The class of m-EP and m-normal matrices

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Abstract

The well-known classes of EP matrices and normal matrices are defined by the matrices that commute with their Moore-Penrose inverse and with their conjugate transpose, respectively. This paper investigates the class of m-EP matrices and m-normal matrices that provide a generalization of EP matrices and normal matrices, respectively, and analyzes both of them for their properties and characterizations.

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1 Introduction and Notation

The symbol $\mathbb{C}^{m\times n}$ stands for the set of $m \times n$ complex matrices. The symbols A^*, $\mathcal{C}(A)$ and $\mathcal{N}(A)$ will denote the conjugate transpose, column space and null space of a matrix $A \in \mathbb{C}^{m\times n}$, respectively. Moreover, I_n will denote the identity matrix of order n.

The symbol $A^\dagger$ will denote the Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$, i.e., the unique matrix $A^\dagger \in \mathbb{C}^{n\times m}$ satisfying the following four Penrose conditions: $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $AA^\dagger = (AA^\dagger)^*$, $A^\dagger A = (A^\dagger A)^*$.

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The orthogonal projectors $AA^\dagger$ and $A^\dagger A$ will be denoted by the symbols P_A and Q_A, respectively. For a given matrix $A \in \mathbb{C}^{n\times n}$, recall that the smallest nonnegative integer m such that $rank(A^m) = rank(A^{m+1})$ is called the index of A and is denoted by $ind(A)$. The Drazin inverse of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $A^d \in \mathbb{C}^{n\times n}$ such that $A^dAA^d = A^d$, $AA^d = A^dA$, $A^{m+1}A^d = A^m$, where $m = ind(A)$. Three generalized inverses were recently introduced for square matrices, namely the core inverse, the DMP inverse and the BT-inverse, the later two being generalizations of the core inverse to matrices of index greater than or equal 2. We wish to mention that the BT-inverse was originally referred as generalized core inverse. Since BT-inverse is not the only generalization of the core inverse known in the literature, we prefer to credit it to the authors Baksalary and Trenkler and, hence, call this generalization the BT-inverse. Let $A \in \mathbb{C}^{n\times n}$. An $n \times n$ matrix X satisfying $AX = P_A$ and $\mathcal{C}(X) \subseteq \mathcal{C}(A)$ is called the core inverse of A [2] (it exists for index 1 matrices and it is unique). If A has index m, the only matrix $X \in \mathbb{C}^{n\times n}$ that satisfies $XAX = X$, $XA = A^dA$ and $A^mX = A^mA^\dagger$ is called the DMP inverse and denoted by $X = A^{d,\dagger}$ [8]. For $m = 1$, the DMP inverse becomes the core inverse [2, 13]. The DMP inverse of a matrix A always exists and satisfies $A^{d,\dagger} = A^dAA^\dagger$ [8]. A matrix $A^\diamond \in \mathbb{C}^{n\times n}$ satisfying $A^\diamond = (AP_A)^\dagger$ is called the BT-inverse of A (it always exists and is unique) [3]. We refer the reader to [2, 3, 4, 5, 12, 15] for properties of these matrices.

We also recall that a square matrix is called normal, EP, partial isometry, SD, bi-EP, bi-normal or bi-dagger if $AA^* = A^*A$, $AA^\dagger = A^\dagger A$, $A^\dagger = A^*$, $A^*A^\dagger = A^\dagger A^*$, $(AA^\dagger)(A^\dagger A) = (A^\dagger A)(AA^\dagger)$, $(AA^*)(A^*A) = (A^*A)(AA^*)$, or $(A^\dagger)^2 = (A^2)^\dagger$, respectively [7, 10]. Some applications of EP matrices can be found for instance in [6, 11].

The main aim of this paper is to investigate the classes of m-EP matrices (square matrices A of index m satisfying $A^mA^\dagger = A^\dagger A^m$) and m-normal matrices, that provide a generalization of EP matrices and normal matrices. We remember that the classes of EP matrices and normal matrices are defined by the square matrices that commute with their Moore-Penrose inverse and with their conjugate transpose, respectively. We note that for a given matrix $A \in \mathbb{C}^{n\times n}$ of index m, Tian showed [14] the equivalence between $A^mA^\dagger = A^\dagger A^m$ and $rank\begin{bmatrix} A^m \\ A^* \end{bmatrix} + rank\begin{bmatrix} A^m & A^* \end{bmatrix} = 2\,rank(A)$ and the equivalence between A^m is EP and $rank\begin{bmatrix} A^m & (A^m)^* \end{bmatrix} = rank(A^m)$. In order to understand more deeply this class of matrices, our task is to provide several properties and characterizations. Additionally, we obtain a characterization of the Drazin inverse and the DMP inverse of m-EP matrices using a Hartwig-Spindelböck decomposition.

2 The class of m-EP matrices

We next study the class of matrices $A \in \mathbb{C}^{n\times n}$ of index m that satisfy the condition that $A^\dagger$ and A^m commute.

Definition 2.1. *A matrix $A \in \mathbb{C}^{n\times n}$ is called m-EP if it satisfies*

$$A^\dagger A^m = A^m A^\dagger$$

where m is the index of A.

Notice that for $m = 1$, the matrices in this class are the class of range hermitian (or EP) matrices. If a square matrix A is m-EP then A^* and UAU^* are also m-EP and $ind(U^*AU) = m$ for any unitary matrix $U \in \mathbb{C}^{n\times n}$. Clearly, any unitary as also any nonsingular matrix is m-EP for $m = 0$. Moreover, any nilpotent matrix is trivially m-EP for m being the nilpotence index of A. We give below a non-trivial example with $m = 2$.

Example 2.2. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $A^\dagger = A^*$, $A^2 = A^3$ and $A^2A^\dagger = A^\dagger A^2$, that is, A is 2-EP. Observe that A is not diagonalizable.

It is well known that an EP matrix may be or not diagonalizable. However, the next result states that m-EP matrices with $m \geq 2$ are always not diagonalizable.

Theorem 2.3. *If $A \in \mathbb{C}^{n\times n}$ is diagonalizable and m-EP then A is EP.*

Proof. Clearly, if $A = 0$ then A is diagonalizable, $0^q 0^\dagger = 0^\dagger 0^q$ for any positive integer q and it is well known that $m = ind(A) = 1$.

Now, let $A \neq 0$ and $r = rank(A)$. Since A is diagonalizable, we can say that $A = Pdiag(d_1, \ldots, d_r, 0, \ldots, 0)P^{-1}$ for some $n \times n$ nonsingular matrix P and non-zero scalars $d_1, \ldots, d_r$. It follows that $rank(A^2) = rank(A)$, so $m = ind(A) \leq 1$. Hence A is EP. $\square$

Lemma 2.4. *Let $A \in \mathbb{C}^{n\times n}$ be an m-EP matrix. The following statements hold.*

(a) If A is m-dagger (i.e., $(A^m)^\dagger = (A^\dagger)^m$) then A^m is EP.

(b) If A is a partial isometry then A^m is normal.

Proof. If A is m-EP then it is easy to see that $A^m(A^\dagger)^m = (A^\dagger)^m A^m$. Thus, it immediately follows that A^m is EP. Second item is trivial. □

We now give a construction that allows us to obtain many more examples of m-EP matrices. For integer $n \geq 2$, let $J_n(0)$ denote the $n\times n$ Jordan block corresponding to the eigenvalue 0 with 1's in super diagonal. Then $J_n(0)$ has index n and $(J_n(0))^\dagger = (J_n(0))^*$ [4, p. 43].

For each fixed $m \in \mathbb{N}$, in the following example we construct m-EP matrices.

Example 2.5. Let $m \geq 2$ be an integer and B be a $p \times p$ EP matrix. The matrix

$$A = J_m(0) \oplus B = \begin{bmatrix} J_m(0) & 0 \\ 0 & B \end{bmatrix}$$

is of index m by [5, Theorem 7.7.4] and satisfies $A^\dagger A^m = A^m A^\dagger$.

Now we give some sufficient conditions for a matrix to be m-EP. Before that result we present a lemma.

Lemma 2.6. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index m and rank $r > 0$. The following statements are equivalent:*

(a) There exists an EP matrix $E \in \mathbb{C}^{n\times n}$ and a nilpotent matrix $M \in \mathbb{C}^{n\times n}$ with nilpotence index m such that $A = E + M$ and $EM = ME = 0$.

(b) There are matrices $C \in \mathbb{C}^{s\times s}$, $T \in \mathbb{C}^{t\times t}$, and $U \in \mathbb{C}^{n\times n}$ such that $A = U(C \oplus T)U^*$ where $s + t = n$, C is nonsingular, T is nilpotent with nilpotence index m, and U is unitary.

Proof. (a) $\Longrightarrow$ (b) Assume that $A = E + M$, with E an EP matrix, M an m-nilpotent matrix and $EM = ME = 0$. The EPness of E assures that [5] there exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $C \in \mathbb{C}^{s\times s}$

such that $E = U(C \oplus 0)U^*$. Partitioning A conformable to the partition of E we have that

$$A = U \begin{bmatrix} X & Y \\ Z & T \end{bmatrix} U^* \qquad \text{and} \qquad M = A - E = U \begin{bmatrix} X - C & Y \\ Z & T \end{bmatrix} U^*.$$

From $ME = 0$ and the non-singularity of C we get $X = C$ and $Z = 0$. Similarly, from $EM = 0$ we arrive at $Y = 0$. Thus, $M = U(0 \oplus T)U^*$. Since M is m-nilpotent, $T^m = 0 \neq T^{m-1}$, that is, T is m-nilpotent. Finally, $A = U(C \oplus T)U^*$.

(b) $\Longrightarrow$ (a) This implication is evident to be checked by writing $A = U(C \oplus 0)U^* + U(0 \oplus T)U^*$ and calling E the first term and M the second one. □

Theorem 2.7. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index m and rank $r > 0$. The following conditions are equivalent.*

(a) The matrix A satisfies any of both equivalent conditions in Lemma 2.6.

(b) A is m-EP and A^m is EP.

Proof. (a) $\Longrightarrow$ (b) Assume that $A = U(C \oplus T)U^*$ where matrices U, C, and T satisfy the conditions indicated in Lemma 2.6. It is easy to see that $A^m = U(C^m \oplus 0)U^*$ and so, A^m is EP, and moreover $A^\dagger = U(C^{-1} \oplus T^\dagger)U^*$. Hence, we arrive at $A^\dagger A^m = A^m A^\dagger$ from which A is m-EP.

(b) $\Longrightarrow$ (a) Suppose that A^m is EP. Then

$$A^m = U(B \oplus 0)U^* \tag{1}$$

for some nonsingular $B \in \mathbb{C}^{s\times s}$ and some unitary $U \in \mathbb{C}^{n\times n}$. Assuming also that A is m-EP and partitioning

$$A^\dagger = U \begin{bmatrix} X & Y \\ Z & V \end{bmatrix} U^*$$

according to the partition of A^m, we obtain $Y = 0$, $Z = 0$ and $XB = BX$ since $A^m A^\dagger = A^\dagger A^m$. Using that $(A^\dagger)^\dagger = A$ we get $A = U(X^\dagger \oplus V^\dagger)U^*$. If we now compute A^m and compare to (1), it is easy to see that $B = (X^\dagger)^m$ and $(V^\dagger)^m = 0$. That is, $X^\dagger$ is nonsingular and $V^\dagger$ is nilpotent as desired. □

Theorem 2.7 extends the useful characterization for EP matrices given in [5, Theorem 4.3.1, p. 74]).

We now remind a canonical form for the class of m-EP matrices using the Hartwig-Spindelböck decomposition [7, 1]. For any matrix $A \in \mathbb{C}^{n\times n}$ of rank $r > 0$ this decomposition is given by

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \tag{2}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = diag\left(\sigma_1 I_{r_1}, \quad \ldots, \quad \sigma_t I_{r_t}\right)$ is a diagonal matrix, the diagonal entries σ_i being singular values of A, $\sigma_1 > \sigma_2 > \ldots > \sigma_t > 0$, $r_1 + r_2 + \ldots + r_t = r$ and $K \in \mathbb{C}^{r\times r},\ L \in \mathbb{C}^{r\times(n-r)}$ satisfy $KK^* + LL^* = I_r$.

Theorem 2.8. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2). Then A is m-EP if and only if the following conditions hold:*

(a) $K^*K(\Sigma K)^{m-1} = (\Sigma K)^{m-1}$,

(b) $L^*\Sigma^{-1}(\Sigma K)^{m-1} = 0$ *(or equivalently* $L^*K(\Sigma K)^{m-2} = 0$*),*

(c) $(\Sigma K)^{m-1}\Sigma L = 0$*, and*

(d) $ind(\Sigma K) = m - 1$.

Proof. Suppose that A is written as in (2). Then

$$A^j = U \begin{bmatrix} (\Sigma K)^j & (\Sigma K)^{j-1}\Sigma L \\ 0 & 0 \end{bmatrix} U^* \tag{3}$$

for all integer $j \geq 1$.

($\Longrightarrow$) Assume that A is m-EP. Condition (d) follows directly from [8, Lemma 2.8]. It is well-known that $A^\dagger$ has the form [2, Formula (1.13)]

$$A^\dagger = U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*.$$

By setting $j = m$ in (3), condition $A^m A^\dagger = A^\dagger A^m$ is equivalent to

$$\begin{aligned} K^*\Sigma^{-1}(\Sigma K)^m &= (\Sigma K)^m K^*\Sigma^{-1} + (\Sigma K)^{m-1}\Sigma LL^*\Sigma^{-1} = (\Sigma K)^{m-1} \\ K^*\Sigma^{-1}(\Sigma K)^{m-1}\Sigma L &= 0, \\ L^*\Sigma^{-1}(\Sigma K)^m &= 0, \\ L^*\Sigma^{-1}(\Sigma K)^{m-1}\Sigma L &= 0. \end{aligned}$$

Clearly, the first condition can be rewritten as in (a). Since ΣK has index $m-1$, post-multiplying third equality by the Drazin inverse of ΣK we get condition $L^*\Sigma^{-1}(\Sigma K)^{m-1} = 0$ which gives (b). Pre-multiplying the second equation by K, the fourth equation by L and adding them, condition (c) is obtained.

($\Longleftarrow$) Assume that conditions (a)-(d) are satisfied. From (3), we can write

$$A^j = U \begin{bmatrix} (\Sigma K)^{j-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma K & \Sigma L \\ Y & Z \end{bmatrix} U^* \tag{4}$$

for all $j \geq 1$ and for some appropriate matrices Y and Z such that $\begin{bmatrix} \Sigma K & \Sigma L \\ Y & Z \end{bmatrix}$ is nonsingular. Notice that both matrices Y and Z exist because the matrix $\begin{bmatrix} \Sigma K & \Sigma L \end{bmatrix}$ has full row rank. Equality (4) implies that $rank(A^j) = rank(\Sigma K)^{j-1}$ for all $j \geq 1$. Now, from $ind(\Sigma K) = m-1$ we get $ind(A) = m$. It remains to show $A^\dagger A^m = A^m A^\dagger$. Under conditions (a)-(c), it can be verified by actual computations. □

The next aim is to show that m-EPness and the fact that A^m is EP are essentially different notions.

Theorem 2.9. *Let A be an m-EP matrix written as in (2). Then*

$$A^m \text{ is } EP \quad \Longleftrightarrow \quad (\Sigma K)^m \text{ is } EP.$$

Proof. If we write matrix A as in (2) then

$$A^m = U \begin{bmatrix} (\Sigma K)^m & (\Sigma K)^{m-1}\Sigma L \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} (\Sigma K)^m & 0 \\ 0 & 0 \end{bmatrix} U^*$$

(because when A is m-EP we can derive that $(\Sigma K)^{m-1}\Sigma L = 0$ holds) and

$$(A^m)^* = U \begin{bmatrix} ((\Sigma K)^m)^* & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Now, $\mathcal{N}(A^m) = \mathcal{N}((A^m)^*)$ if and only if $\mathcal{N}((\Sigma K)^m) = \mathcal{N}(((\Sigma K)^m)^*)$. □

In order to obtain the Drazin inverse of m-EP matrices we need the following properties.

Proposition 2.10. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2). If A is m-EP then the following properties hold:*

(a) $(\Sigma K)^m(K^*\Sigma^{-1}) = (\Sigma K)^{m-1}$.

(b) $(K^*\Sigma^{-1})(\Sigma K)^m = (\Sigma K)^{m-1}$.

(c) $(\Sigma K)^{m-1+q}(K^*\Sigma^{-1})^q = (K^*\Sigma^{-1})^q(\Sigma K)^{m-1+q} = (\Sigma K)^{m-1}$ *for all integer* $q \geq 1$.

Proof. (a) By Theorem 2.8 (c), $(\Sigma K)^m(K^*\Sigma^{-1}) = (\Sigma K)^{m-1}(\Sigma K)(K^*\Sigma^{-1}) = (\Sigma K)^{m-1}\Sigma(I_r - LL^*)\Sigma^{-1} = (\Sigma K)^{m-1}$.

(b) By Theorem 2.8 (a), $(K^*\Sigma^{-1})(\Sigma K)^m = K^*K(\Sigma K)^{m-1} = (\Sigma K)^{m-1}$.

(c) It follows by induction on q using (a) and (b).

□

Theorem 2.11. *Let $A \in \mathbb{C}^{n\times n}$ be an m-EP matrix written as in (2). Then*

$$A^d = U \begin{bmatrix} (\Sigma K)^{m-1}(K^*\Sigma^{-1})^m & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} (K^*\Sigma^{-1})^m(\Sigma K)^{m-1} & 0 \\ 0 & 0 \end{bmatrix} U^*$$

Proof. By Proposition 2.10 (a) and (b), $(\Sigma K)^m$ and $K^*\Sigma^{-1}$ commute. So,

$$(\Sigma K)^{m-1}(K^*\Sigma^{-1})^m(\Sigma K)^m(K^*\Sigma^{-1})^m = (\Sigma K)^{m-1+m}(K^*\Sigma^{-1})^{m+m},$$

and using Proposition 2.10 (c), we get $(\Sigma K)^{m-1+m}(K^*\Sigma^{-1})^m(K^*\Sigma^{-1})^m = (\Sigma K)^{m-1}(K^*\Sigma^{-1})^m$. Hence, $A^dAA^d = A^d$. Similarly, we obtain $A^dA = AA^d$ and $A^{m+1}A^d = A^m$. □

Now, expressions for A^d and $A^\dagger$ of an *m-EP* matrix allow us to ensure the equality between the DMP inverse and the Drazin inverse of A.

Corollary 2.12. *Let $A \in \mathbb{C}^{n\times n}$ be an m-EP matrix written as in (2). Then* $A^{d,\dagger} = A^d$.

Next result shows that condition "$(\Sigma K)^2$ is *EP*" fulfils vacuously in Theorem 2.9 for $m = 2$.

Proposition 2.13. *Let $A \in \mathbb{C}^{n\times n}$ be a 2-EP matrix. Then A^2 is an EP matrix and A is bi-dagger.*

Proof. Assume that A is a 2-EP matrix written as in (2). By Theorem 2.8 we get: (1) $K^*K\Sigma K = \Sigma K$, (2) $L^*K = 0$, (3) $K\Sigma L = 0$. In addition, by $L^*K = 0$ we have that: (4) K is a partial isometry since $K = (KK^* + LL^*)K = KK^*K$.

On the other hand, $A^2 = U \begin{bmatrix} (\Sigma K)^2 & \Sigma K \Sigma L \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} (\Sigma K)^2 & 0 \\ 0 & 0 \end{bmatrix} U^*$. We claim that

$$(A^2)^\dagger = U \begin{bmatrix} (K^*\Sigma^{-1})^2 & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{5}$$

or equivalently $((\Sigma K)^2)^\dagger = (K^*\Sigma^{-1})^2$. In fact, we will demonstrate the four Penrose equations:

(i) By (1) we have $(\Sigma K)^2(K^*\Sigma^{-1})^2(\Sigma K)^2 = (\Sigma K)^2 K^*\Sigma^{-1}(K^*K\Sigma K) = (\Sigma K)^2 K^*\Sigma^{-1}(\Sigma K) = \Sigma K \Sigma (KK^*K) = (\Sigma K)^2$ since K is a partial isometry.

(ii) By (1) we have $(K^*\Sigma^{-1})^2(\Sigma K)^2(K^*\Sigma^{-1})^2 = K^*\Sigma^{-1}(K^*K\Sigma K)(K^*\Sigma^{-1})^2 = K^*\Sigma^{-1}(\Sigma K)(K^*\Sigma^{-1})^2 = (K^*KK^*)\Sigma^{-1}K^*\Sigma^{-1} = (K^*\Sigma^{-1})^2$ since K is a partial isometry.

(iii) By using (3) and (1) we have $(\Sigma K)^2(K^*\Sigma^{-1})^2 = \Sigma K \Sigma (KK^*)\Sigma^{-1}K^*\Sigma^{-1}$ $= \Sigma K \Sigma (I_r - LL^*)\Sigma^{-1}K^*\Sigma^{-1} = \Sigma KK^*\Sigma^{-1} - \Sigma(K\Sigma L)L^*\Sigma^{-1}K^*\Sigma^{-1}$ $= (\Sigma K)K^*\Sigma^{-1} = (K^*K\Sigma K)K^*\Sigma^{-1} = K^*K\Sigma(KK^*)\Sigma^{-1} = K^*K\Sigma(I_r - LL^*)\Sigma^{-1} = K^*K - K^*(K\Sigma L)L^*\Sigma^{-1} = K^*K$.

(iv) By (1) we have $(K^*\Sigma^{-1})^2(\Sigma K)^2 = K^*\Sigma^{-1}(K^*K\Sigma K) = K^*\Sigma^{-1}(\Sigma K) = K^*K$.

Hence, $A^2(A^2)^\dagger = (A^2)^\dagger A^2$, that is A^2 is EP. Even more, it can be also proved that A is bi-dagger. By the expression of the Moore-Penrose inverse [2] it then follows

$$(A^\dagger)^2 = U \begin{bmatrix} (K^*\Sigma^{-1})^2 & 0 \\ L^*\Sigma^{-1}K^*\Sigma^{-1} & 0 \end{bmatrix} U^*,$$

and we can show that $L^*\Sigma^{-1}K^* = 0$. In fact, pre-multiplying $\Sigma^{-1}K^*K\Sigma K = K$ by L^* and post-multiplying it by K^*, we get (i) $L^*\Sigma^{-1}K^*K\Sigma KK^* = L^*KK^* = 0$. Now, pre-multiplying and post-multiplying (3) by $L^*\Sigma^{-1}K^*$ and L^*, respectively, we have (ii) $L^*\Sigma^{-1}K^*K\Sigma LL^* = 0$. Adding (i) and (ii) we get $L^*\Sigma^{-1}K^*K = 0$ and finally using (4) we arrive at $L^*\Sigma^{-1}K^* = 0$. It follows that $(A^2)^\dagger = (A^\dagger)^2$. $\square$

Notice that if we first establish that A is bi-dagger in Proposition 2.13, it then follows that A^2 is EP from Lemma 2.4.

Proposition 2.14. *Let $A \in \mathbb{C}^{n\times n}$ be a 2-EP matrix. Then A is bi-EP.*

Proof. Let A be a 2-EP matrix. By Proposition 2.13, A is bi-dagger. It then implies that $(A^\dagger)^2 = (A^\dagger)^2((A^\dagger)^2)^\dagger(A^\dagger)^2 = (A^\dagger)^2A^2(A^\dagger)^2$. Thus, $A(A^\dagger)^2A = A(A^\dagger)^2A^2(A^\dagger)^2A = AA^\dagger(A^\dagger A^2)(A^\dagger)^2A = AA^\dagger(A^2A^\dagger)(A^\dagger)^2A$. Using the definition of the Moore-Penrose inverse, $AA^\dagger(A^2A^\dagger)(A^\dagger)^2A = (AA^\dagger A)AA^\dagger(A^\dagger)^2A = A^2A^\dagger(A^\dagger)^2A = A^\dagger(A^2A^\dagger)A^\dagger A = A^\dagger A^\dagger A(AA^\dagger A) = A^\dagger(A^\dagger A^2) = A^\dagger A^2A^\dagger$. $\square$

Related to a generalization of core inverse introduced by Baksalary and Trenkler in [3] we have the following result that can be easily shown.

Proposition 2.15. *Let $A \in \mathbb{C}^{n\times n}$. Then the following statements hold:*

(a) A is 2-EP if and only if $A^\diamond = (Q_AA)^\dagger$.

(b) If A is 2-EP then $(A^\diamond)^\dagger$ and $A^\dagger$ commute.

It is remarkable that a formula similar to (5) can be established in general for a m-EP matrix A when $m > 2$. It reads like

$$(A^m)^\dagger = U\begin{bmatrix} ((\Sigma K)^m)^\dagger & 0 \\ 0 & 0 \end{bmatrix}U^* \tag{6}$$

and can be obtained computing A^m and using Theorem 2.8 (c). However, using the expression for $A^\dagger$ given in [2], the formula for $(A^\dagger)^m$ now adds a not necessarily zero block in position $(2,1)$ as follows:

$$(A^\dagger)^m = U\begin{bmatrix} (K^*\Sigma^{-1})^m & 0 \\ L^*\Sigma^{-1}(K^*\Sigma^{-1})^{m-1} & 0 \end{bmatrix}U^*. \tag{7}$$

Proposition 2.16. *Let $A \in \mathbb{C}^{n\times n}$ be a SD and m-EP matrix. Then the following statements hold:*

(a) A is m-dagger if and only if $((\Sigma K)^m)^\dagger = (K^*\Sigma^{-1})^m$. In this case, K^m is a partial isometry.

(b) A^m is a partial isometry if and only if $(\Sigma K)^m$ is a partial isometry.

Proof. Let A be a SD and m-EP matrix written as in (2). Since A is m-EP, Theorem 2.8 implies that $K^*K(\Sigma K)^{m-1} = (\Sigma K)^{m-1}$, $L^*\Sigma^{-1}(\Sigma K)^{m-1} = 0$, $(\Sigma K)^{m-1}\Sigma L = 0$, and $ind(\Sigma K) = m-1$. By Corollary 6 in [7], it follows that Σ and K commute and, moreover, Σ^{-1} and K commute and also Σ^{-1} and K^* commute. Now, the above conditions can be re-written as $K^*K^m = K^{m-1}$, $L^*K^{m-1} = 0$, $K^{m-1}L = 0$. Substituting $L^*(K^*)^{m-1} = 0$ in (7) we get

$$(A^\dagger)^m = U \begin{bmatrix} (K^*\Sigma^{-1})^m & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

(a) By using (6), we can establish that A is m-dagger if and only if $((\Sigma K)^m)^\dagger = (K^*\Sigma^{-1})^m$. Now, pre- and post-multiplying both sides by $\Sigma^m K^m$ and using the non-singularity of Σ we have $K^m = K^m(K^*)^m K^m$, hance K^m is a partial isometry.
(b) It follows from comparing expressions for $(A^m)^\dagger$ and $(A^m)^*$. $\square$

We now give an example of a matrix that shows that the concepts 'A is m-EP' and 'A^m is EP' are really different for $m \geq 3$ (and also different from that of m-dagger one).

Example 2.17. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

of index 3. In this case, it can be checked that

$$A^\dagger = \begin{bmatrix} 0 & 0 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & -1/2 & 1/2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 1/2 & -1/2 & 0 & 0 & 0 & 1 \\ 1/2 & -1/2 & 0 & 0 & -1 & 2 \end{bmatrix},$$

and $A^3A^\dagger = A^\dagger A^3$, so A is 3-EP. However, using that

$$A^3 = \begin{bmatrix} -8 & 8 & 0 & 0 & 5 & 1 \\ 8 & -8 & 0 & 0 & -5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 6 & -6 & 0 & 0 & -1 & -3 \\ 6 & -6 & 0 & 0 & -2 & -2 \end{bmatrix}$$

and

$$(A^3)^\dagger = \begin{bmatrix} 1/4 & -1/4 & 0 & -11/12 & -1/12 & 5/6 \\ -1/4 & 1/4 & 0 & 11/12 & 1/12 & -5/6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3/4 & -3/4 & 0 & -7/3 & -1/6 & 13/6 \\ 3/4 & -3/4 & 0 & -3 & -1/2 & 5/2 \end{bmatrix},$$

a simple computation shows that the equality $A^3(A^3)^\dagger = (A^3)^\dagger A^3$ does not hold, so A^3 is not EP.

More generally, for $m > 3$ the matrix $B = A \oplus J_m(0)$ of index m satisfies that A is m-EP but A^m is not EP. In addition, it can be checked that neither A is 3-dagger nor B is m-dagger for $m > 3$ (see Lemma 2.4).

Remark 2.18. For $m \geq 3$, it is not possible to deduce (ii)-(iv) from Proposition 2.13 assuming that A is m-EP. A similar proof allows us to prove only the following statements: $(\Sigma K)^m (K^*\Sigma^{-1})^m (\Sigma K)^m = (\Sigma K)^m$, $(\Sigma K)^m (K^*\Sigma^{-1})^m = (\Sigma K)^{m-1}(K^*\Sigma^{-1})^{m-1}$ and $(K^*\Sigma^{-1})^m(\Sigma K)^m = (K^*\Sigma^{-1})^{m-1}(\Sigma K)^{m-1}$.

Proposition 2.19. *If A is m-EP with the decomposition as in (2) then K is a partial isometry of $\mathcal{C}((\Sigma K)^{m-2})$ into $\mathbb{C}^r$, that is $KK^*K(\Sigma K)^{m-2} = K(\Sigma K)^{m-2}$.*

Proof. It follows pre-multiplying expression in Theorem 2.8 (b) by L and using that $LL^* = I - KK^*$. □

We close this section with the following remark. In [8], the authors proved that if A is m-EP then $ind(\Sigma K) = m - 1$. Can we assure that ΣK is always an $(m-1)$-EP matrix? Example 2.17 illustrates that this is not the case.

3 More properties of m-EP matrices

Next, we state some links between m-EP matrices, Moore-Penrose inverses, Drazin inverses, and DMP inverses.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index m. Then A is m-EP if and only if $A^\dagger A^{m+1} = A^{m+1}A^\dagger = A^m$.*

Proof. 'Only if' part is easy. 'If' part. Let $A^\dagger A^{m+1} = A^{m+1}A^\dagger = A^m$. Consider $A^{m+1}A^\dagger = A^m$. Pre-multiplying by A^d gives $A^m A^\dagger = A^d A^m$. Post-multiplying $A^\dagger A^{m+1} = A^m$ by A^d gives $A^\dagger A^m = A^m A^d$. But $A^d A^m = A^m A^d$. Hence $A^\dagger A^m = A^m A^\dagger$. □

Corollary 3.2. *If $A \in \mathbb{C}^{n\times n}$ is an m-EP matrix then $A^d = A^m(A^\dagger)^{m+1} = (A^\dagger)^{m+1}A^m$.*

Proof. Firstly we define $X = A^m(A^\dagger)^{m+1}$. Taking into account the definition of m-EP matrix and applying repeatedly Theorem 3.1 we have

(a) $XAX = A^m(A^\dagger)^{m+1}AA^m(A^\dagger)^{m+1} = A^{2m}(A^\dagger)^{m+1}A(A^\dagger)^{m+1} = A^m A^\dagger A A^\dagger (A^\dagger)^m = A^m A^\dagger (A^\dagger)^m = A^m(A^\dagger)^{m+1} = X$.

(b) $A^{m+1}X = A^{m+1}A^m(A^\dagger)^{m+1} = A^m$.

(c) $AX = AA^m(A^\dagger)^{m+1} = A^m(A^\dagger)^m = (A^\dagger)^m A^m = (A^\dagger)^m A^\dagger A^{m+1} = (A^\dagger)^{m+1}A^m A = A^m(A^\dagger)^{m+1}A = XA$.

By the uniqueness of the Drazin inverse, $X = A^d$. Now, by using the definition of m-EP matrix we arrive at $A^m(A^\dagger)^{m+1} = (A^\dagger)^{m+1}A^m$. □

We can conclude that when A is m-EP then the (two unknowns) equation $A^m X = Y A^m$ holds for any pair of $X, Y \in \{A^\dagger, A^d, A^{d,\dagger}\}$.

Theorem 3.3. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index m. Then A is m-EP if and only if $A^{d,\dagger}A^m = A^m A^{d,\dagger} = A^d A^m = A^m A^d = A^\dagger A^m = A^m A^\dagger$.*

Proof. ($\Longrightarrow$) We first prove that if A is m-EP then $A^{d,\dagger}A^m = A^m A^{d,\dagger}$ and $A^m A^\dagger = A^d A^m$. In fact, pre-multiplying $A^\dagger A^m = A^m A^\dagger$ by $A^d A$, we have $A^{d,\dagger}A^m = A^m A^{d,\dagger}$. Since $A^m A^{d,\dagger} = A^m A^\dagger$, the other half of the statement follows. Now, premultiplying $A^\dagger A^m = A^m A^\dagger$ by $A^d A$ we get $A^{d,\dagger}A^m = A^d A A^\dagger A^m = A^d A A^m A^\dagger = A^d A^{m+1} A^\dagger = A^m A^\dagger$. Now, the result follows by Theorem 3.1.

($\Longleftarrow$) is trivial. □

Proposition 3.4. *Let $A \in \mathbb{C}^{n\times n}$ be m-EP and a partial isometry with $m \geq 2$. Then (a) $(A^\dagger)^m A = A(A^\dagger)^m$ and (b) $(A^\dagger)^{m+q}A = A(A^\dagger)^{m+q} = (A^\dagger)^{m+q-1}$ for all integer $q \geq 1$.*

Proof. (a) We have $(A^\dagger)^m A = (A^*)^m A = (A^* A^m)^* = (A^\dagger A^m)^* = (A^m A^\dagger)^* = (A^{m-1}AA^\dagger)^* = AA^\dagger(A^{m-1})^* = AA^\dagger(A^*)^{m-1} = AA^\dagger(A^\dagger)^{m-1} = A(A^\dagger)^m$.

(b) It can be proved by induction on q using (a). □

Proposition 3.5. *Let $A \in \mathbb{C}^{n\times n}$ be m-EP and a partial isometry. Then $(A^{m+q})^\dagger = A^\dagger(A^{m+q-1})^\dagger$ for any integer $q \geq 1$.*

Proof. In order to demonstrate that the reverse order law $(A^{m+1})^\dagger = A^\dagger(A^m)^\dagger$ holds we have to show: $\mathcal{C}(AA^*(A^m)^*) \subseteq \mathcal{C}((A^m)^*)$ and $\mathcal{C}((A^m)^*A^mA) \subseteq \mathcal{C}(A)$ (see [5], Theorem 1.4.2, Page 23). To prove the first inclusion, we apply Proposition 3.4 (a) obtaining $\mathcal{C}(A(A^*)^{m+1}) = \mathcal{C}(A(A^\dagger)^{m+1}) = \mathcal{C}(A(A^\dagger)^m A^\dagger) = \mathcal{C}((A^\dagger)^m AA^\dagger) = \mathcal{C}((A^\dagger)^m) = \mathcal{C}((A^*)^m)$. So, $\mathcal{C}(AA^*(A^m)^*) \subseteq \mathcal{C}((A^m)^*)$.

We now observe that if A is m-EP then it is easy to see that $A^m(A^\dagger)^m = (A^\dagger)^m A^m$. Then, second inclusion follows as $\mathcal{C}((A^m)^*A^mA) = \mathcal{C}((A^\dagger)^m A^m A) = \mathcal{C}(A^m(A^\dagger)^m A) \subseteq \mathcal{C}(A)$. Thus, the assertion has been proved for $q = 1$. Applying Proposition 3.4 (b), the equality for the remaining cases follow by induction. □

It is well known that $(A^\dagger)^2 = (A^2)^\dagger$ does not hold in general. In [7], Hartwig and Spindelböck studied some conditions for a square matrix A to be bi-dagger. We show that this equality holds for m-EP partial isometries and that all properties indicated in their paper are valid for this class of matrices. Note that, when A is a 2-EP matrix, A is not a q-EP matrix for any integer $q \geq 3$. Despite this, we can show the following.

Theorem 3.6. *Let $A \in \mathbb{C}^{n\times n}$ be a m-EP matrix. The following hold:*

(i) $A^\dagger A^{m+q} = A^{m+q}A^\dagger = A^{m+q-1}$ *for all integer $q \geq 1$.*

(ii) $A^\dagger A^{m+1}A^\dagger = A^\dagger A^m$ *and* $A^\dagger A^{m+q}A^\dagger = A^{m+q-2}$ *for all integer $q \geq 2$.*

(iii) If A is a partial isometry then $(A^{m+q})^\dagger = (A^\dagger)^{m+q}$ for all integer $q \geq 0$, so A^{m+q} is partial isometry for each integer $q \geq 0$.

Proof. (i) It follows by induction on q using Theorem 3.1.

(ii) Notice that $A^\dagger A^{m+1}A^\dagger = A^\dagger A^m$, since $A^\dagger A^{m+1}A^\dagger = (A^\dagger A^m)AA^\dagger = A^m A^\dagger AA^\dagger = A^\dagger A^m$. Now, using (i) twice we have $A^\dagger A^{m+q}A^\dagger = A^{m+q-1}A^\dagger = A^{m+q-2}$.

(iii) Firstly we prove the assertion for $q = 0$. From the condition $A^\dagger A^m = A^m A^\dagger$, it is easy to see $(A^\dagger)^m A^m = A^m (A^\dagger)^m$. Premultiplying $(A^\dagger)^m A^m = A^m (A^\dagger)^m$ by A^m, and using Theorem 3.1 repeatedly we have the following $A^m (A^\dagger)^m A^m = A^m A^m (A^\dagger)^m = A^{m-1}(A^{m+1} A^\dagger)(A^\dagger)^{m-1} = A^{m-1} A^m (A^\dagger)^{m-1} = A^{m-2}(A^{m+1}A^\dagger)(A^\dagger)^{m-2} = A^{m-2} A^m (A^\dagger)^{m-2} = \cdots = A^{m+1} A^\dagger = A^m$.

Further, as before, from $(A^\dagger)^m A^m = A^m (A^\dagger)^m$ we have $(A^\dagger)^m A^m (A^\dagger)^m = A^m (A^\dagger)^{m+m}$. Now, using Proposition 3.4 repeatedly, we get the following $A^m (A^\dagger)^{m+m} = A^{m-1}(A(A^\dagger)^{m+m}) = A^{m-1}(A^\dagger)^{m+m-1} = A^{m-2}(A(A^\dagger)^{m+m-1}) = A^{m-2}(A^\dagger)^{m+m-2} = \cdots = (A^\dagger)^m$. Thus, $(A^\dagger)^m A^m (A^\dagger)^m = (A^\dagger)^m$.

Finally, we show that $A^m (A^\dagger)^m$ and $(A^\dagger)^m A^m$ are hermitian. Since $A^\dagger = A^*$, we have $A^m (A^\dagger)^m = A^m (A^*)^m = A^m (A^m)^*$, which is hermitian. Similarly, $(A^\dagger)^m A^m$ is hermitian. Hence, $(A^m)^\dagger = (A^\dagger)^m$.

The remaining cases can be showed by induction using Proposition 3.5. □

Finally, some geometrical facts can be deduced for m-EP matrices. Recall the Sylvester rank formula for $M, N \in \mathbb{C}^{n\times n}$ [9]:

$$rank(MN) = rank(N) - \dim(\mathcal{N}(M) \cap \mathcal{C}(N)).$$

Proposition 3.7. *Suppose that A is m-EP. Then the following facts hold:*

(a) $\mathcal{C}(A^m A^\dagger) = \mathcal{C}(A^m) = \mathcal{C}(A^m A^*) = \mathcal{C}(A^\dagger A^m)$. *Consequently,* $rank(A^m A^\dagger) = rank(A^m) = rank(A^m A^*) = rank(A^\dagger A^m) = rank(A^m)^*$.

(b) $\mathcal{N}(A^*) \cap \mathcal{C}(A^m) = \{0\}$.

(c) $\mathcal{N}(A) \cap \mathcal{C}(A^* A^m) = \{0\}$.

(d) $\mathcal{N}(A^m A^*) \cap \mathcal{C}(A) \neq \{0\}$.

(e) $\mathcal{C}(A^m) \subseteq \mathcal{C}(A^*) \cap \mathcal{C}(A)$.

Proof. (a) On one hand $\mathcal{C}(A^m A^\dagger) \subseteq \mathcal{C}(A^m) = \mathcal{C}(A^{m-1}A) = \mathcal{C}(A^{m-1}AA^\dagger A) \subseteq \mathcal{C}(A^m A^\dagger)$, and on the other hand $\mathcal{C}(A^m A^\dagger) = \mathcal{C}(A^m A^*)$ since $A^\dagger = A^*(AA^*)^\dagger$ and $A^* = A^\dagger A A^*$. The last equality follows by definition.

(b) Using (a) and the Sylvester rank formula we have $rank(A^m) = rank(A^\dagger A^m) = rank(A^m) - \dim(\mathcal{N}(A^\dagger) \cap \mathcal{C}(A^m))$. Then, $\dim(\mathcal{N}(A^\dagger) \cap \mathcal{C}(A^m)) = 0$ and thus $\mathcal{N}(A^*) \cap \mathcal{C}(A^m) = \mathcal{N}(A^\dagger) \cap \mathcal{C}(A^m) = \{0\}$.

(c) Using that $A^\dagger = (A^*A)^\dagger A^*$, we have $A^\dagger A^m = [(A^*A)^\dagger][A^*A^m]$. The Sylvester formula and item (a) give $\mathcal{N}(A^*A)^\dagger \cap \mathcal{C}(A^*A^m) = \{0\}$. Since $\mathcal{N}(A^*A)^\dagger = \mathcal{N}(A^*A)^* = \mathcal{N}(A^*A) = \mathcal{N}(A)$ we arrive at $\mathcal{N}(A) \cap \mathcal{C}(A^*A^m) = \{0\}$.

(d) Using that $A^\dagger = A^*(AA^*)^\dagger$ we have $A^m A^\dagger = [A^m A^*][(AA^*)^\dagger]$. The Sylvester formula gives $rank(A^m) = rank(A) - dim[\mathcal{N}(A^m A^*) \cap \mathcal{C}(A)]$ where we have used that $\mathcal{C}(B^\dagger) = \mathcal{C}(B^*)$. Since $rank(A^{m+1}) = rank(A^m) < \cdots < rank(A^2) < rank(A)$ we get $dim[\mathcal{N}(A^m A^*) \cap \mathcal{C}(A)] \neq 0$.

(e) The orthogonal spaces to $\mathcal{C}(A^m) = \mathcal{C}(A^\dagger A^m)$ yield the following $\mathcal{N}((A^m)^*) = \mathcal{N}((A^{m-1})^* A^\dagger A) \supseteq \mathcal{N}(A)$, and computing again the orthogonal spaces we arrive at the result.

$\square$

4 The m-normal class of matrices

In this section we study an interesting particular case of m-EP matrices. The class of square matrices A of index m satisfying $A^*A^m = A^m A^*$ will be called m-normal matrices. We observe that:

(1) If N is any $p \times p$ normal matrix then

$$\begin{bmatrix} J_m(0) & 0 \\ 0 & N \end{bmatrix} \tag{8}$$

is m-normal.

(2) A matrix can be m-normal without being a partial isometry, e.g., $A = \begin{bmatrix} -a & a \\ -a & a \end{bmatrix}$, where $a \in \mathbb{R}, a \neq 0, 1$.

For some related results on m-normal matrices we refer the reader to [11] where some characterizations are given in the setting of rings. However, we point out that in [11] m does not correspond necessarily to the index.

In order to obtain a characterization for m-normal matrices, we take a matrix $A \in \mathbb{C}^{n \times n}$ of rank $r > 0$ in the Hartwig-Spindelböck decomposition, i.e.,

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \tag{9}$$

as in (2). Then A is m-normal if and only if

(a) $(\Sigma L)^*(\Sigma K)^{m-1} = 0$
(b) $(\Sigma K)^{m-1}(\Sigma L) = 0$
(c) $(\Sigma K)^{m-1}\Sigma^2 = (\Sigma K)^*(\Sigma K)^m$
(d) $ind(\Sigma K) = m - 1$.

This last result can be proved as the previous one for m-EP matrices and its proof is omitted.

Consider the following matrix A decomposed in its Jordan canonical form:

$$A = \begin{bmatrix} -a & a \\ -a & a \end{bmatrix} = \begin{bmatrix} -a & 1 \\ -a & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1/a \\ 1 & -1 \end{bmatrix} \qquad a \in \mathbb{R}, a \neq 0.$$

It can be seen that $m = 2$, $A^*A^2 = A^2A^*$ and $A^\dagger A^2 = A^\dagger A^2$ using that

$$A^\dagger = \frac{1}{4a^2} \begin{bmatrix} -a & -a \\ a & a \end{bmatrix}.$$

Indeed, every Jordan block $J_m(0)$ is m-normal and m-EP.

However, the matrix

$$A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ -\sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is 2-EP (to compute $A^\dagger$ observe that the 2×2 sub-matrix in the N-W corner is nonsingular and the 2×2 sub-matrix in the S-E is $J_2(0)$) but A is not 2-normal.

Since a normal matrix is EP, our next result is not unexpected. It follows that the class of m-normal matrices is a subclass of class of m-EP matrices.

Theorem 4.1. *If $A \in \mathbb{C}^{n\times n}$ is a m-normal matrix then A is a m-EP matrix.*

Proof. Let A be m-normal. So, $A^*A^m = A^mA^*$. Post-multiplying both sides by $AA^\dagger$ we have $A^*A^mAA^\dagger = A^mA^*AA^\dagger$, that is, $A^*A^{m+1}A^\dagger = A^mA^*(AA^\dagger)^*$ or equivalently $A^*A^{m+1}A^\dagger = A^mA^*$. Pre-multiplying both sides by $(A^\dagger)^*$ we get $(AA^\dagger)^*A^{m+1}A^\dagger = (A^\dagger)^*A^*A^m$, that is, $AA^\dagger A^{m+1}A^\dagger = AA^\dagger A^m$ or equivalently $A^{m+1}A^\dagger = A^m$. Similarly by pre-multiplying both sides of $A^*A^m = A^mA^*$ by $A^\dagger A$ and then post-multiplying by $(A^\dagger)^*$ and using $A^mA^* = A^*A^m$, we have $A^\dagger A^{m+1} = A^m$. Thus m-normality implies m-EPness. $\square$

We close this paper noticing that a partial isometry is m-EP if and only if it is m-normal.

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