemma 3.8. *For* $a, b \in P$*, suppose* $a \lhd b$*, then there exists a lower homomorphism* $h : (P, \prec) \to (C_I, \prec_I)$*, such that* $a \leq g(0)$ *and* $g(I) \leq b$*.*

Proof. The upper adjoint $g : (C_I, \prec_I) \to (P, \prec)$ is first defined. Since $a \lhd b$, then there exists a scale $\langle c_\alpha \rangle$, such that $a \leq c_0$, $c_1 \leq b$ and $c_\alpha \prec c_\beta$ for $\alpha < \beta$. This implies $\{c_\alpha\}$ is an increasing function of α.

For $[\alpha, \beta] \in C_I$, $g([\alpha, \beta]) = e_\alpha \wedge d_\beta$, where $e_\alpha = \vee_{r \geq \alpha} c_r$, $d_\alpha = \vee_{r \leq \alpha} c_r$. By [5] Theorem 3, we obtain $e_\alpha, d_\alpha \in M(\prec)$.

(1) For $(C_I, \prec_I)$, the closed interval is $[\alpha, \beta]$, and the elementary closed set $F_\lambda = \bigcup_{i=1}^{n} [\alpha_i, \beta_i]$, the closed set $F = \cap F_\lambda$. Since for every $S \in M(\prec_I)$, by [7] Lemma 4.5, $\vee S = \cup S$. So for every $S \in M(\prec_I)$, we have $g(S) \in M(\prec)$.

(2) By $\vee S = \cup S$, we obtain $g(\vee S) = g(\cup S) = \vee g(S)$ for every $S \in M(\prec_I)$,

(3) Since for $S \subseteq C_I$, $\wedge S = \cap S$, we know that g also preserves arbitrary meets, i.e., $g(\wedge S) = \wedge g(S)$.

By the above proof, g is an upper adjoint. Thus $h : (P, \prec) \to (C_I, \prec_I)$ is a lower homomorphism.

$$g(I) = g([0, 1]) = e_0 \wedge d_1 \leq b$$

$$g(0) = g(\{0\}) = e_0 \wedge d_0 \geq a.$$

□

Proposition 3.9 (Tychonoff embedding theorem)**.**

Suppose $(P, \prec)$ is an additive generalized algebraic lattice, then $(P, \prec)$ is regular, completely regular iff $(P, \prec)$ is isomorphic to a lower sublattice of $(\Pi C_I, \prec_\Pi)$.

Proof. By Proposition 3.6, $(\Pi C_I, \prec_\Pi)$ is regular, completely regular, and every lower sublattice of $(\Pi C_I, \prec_\Pi)$ is also regular, completely regular, so the proof is trivial

On the other hand, suppose $(P, \prec)$ is an additive generalized algebraic lattice, let $S = \{(g_s, h_s) \mid g_s : (C_I, \prec_I) \to (P, \prec)$ is an upper adjoint, $h_s : (P, \prec) \to (C_I, \prec_I)$ is a lower homomorphism of $g_s\}$, $S \neq \varnothing$

Taking: H: $(P, \prec) \to (\Pi C_I, \prec_\Pi)$ the direct product of $(C_I, \prec_I)$ by index set of $S, \forall a \in P$, $H(a) = \Pi h_s(a)$.

By the property of $\{h_s\}$, H is also a lower homomorphism, so $G : (\Pi C_I, \prec_\Pi) \to (P, \prec)$ is the upper adjoint of H.

We show $(P, \prec)$ is isomorphic to a lower sublattice of $(\Pi C_I, \prec_\Pi)$, it suffices to prove H is one-to-one on $K(\prec)$.

$\forall x, y \in K(\prec)$, $x \neq y$, then we may assume $x \not\prec y$. Since $(P, \prec)$ is regular, so $x \wedge y = 0$, which follows that $H(x) \neq H(y)$.

Thus $(P, \prec)$ is isomorphic a lower sublattice of $(\Pi C_I, \prec_\Pi)$, which generated by $H(K(\prec))$, and $H(K(\prec)) \subseteq K(\prec_\Pi)$. □

Proposition 3.10 (Stone compactification)**.**

Suppose $(P, \prec)$ is regular, completely regular, then there exists a regular, completely regular compact additive generalized algebraic lattice $(\beta P, \prec_\beta)$, such that $(P, \prec)$ is isomorphic to a dense lower sublattice of $(\beta P, \prec_\beta)$.

Proof. By Proposition 3.9, $(P, \prec)$ is isomorphic to a lower sublattice of $(\Pi C_I, \prec_\Pi)$. Let $(\beta P, \prec_\beta)$ be the closure of the lower sublattice, and the compactness of $(\beta P, \prec_\beta)$ follows from Proposition 3.7. □

In general, $(\beta P, \prec_\beta)$ is said to be a Stone compactification of $(P, \prec)$.

Note 4. *Clearly, if the generalized way-below relation $\prec$ satisfies the interpolation property, then $(P, \prec)$ is completely regular by The Choice Axiom.*

As the end of this paper, we embark on an alternative description of $(\beta P, \prec_\beta)$ by means of ideals of $(P, \prec)$.

Definition 3.11. *$I \subseteq P$ is said to be an ideal if (1) for any finite $E \subseteq I$, $\vee E \in I$, (2) $z \in I$, $x \leq z$ implies $x \in I$.*

$Idl(P)$ denotes all ideals of P, and certainly $Idl(P)$ is a complete lattice, the order is the inclusion order.

$$\forall I \in Idl(P), \downarrow I = \{J \mid J \leq I\}, \text{where } J \leq I \text{ iff } J \subseteq I$$

Definition 3.12. *For $I, J \in Idl(P)$, a binary relation on $Idl(P)$ is defined as: $I \prec^* J$ if and only if $\vee I \prec \vee J$ holds on $(P, \prec)$.*

Lemma 3.13 ([15])**.** *$\prec^*$ is a generalized way-below relation on $Idl(P)$.*

Proof. (1) $I \prec^* J$ if and only if $\vee I \prec \vee J$ holds on $(P, \prec)$. Then $\forall a \in I$, $a \leq \vee I \prec \vee J$, so $a \in J$. that is, $I \subseteq J$, thus $I \leq J$.

(2) $I_1 \leq I_2 \prec^* I_3 \leq I_4$, which implies that $\vee I_1 \leq \vee I_2 \prec \vee I_3 \leq \vee I_4$ holds on $(P, \prec)$. So we have $\vee I_1 \prec \vee I_4$, thus $I_1 \prec^* I_4$. □

Lemma 3.14. *$Idl(P)$ is algebraic.*

Proof. For $I, J \in Idl(P)$, $I \prec^* J$ implies $\vee I \prec \vee J$ on $(P, \prec)$. Since $(P, \prec)$ is algebraic, there exists $c \in K(\prec)$, such that $\vee I \leq c \leq \vee J$. Furthermore $\downarrow c \in Idl(P)$.

By $c \in K(\prec)$, so $c \prec c$ on $(P, \prec)$, hence $\downarrow c \prec^* \downarrow c$ on $Idl(P)$. i.e, $\downarrow c \in K(\prec^*)$ by Definition 3.11.

Considering $I \leq \downarrow c \leq J$ and $\downarrow c \in K(\prec^*)$, thus $(Idl(P), \prec^*)$ is algebraic. □

Lemma 3.15. *$Idl(P)$ is continuous.*

Proof. It is trivial ([4]). □

Lemma 3.16. *$Idl(P)$ is additive.*

Proof. For $I \prec^* J_1 \vee J_2$, where $I, J_1, J_2 \in Idl(P)$, then on $(P, \prec)$, $\vee I \prec \vee(J_1 \vee J_2) = (\vee J_1) \vee (\vee J_2)$ holds. Since $(P, \prec)$ is additive, it follows that $\vee I \prec \vee J_1$ or $\vee I \prec \vee J_2$, thus $I \prec^* J_1$ or $I \prec^* J_2$, which proves Lemma 3.16. □

Proposition 3.17. *$(Idl(P), \prec^*)$ is an additive generalized algebraic lattice.*

Proof. By Lemmas 3.14, 3.15, 3.16. □

Lemma 3.18. *For any regular $(P, \prec)$, $Idl(P)$ is also regular.*

Proof. It is obvious that on $(Idl(P), \prec^*)$, $K(\prec^*) = \{\downarrow x \mid x \in K(\prec)\}$. Then $\forall \downarrow x \in K(\prec^*)$, $\forall J \in Idl(P)$, if $\downarrow x \not\prec^* J$, which implies $x \not\prec \vee J$ by Definition 3.11.

Since $(P, \prec)$ is regular, $x \in K(\prec)$, $\vee J \in P$, $x \not\prec \vee J$, then $x \wedge (\vee J) = 0$. So we obtain that $\downarrow x \wedge J = 0$. It follows that $Idl(P)$ is regular. □

For $I \in Idl(P)$, I is called completely regular, if $\forall a \in I$, there exists $b \in I$, such that $a \lhd b$. Let $R(P) = \{I \text{ is completely regular in } Idl(P)\}$, then we have

Lemma 3.19. *Suppose $(P, \prec)$ is completely regular, then $(R(P), \prec^*)$ is also completely regular.*

Proof. It is trivial. □

Lemma 3.20. *Suppose $(P, \prec)$ is compact, then $(R(P), \prec^*)$ is also compact.*

Proof. For a family $\{I_\alpha | \alpha \in \Lambda\}$ satisfying $\wedge I_\alpha = 0$. Since $(P, \prec)$ is a complete lattice, $I_\alpha = \vee\{\downarrow x \mid x \in I_\alpha\}$, we may assume $I_\alpha = \downarrow a_\alpha$. Then $\wedge I_\alpha = \wedge(\downarrow a_\alpha) = \downarrow (\wedge a_\alpha)$, thus $\downarrow (\wedge a_\alpha) = 0$, it follows that $\wedge a_\alpha = 0$.

Furthermore $(P, \prec)$ is compact, by Definition 3.4, there exist $a_1, a_2, \cdots, a_m$ satisfying $\bigwedge_{i=1}^{m} a_i = 0$. By this, it is easy to prove $\bigwedge_{i=1}^{m} (\downarrow a_i) = 0$. that is, $\bigwedge_{i=1}^{m} I_i = 0$. Thus $(R(P), \prec^*)$ is compact. □

Proposition 3.21. *Suppose $(P, \prec)$ is compact, regular and completely regular, then $(P, \prec)$ and $R(P)$ are isomorphic.*

Proof. By Lemmas 3.18, 3.19, 3.20, and $h : P \to R(P)$, $\forall a \in P$, $h(a) = \Downarrow a = \{b \mid b \prec a\}$, certainly $h(a) \in R(P)$.

$a \prec b$ holds on $(P, \prec)$ if and only if $h(a) \prec^* h(b)$ holds on $(R(P), \prec^*)$. Since $(P, \prec)$ is continuous, $\forall a \in P$, $a = \vee \Downarrow a$, so $(P, \prec)$ is embedded into $R(P)$, and h preserves the generalized way-below relation. It is trivial to prove h is one-to-one. □

By Proposition 3.10, suppose $(P, \prec)$ is compact, regular, completely regular, then $(\beta P, \prec_\beta)$ and $(R(P), \prec^*)$ are also isomorphic

By Propositions 3.10 and 3.21, the following theorem is also obtained.

Theorem 3.22. *An additive generalized algebraic lattice $(P, \prec)$ has a Stone compactification iff it is regular, completely regular.*

Note 5.

(1) *According to* [3]*, the class of generalized continuous lattices includes completely distributive lattices and traditional continuous lattices* ([15]) *as its special cases.*
(2) *According to* [4]*, the traditional algebraic lattice is generalized algebraic lattice* ([4])*, and completely distributive lattice is also generalized algebraic lattice* ([4])*.*

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