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Research Article

Random First-Order Linear Discrete Models and Their Probabilistic Solution: A Comprehensive Study

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This paper presents a complete stochastic solution represented by the first probability density function for random first-order linear difference equations. The study is based on Random Variable Transformation method. The obtained results are given in terms of the probability density functions of the data, namely, initial condition, forcing term, and diffusion coefficient. To conduct the study, all possible cases regarding statistical dependence of the random input parameters are considered. A complete collection of illustrative examples covering all the possible scenarios is provided.

1. Introduction and Motivation

The birth/death rates of species in biology, the volatility of assets in finance, the transmission rates of the spread of epidemics or social addictions in epidemiology, the diffusion and advection coefficients of mass transport processes in physics, and so forth are quantities that, in practice, involve uncertainty. Thus, their deterministic modelling is clearly limited. This motivates the search of mathematical models that consider randomness in their formulation. Deterministic differential and difference equations have been demonstrated to be useful mathematical representations for modelling numerous real problems. The consideration of randomness in these types of equations is a relatively recent research area whose main goal is to extend classical deterministic results to the random scenario. Regarding continuous models, most of the contributions have focussed on Itô-type stochastic differential equations. In this class of differential equations, uncertainty is considered through a Gaussian and stationary stochastic process (SP) called white noise, which is the derivative of the Wiener SP [1–3]. Autoregressive (AR) models can be interpreted as their discrete counterpart. These types of models are extensively used in time series analysis in statistics [4]. Some recent interesting models based on Itô-type stochastic differential equations include [5–7], for instance. Complementary to these approaches, uncertainty can be directly introduced in differential and difference equations by assuming that coefficients, source term, and/or initial/boundary conditions are random variables and/or stochastic processes. Under this approach, the probability distributions associated with RVs and SPs are not required to be Gaussian. This approach leads to the area usually referred to as random differential/difference equations [8], [9, p. 66]. Intensive studies on random differential and difference equations have been undertaken only over the last few decades. Currently, they are exerting a profound influence on the analysis of many problems in engineering and science [10, 11]. Most of these contributions are based on mean square calculus [8, 12–14].

The solution of a random difference equation is a discrete SP, say $\{Z_n : n \geq 0\}$. In dealing with random difference equations, the main goals are computing the solution SP and its statistical characteristics, such as the mean function, $\mathbb{E}[Z_n]$, and the variance function, $\mathbb{V}[Z_n] = \mathbb{E}[(Z_n)^2] - (\mathbb{E}[Z_n])^2$. Despite being more complicated, the computation of the first probability density function (1-PDF), $f_1(z,n)$, associated with solution Z_n is more convenient since from 1-PDF, besides determining the mean and the variance functions, one can also compute higher-order statistical moments of Z_n:

$$\mathbb{E}\left[(Z_n)^k\right] = \int_{-\infty}^{\infty} z^k f_1(z,n)\,dz, \quad n,k = 0,1,2,\ldots. \tag{1}$$

In the context of random differential equations, a number of contributions have dealt with the computation of the 1-PDF in specific problems appearing in physics [15–17] or in mathematics [18, 19]. A comprehensive study for the random first-order linear differential equation under general hypotheses has been recently published by the authors of [20]. The unifying element of all these contributions is the Random Variable Transformation (RVT) method. This technique permits, under certain hypotheses that will be specified later, the computation of the PDF of a random variable (RV) resulting after mapping another RV whose PDF is known [21–23]. Although RVT technique is a classical probability result, it must be pointed out that its application to study differential equations with randomness is likely due to Professor M. El-Tawil. He was the first author who had used RVT technique to present some approximate solutions of random differential equations. In [24], the RVT method together with different numerical schemes (finite difference and Runge-Kutta) is implemented to get the 1-PDF of the solution SP in solving both partial and/or ordinary (first- and/or second-order) random differential equations. His contribution includes the definition of probabilistic error and a formula for its computation as well.

As continuation of the study initiated in [20], the aim of this paper is to develop a comprehensive study to determine the 1-PDF of the solution discrete SP of the following random initial value problem:

$$\begin{aligned} Z_{n+1} &= AZ_n + B, \quad n = 0, 1, 2, \ldots, \\ Z_0 &= \Gamma_0, \end{aligned} \tag{2}$$

taking advantage of RVT method. All the input parameters, Γ_0, B, and A, are assumed to be continuous RVs defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Although RVT method is the unifying technique used to conduct the analysis of model (2) and also the one studied in [20], it is important to point out that both problems have distinctly different nature. Indeed, model (2) is discrete whereas problem faced in [20] is continuous counterpart. As we will see later, Proposition 1 constitutes the key result we have had to establish to conduct the analysis of problem (2). Its formulation is based on RVT method. Throughout the paper, significant differences between the new analytical expressions and graphical behaviour of the 1-PDF of the solution of (2) and its continuous counterpart will be also exhibited.

As it also happens in the deterministic framework, in general, the study of random difference equations has been less prolific than of random differential equations. In [25], the authors study the mean square exponential stability of impulsive stochastic difference equations. In [26], one studies random matrix linear difference equations assuming that diffusion coefficient A in (2) is a deterministic matrix rather than a RV. In [26], the authors focus on the computation of the mean vector and variance-covariance matrix of the solution discrete SP instead of the 1-PDF. In this paper, we present a comprehensive study of model (2) assuming randomness in all the inputs, Γ_0, B, and A. This includes the general case where inputs are statistically dependent. From a statistical point of view, (2) is a generalization of autoregressive model of order 1, AR(1), where uncertainty is enclosed in the term B through white noise.

For the sake of clarity in the presentation and in order to facilitate the comparison of the results obtained in this paper against the ones achieved in [20], we will keep the notation used in both contributions identical. Hence, the domain of the random inputs Γ_0, B, and A will be denoted by

$$\begin{aligned} D_{\Gamma_0} &= \{\gamma_0 = \Gamma_0(\omega),\ \omega \in \Omega : \gamma_{0,1} \le \gamma_0 \le \gamma_{0,2}\}, \\ D_B &= \{b = B(\omega),\ \omega \in \Omega : b_1 \le b \le b_2\}, \\ D_A &= \{a = A(\omega),\ \omega \in \Omega : a_1 \le a \le a_2\}, \end{aligned} \tag{3}$$

respectively. From this point forward, we will omit the ω-notation when writing RVs. In this manner, for instance, we will write Γ_0 rather than $\Gamma_0(\omega)$. The same can be said for the notation of the PDFs that appear throughout this paper. For example, $f_{\Gamma_0}(\gamma_0)$ will denote the PDF of RV Γ_0; $f_{B,A}(b, a)$ will denote the joint PDF of RVs B and A; we will write $f_{\Gamma_0,B,A}(\gamma_0, b, a)$ for the joint PDF of the random vector (Γ_0, B, A), and so on. As usual, we will assume that any PDF is null outside its domain.

Based on the same arguments exhibited in [20, Section 1], we will distinguish the thirteen cases listed in Table 1 to conduct our study. These casuistries consider whether the random difference equation (2) is homogeneous or nonhomogeneous as well as all possible cases regarding the random or deterministic nature of the input parameters Γ_0, B, and A. Note that, by splitting the study in all these cases, the comparison of the results concerning the discrete problem (2) against its continuous counterpart is facilitated. Examples have also been devised with the same aim. Even more, in the majority of the examples, the same statistical distributions have been taken as in [20] to highlight better analogies and differences between both models.

The paper is organized as follows. Section 2 is addressed to introduce the preliminaries related to RVT technique required to conduct our study. In this section, we establish a key result related to the PDF of the power transformation of RVs which will be crucial to deal with Case I.2 of Table 1, where uncertainty just enters in model (2) through the RV A. Section 3 is divided into three subsections where the 1-PDF of the discrete solution SP of (2) is determined for each one of Cases I, II, and III listed in Table 1. Illustrative examples covering the thirteen cases are provided throughout the paper. In the last section, we present our conclusions. Finally, we present an appendix where the main obtained results are collected in order to facilitate their practical use.

2. Preliminaries

As it has been pointed out in the previous section, the goal of this paper is to compute the 1-PDF $\{f_1(z, n) : n \ge 0\}$ of the solution SP of problem (2) in each one of the cases listed in Table 1. The key result to achieve this goal is the RVT method. This is a probabilistic technique that allows us to calculate the PDF of a random variable/vector which is obtained after mapping another random variable/vector whose PDF is known. Depending on the type of mapping as

Table 1: List of the thirteen different cases considered to conduct the full study. This classification is made regarding whether the discrete initial value problem is homogeneous (H) or nonhomogeneous (NH) and the way that uncertainty is considered (one random variable or a random vector in two or three dimensions).

Type	Discrete initial value problem	Case
H	$\begin{cases} Z_{n+1} = AZ_n, \quad n = 0, 1, 2, \ldots \\ Z_0 = \Gamma_0 \end{cases}$ (I)	I.1 Γ_0 is a random variable I.2 A is a random variable I.3 (Γ_0, A) is a random vector
NH	$\begin{cases} Z_{n+1} = Z_n + B, \quad n = 0, 1, 2, \ldots \\ Z_0 = \Gamma_0 \end{cases}$ (II)	II.1 Γ_0 is a random variable II.2 B is a random variable II.3 (Γ_0, B) is a random vector
	$\mathbb{P}[\{\omega \in \Omega : A(\omega) = 1\}] = 0$ $\begin{cases} Z_{n+1} = AZ_n + B, \quad n = 0, 1, 2, \ldots \\ Z_0 = \Gamma_0 \end{cases}$ (III)	III.1 Γ_0 is a random variable III.2 B is a random variable III.3 A is a random variable III.4 (Γ_0, B) is a random vector III.5 (Γ_0, A) is a random vector III.6 (B, A) is a random vector III.7 (Γ_0, B, A) is a random vector

well as its dimension, several versions of RVT method can be established. Throughout this paper, the general scalar version and its specialization to the linear case, as well as the general multidimensional version, will be required. These results are stated in [20, Theorem 1, Proposition 2 and Theorem 4], respectively.

Next, we will establish the following result concerning the PDF of a RV which is obtained after mapping another RV via a power transformation. This result will play a relevant role in the analysis of Case I.2. listed in Table 1. It is important to underline the notion that power transformation is a distinctive feature to describe the solution SP of the discrete model (2) against the exponential transformation which appears when its continuous counterpart is dealt with. In this sense, the next result plays the same role as [20, Proposition 3] performed there.

Proposition 1 (RVT technique: power transformation). *Let X be a continuous RV with domain $D_X = \{x : x_1 \le x \le x_2\}$ and PDF $f_X(x)$. Let one denote by $\delta(\cdot)$ the Dirac delta function. Then, the PDF $f_Y(y)$ of the power transformation $Y = kX^n$, with $n \ge 0$ and $k \ne 0$, is given by the following:*

(i) If $n = 0$,

$$f_Y(y) = \delta(y - k), \quad -\infty < y < +\infty. \tag{4}$$

(ii) If $n = 1$,

$$f_Y(y) = \frac{1}{|k|} f_X\left(\frac{y}{k}\right), \quad \begin{cases} kx_1 \le y \le kx_2, & \text{if } k > 0, \\ kx_2 \le y \le kx_1, & \text{if } k < 0. \end{cases} \tag{5}$$

(iii) If n is even, then one has the following:

Case 1 ($x_1 \ge 0$):

$$f_Y(y) = \frac{1}{|k| n} \left| \left(\frac{y}{k}\right)^{(1-n)/n} \right| f_X\left(+\sqrt[n]{\frac{y}{k}}\right), \quad \begin{cases} k(x_1)^n \le y \le k(x_2)^n, & \text{if } k > 0, \\ k(x_2)^n \le y \le k(x_1)^n, & \text{if } k < 0. \end{cases} \tag{6}$$

Case 2 ($x_2 \le 0$):

$$f_Y(y) = \frac{1}{|k| n} \left| \left(\frac{y}{k}\right)^{(1-n)/n} \right| f_X\left(-\sqrt[n]{\frac{y}{k}}\right), \quad \begin{cases} k(x_2)^n \le y \le k(x_1)^n, & \text{if } k > 0, \\ k(x_1)^n \le y \le k(x_2)^n, & \text{if } k < 0. \end{cases} \tag{7}$$

Case 3 ($x_1 x_2 < 0$).

Case 3.1 ($x_2 \ge |x_1|$):

$$f_Y(y) = f_Y^1(y) + f_Y^2(y), \tag{8}$$

where

$$f_Y^1(y) = \frac{1}{|k| n} \left| \left(\frac{y}{k}\right)^{(1-n)/n} \right| \left\{ f_X\left(-\sqrt[n]{\frac{y}{k}}\right) + f_X\left(+\sqrt[n]{\frac{y}{k}}\right) \right\}, \tag{9}$$

on the domain

$$\begin{cases} 0 < y \le k(x_1)^n, & \text{if } k > 0, \\ k(x_1)^n \le y < 0, & \text{if } k < 0, \end{cases}$$

$$f_Y^2(y) = \frac{1}{|k| n} \left| \left(\frac{y}{k}\right)^{(1-n)/n} \right| f_X\left(+\sqrt[n]{\frac{y}{k}}\right), \quad \begin{cases} k(x_1)^n < y \le k(x_2)^n, & \text{if } k > 0, \\ k(x_2)^n \le y < k(x_1)^n, & \text{if } k < 0. \end{cases} \tag{10}$$

Case 3.2 ($x_2 < |x_1|$):

$$f_Y(y) = f_Y^1(y) + f_Y^2(y), \tag{11}$$

where

$$f_Y^1(y) = \frac{1}{|k|\,n}\left|\left(\frac{y}{k}\right)^{(1-n)/n}\right|\left\{f_X\left(-\sqrt[n]{\frac{y}{k}}\right) + f_X\left(+\sqrt[n]{\frac{y}{k}}\right)\right\}, \tag{12}$$

on the domain

$$\begin{cases} 0 < y \le k(x_2)^n, & \text{if } k > 0, \\ k(x_2)^n \le y < 0, & \text{if } k < 0, \end{cases}$$

$$f_Y^2(y) = \frac{1}{|k|\,n}\left|\left(\frac{y}{k}\right)^{(1-n)/n}\right| f_X\left(-\sqrt[n]{\frac{y}{k}}\right), \tag{13}$$

$$\begin{cases} k(x_2)^n < y \le k(x_1)^n, & \text{if } k > 0, \\ k(x_1)^n \le y < k(x_2)^n, & \text{if } k < 0. \end{cases}$$

(iv) If $n \ge 3$ and is odd, then one has the following:

Case 1 ($x_1 > 0$ or $x_2 < 0$):

$$f_Y(y) = \frac{1}{|k|\,n}\left(\frac{y}{k}\right)^{(1-n)/n} f_X\left(\sqrt[n]{\frac{y}{k}}\right), \quad \begin{cases} k(x_1)^n \le y \le k(x_2)^n, & \text{if } k > 0, \\ k(x_2)^n \le y \le k(x_1)^n, & \text{if } k < 0. \end{cases} \tag{14}$$

Case 2 ($x_1 x_2 \le 0$):

$$f_Y(y) = \frac{1}{|k|\,n}\left(\frac{y}{k}\right)^{(1-n)/n} f_X\left(\sqrt[n]{\frac{y}{k}}\right), \quad \begin{cases} k(x_1)^n \le y < 0,\ 0 < y \le k(x_2)^n, & \text{if } k > 0, \\ k(x_2)^n \le y < 0,\ 0 < y \le k(x_1)^n, & \text{if } k < 0. \end{cases} \tag{15}$$

If $k = 0$, for $n \ge 0$,

$$f_Y(y) = \delta(y), \quad -\infty < y < +\infty. \tag{16}$$

Proof. (i) If $n = 0$, then $Y = k$ w.p. 1 and its PDF is given by

$$f_Y(y) = \delta(y - k), \quad -\infty < y < +\infty. \tag{17}$$

(ii) If $n = 1$, the mapping r is monotone on the whole domain of RV X; then the inverse function of r takes the form

$$x = s(y) = \frac{y}{k}, \tag{18}$$

whose derivative is given by

$$s'(y) = \frac{1}{k}. \tag{19}$$

Then, applying expression [20, Eq. (3)] and taking into account (18)-(19), one gets

$$f_Y(y) = \frac{1}{|k|} f_X\left(\frac{y}{k}\right), \quad \begin{cases} kx_1 \le y \le kx_2, & \text{if } k > 0, \\ kx_2 \le y \le kx_1, & \text{if } k < 0. \end{cases} \tag{20}$$

(iii) Let us assume n is even. We distinguish three cases depending on the domain $D_X = \{x : x_1 \le x \le x_2\}$ of the RV X. In order to avoid any confusion, it is important to note that these cases are mutually exclusive.

Case 1 ($x_1 \ge 0$). Assuming $k > 0$, the mapping r is monotone in D_X. Hence, the inverse function of r, denoted by $s(y)$, takes the form

$$x = s(y) = +\sqrt[n]{\frac{y}{k}}, \tag{21}$$

and its derivative, $s'(y)$, is given by

$$s'(y) = +\frac{1}{kn}\left(\frac{y}{k}\right)^{(1-n)/n}. \tag{22}$$

Then, applying [20, Eq. (3)] and taking into account (21)-(22), one gets

$$f_Y(y) = \frac{1}{kn}\left|\left(\frac{y}{k}\right)^{(1-n)/n}\right| f_X\left(+\sqrt[n]{\frac{y}{k}}\right), \quad k(x_1)^n \le y \le k(x_2)^n. \tag{23}$$

When $k < 0$, the reasoning is analogous. Then, considering expression (23) when $k > 0$, both cases for the sign of k can be expressed as follows:

$$f_Y(y) = \frac{1}{|k|\,n}\left|\left(\frac{y}{k}\right)^{(1-n)/n}\right| f_X\left(+\sqrt[n]{\frac{y}{k}}\right), \tag{24}$$

on the domain

$$\begin{cases} k(x_1)^n \le y \le k(x_2)^n, & \text{if } k > 0, \\ k(x_2)^n \le y \le k(x_1)^n, & \text{if } k < 0. \end{cases} \tag{25}$$

Case 2 ($x_2 \le 0$). In this case, the mapping r is monotone in D_X. Using an analogous development as we did in Case 1, one obtains

$$f_Y(y) = \frac{1}{|k|\,n}\left|\left(\frac{y}{k}\right)^{(1-n)/n}\right| f_X\left(-\sqrt[n]{\frac{y}{k}}\right), \tag{26}$$

on the domain

$$\begin{cases} k(x_2)^n \le y \le k(x_1)^n, & \text{if } k > 0, \\ k(x_1)^n \le y \le k(x_2)^n, & \text{if } k < 0. \end{cases} \tag{27}$$

Case 3 ($x_1 x_2 < 0$). We will consider two subcases: $x_2 \geq |x_1|$ and $x_2 < |x_1|$. We split each subcase into appropriate subintervals in order to apply [20, Theorem 1] in order to compute the PDF.

Case 3.1 ($x_2 \geq |x_1|$). Let us consider the piece $[x_1, -x_1]$. On the subinterval $[x_1, 0]$, the mapping r (denoted by r_1, for the sake of clarity) is monotone and its inverse s_1 is

$$x = s_1(y) = -\sqrt[n]{\frac{y}{k}}, \tag{28}$$

whose derivative, $s_1'(y)$, for $y \neq 0$, is given by

$$s_1'(y) = -\frac{1}{kn}\left(\frac{y}{k}\right)^{(1-n)/n}. \tag{29}$$

On the other hand, on the piece $[0, -x_1]$, its corresponding mapping r_2 is monotone and its inverse s_2 is

$$x = s_2(y) = +\sqrt[n]{\frac{y}{k}}, \tag{30}$$

and, for $y \neq 0$,

$$s_2'(y) = +\frac{1}{kn}\left(\frac{y}{k}\right)^{(1-n)/n}. \tag{31}$$

Notice that $s_1'(y) \neq 0$ and $s_2'(y) \neq 0$ if $y \neq 0$. Then, applying [23, Theorem 2.1.8] and taking into account (28)–(31), one gets

$$\begin{aligned} f_Y^1(y) &= f_X(s_1(y))\left|s_1'(y)\right| + f_X(s_2(y))\left|s_2'(y)\right| \\ &= \frac{1}{|k|\,n}\left|\left(\frac{y}{k}\right)^{(1-n)/n}\right|\left\{f_X\left(-\sqrt[n]{\frac{y}{k}}\right) + f_X\left(+\sqrt[n]{\frac{y}{k}}\right)\right\}, \end{aligned} \tag{32}$$

on the domain

$$\begin{cases} 0 < y \leq k(x_1)^n, & \text{if } k > 0, \\ k(x_1)^n \leq y < 0, & \text{if } k < 0. \end{cases} \tag{33}$$

As usual, we assume $f_Y^1(y) \equiv 0$ outside domain (33).

To complete the computation of PDF $f_Y(y)$ on the whole domain, finally we consider the subinterval $]-x_1, x_2]$, where the RV X is positive. Hence, we are in Case 1 and according to (24) it follows that

$$f_Y^2(y) = \frac{1}{|k|\,n}\left|\left(\frac{y}{k}\right)^{(1-n)/n}\right| f_X\left(+\sqrt[n]{\frac{y}{k}}\right), \tag{34}$$

on the domain

$$\begin{cases} k(-x_1)^n < y \leq k(x_2)^n, & \text{if } k > 0, \\ k(x_2)^n \leq y < k(-x_1)^n, & \text{if } k < 0. \end{cases} \tag{35}$$

Again, as usual, we assume $f_Y^2(y) \equiv 0$ outside domain (35). Notice that one satisfies

$$\begin{aligned} &\int_0^{k(x_1)^n} f_Y^1(y)\,dy + \int_{k(x_1)^n}^{k(x_2)^n} f_Y^2(y)\,dy = 1 \quad \text{if } k > 0, \\ &\int_{k(x_2)^n}^{k(x_1)^n} f_Y^2(y)\,dy + \int_{k(x_1)^n}^{0} f_Y^1(y)\,dy = 1 \quad \text{if } k < 0. \end{aligned} \tag{36}$$

To summarize, from (32)–(35), the complete PDF of $Y = kX^n$ in this case is given by

$$\begin{gathered} f_Y(y) = f_Y^1(y) + f_Y^2(y), \\ \begin{cases} 0 < y \leq k(x_2)^n, & \text{if } k > 0, \\ k(x_2)^n \leq y < 0, & \text{if } k < 0. \end{cases} \end{gathered} \tag{37}$$

Case 3.2 ($x_2 < |x_1|$). Let us consider the piece $[-x_2, x_2]$. Following analogous reasoning as in Case 3.1, according to (32), one obtains the piece of the PDF of the power transformation Y:

$$\begin{aligned} &f_Y^1(y) \\ &= \frac{1}{|k|\,n}\left|\left(\frac{y}{k}\right)^{(1-n)/n}\right|\left\{f_X\left(-\sqrt[n]{\frac{y}{k}}\right) + f_X\left(+\sqrt[n]{\frac{y}{k}}\right)\right\}, \end{aligned} \tag{38}$$

on the domain

$$\begin{cases} 0 < y \leq k(x_2)^n, & \text{if } k > 0, \\ k(x_2)^n \leq y < 0, & \text{if } k < 0. \end{cases} \tag{39}$$

We assume $f_Y^1(y) \equiv 0$ outside domain (39).

We complete the computation of PDF $f_Y(y)$ on the whole domain considering the subinterval $[x_1, -x_2[$. In this subinterval, X is negative. As it was shown in Case 2, the PDF is given by

$$f_Y^2(y) = \frac{1}{|k|\,n}\left|\left(\frac{y}{k}\right)^{(1-n)/n}\right| f_X\left(-\sqrt[n]{\frac{y}{k}}\right), \tag{40}$$

on the domain

$$\begin{cases} k(x_2)^n < y \leq k(x_1)^n, & \text{if } k > 0, \\ k(x_1)^n \leq y < k(x_2)^n, & \text{if } k < 0. \end{cases} \tag{41}$$

We assume $f_Y^2(y) \equiv 0$ outside domain (41).

To summarize, from (38)–(41), the complete PDF of $Y = kX^n$ in this case is

$$\begin{gathered} f_Y(y) = f_Y^1(y) + f_Y^2(y), \\ \begin{cases} 0 < y \leq k(x_1)^n, & \text{if } k > 0, \\ k(x_1)^n \leq y < 0, & \text{if } k < 0. \end{cases} \end{gathered} \tag{42}$$

(iv) Let us assume that $n \geq 3$ and is odd. The mapping r is monotone on the whole domain of RV X; then the inverse function of r takes the form

$$x = s(y) = \sqrt[n]{\frac{y}{k}}, \tag{43}$$

whose derivative, for $y \neq 0$, is given by

$$s'(y) = \frac{1}{kn}\left(\frac{y}{k}\right)^{(1-n)/n}. \tag{44}$$

Notice that $s'(y) \neq 0$ if $y \neq 0$. Therefore, we distinguish two cases depending on the domain $D_X = \{x : x_1 \leq x \leq x_2\}$ of the RV X.

Case 1 ($x_1 > 0$ or $x_2 < 0$). Applying [20, Theorem 1] and taking into account (43)-(44), one gets

$$f_Y(y) = \frac{1}{|k|\, n} \left(\frac{y}{k}\right)^{(1-n)/n} f_X\left(\sqrt[n]{\frac{y}{k}}\right), \quad \begin{cases} k(x_1)^n \leq y \leq k(x_2)^n, & \text{if } k > 0, \\ k(x_2)^n \leq y \leq k(x_1)^n, & \text{if } k < 0. \end{cases} \tag{45}$$

Case 2 ($x_1 x_2 \leq 0$). Applying [20, Theorem 1] and taking into account (43)-(44), one gets

$$f_Y(y) = \frac{1}{|k|\, n} \left(\frac{y}{k}\right)^{(1-n)/n} f_X\left(\sqrt[n]{\frac{y}{k}}\right), \quad \begin{cases} k(x_1)^n \leq y < 0,\ 0 < y \leq k(x_2)^n, & \text{if } k > 0, \\ k(x_2)^n \leq y < 0,\ 0 < y \leq k(x_1)^n, & \text{if } k < 0. \end{cases} \tag{46}$$

Finally, if $k = 0$ and $n \geq 0$, then $Y = 0$ with probability 1 (w.p. 1) and its PDF is given by

$$f_Y(y) = \delta(y), \quad -\infty < y < +\infty. \tag{47}$$

□

3. Case Study: Homogeneous Discrete Initial Value Problem (I)

This section is addressed to compute the 1-PDF $\{f_1(z,n) : n \geq 0\}$ of the solution discrete stochastic process $\{Z_n : n \geq 0\}$ of the homogeneous discrete initial value problem (I) in all different cases collected in Table 1. In this case, the solution $\{Z_n : n \geq 0\}$ can be expressed as follows:

$$Z_0 = \Gamma_0, \quad Z_n = A^n \Gamma_0, \quad n = 1, 2, \ldots. \tag{48}$$

3.1. *Case I.1: Γ_0 Is a Random Variable.*

For the sake of clarity in the presentation, we rewrite solution (48) by using the lowercase letter a in order to indicate the deterministic character of parameter A:

$$Z_0 = \Gamma_0, \quad Z_n = a^n \Gamma_0, \quad n = 1, 2, \ldots. \tag{49}$$

Let $n \geq 0$ be an arbitrary and fixed integer. Let us assume $a \neq 0$ and denote $Z = Z_n = a^n \Gamma_0$. By applying [20, Proposition 2] to

$$Y = Z, \quad X = \Gamma_0, \quad \alpha = a^n \neq 0, \quad \beta = 0, \tag{50}$$

one gets the 1-PDF

$$f_1(z,n) = \frac{1}{|a|^n} f_{\Gamma_0}\left(\frac{z}{a^n}\right), \quad n = 0, 1, 2, \ldots, \ z \in \mathbb{R}. \tag{51}$$

Note that if $a = 0$, from (49), it follows that $Z_n = 0$ w.p. 1 for each $n \geq 1$ and $Z_0 = \Gamma_0$. So, the 1-PDF for the trivial case, $a = 0$, can be written as

$$f_1(z,n) = \begin{cases} f_{\Gamma_0}(z), & n = 0, \\ \delta(z), & n = 1, 2, \ldots, \end{cases} \quad z \in \mathbb{R}. \tag{52}$$

In order to facilitate the comparison of the 1-PDF of the solution of problem (49) against its continuous counterpart provided in [20], in the following example, Γ_0 is assumed to be a Gaussian RV. Note that we are going to consider a standard distribution, although the method is also able to be applied to nonstandard distributions.

Example 1. Let us assume $a \neq 0$ and consider Γ_0 a Gaussian distribution, $\Gamma_0 \sim N(\mu; \sigma^2)$. Hence, applying (51), the 1-PDF of $\{Z_n : n \geq 0\}$ is given by

$$f_1(z,n) = \frac{1}{|a|^n \sqrt{2\pi\sigma^2}} e^{-(z/a^n - \mu)^2/2\sigma^2}, \quad n = 0, 1, 2, \ldots, \ z \in \mathbb{R}. \tag{53}$$

It can be checked that $f_1(z,n)$ is a PDF for each $n \geq 0$. Figure 1 shows $f_1(z,n)$ for $n \in \{0, \ldots, 10\}$, in the particular case that $\Gamma_0 \sim N(0;1)$: $a = 10/9$ (a) and $a = 9/10$ (b). Note the different behavior of 1-PDF depending on the modulus of the parameter a. This is in agreement with the expectation and variance of the solution which are given, respectively, by

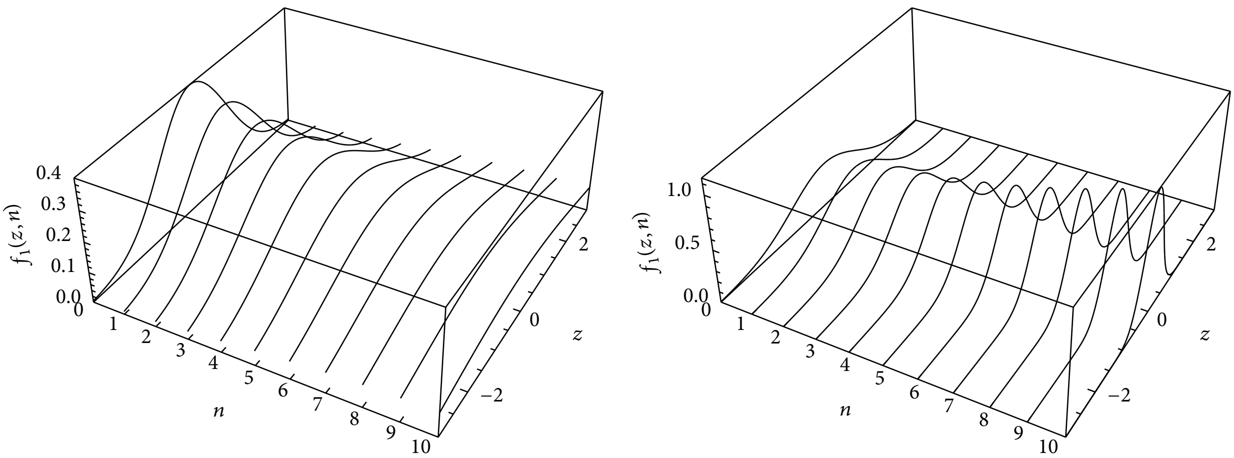
$$\mathbb{E}[Z_n] = 0, \quad \mathbb{V}[Z_n] = a^{2n}\sigma^2, \quad n = 0, 1, 2, \ldots. \tag{54}$$

Indeed, in Figure 1, we observe that, for each n, the 1-PDF is symmetric about $z = 0$, whereas, in the case that $a = 10/9 > 1$ ($a = 9/10 \in]-1, 1[$), it becomes flat (sharp) as n increases. This means that its variability around zero, that is, the variance, tends to infinity (zero).

3.2. *Case I.2: A Is a Random Variable.*

In order to emphasize the deterministic nature of the initial condition Γ_0, we recast (48) by using the lowercase letter γ_0:

$$Z_n = A^n \gamma_0, \quad n = 0, 1, 2, \ldots. \tag{55}$$



(a) (b)

Figure 1: $f_1(z,n)$, $n \in \{0,1,\ldots,10\}$, in Example 1, where $\Gamma_0 \sim N$ ($\mu = 0$; $\sigma^2 = 1$); $a = 10/9$ (a) and $a = 9/10$ (b).

Let $n \geq 0$ be an arbitrary and fixed integer and denote $Z = Z_n = A^n\gamma_0$. The RV Z represents power transformation of RV A; that is, Z can be written as $Z = kA^n$. By applying Proposition 1 to $Y = Z$, $k = \gamma_0$, and $X = A$, one obtains the 1-PDF $f_1(z,n)$. For the sake of clarity, we do not provide the corresponding explicit expression for $f_1(z,n)$ since it just consists of substituting the previous identification. Below, we show an illustrative example.

Example 2. Let us assume that A has a uniform distribution on the interval $[-2,4]$, $A \sim \mathrm{Un}([-2,4])$, and $\gamma_0 = 1 > 0$. Therefore, according to Proposition 1, the 1-PDF of $\{Z_n : n \geq 0\}$ is given by

$$f_1(z,n) = \begin{cases} \delta(z-1), & n = 0,\ z \in \mathbb{R}, \\ \dfrac{1}{6}, & n = 1,\ -2 \leq z \leq 4, \\ \dfrac{1}{3n} z^{(1-n)/n}, & n \text{ even},\ 0 < z \leq 2^n, \\ \dfrac{1}{6n} z^{(1-n)/n}, & n \text{ even},\ 2^n < z \leq 4^n, \\ \dfrac{1}{6n} z^{(1-n)/n}, & n \text{ odd},\ n \geq 3,\ (-2)^n \leq z < 0,\ 0 < z \leq 4^n. \end{cases} \quad (56)$$

It can be checked that $f_1(z,n)$ is a PDF for each $n = 0,1,2,\ldots$. Figure 2 shows $f_1(z,n)$ at different values of n.

This example exhibits a different behaviour of the 1-PDF of the solution of (55) depending on whether n is odd or even.

3.3. Case I.3: (Γ_0, A) Is a Random Vector. Throughout this case, the joint PDF of the random vector (Γ_0, A) will be denoted by $f_{\Gamma_0,A}(\gamma_0, a)$. Let $n \geq 0$ be an arbitrary and fixed integer and denote $Z = Z_n = A^n\Gamma_0$. To compute the PDF of Z, first we will determine the joint PDF of the RVs Z and A by applying [20, Theorem 4] to the two-dimensional RV $\mathbf{Y} = \mathbf{r}(\mathbf{X})$ with

$$\mathbf{X} = \begin{bmatrix} \Gamma_0 \\ A \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} r_1(\Gamma_0, A) \\ r_2(\Gamma_0, A) \end{bmatrix} = \begin{bmatrix} A^n\Gamma_0 \\ A \end{bmatrix}. \quad (57)$$

From (57), the inverse transformation of $\mathbf{r}(\mathbf{X})$, $\mathbf{X} = \mathbf{r}^{-1}(\mathbf{Y}) = \mathbf{s}(\mathbf{Y})$, takes the form

$$\mathbf{X} = \begin{bmatrix} \Gamma_0 \\ A \end{bmatrix} = \begin{bmatrix} s_1(Y_1, Y_2) \\ s_2(Y_1, Y_2) \end{bmatrix} = \begin{bmatrix} \dfrac{Y_1}{(Y_2)^n} \\ Y_2 \end{bmatrix}. \quad (58)$$

Taking into account the fact that $\partial s_2(y_1, y_2)/\partial y_1 = 0$, the involved Jacobian simplifies to $|J_2| = 1/|y_2|^n > 0$. Therefore, the joint PDF $f_{\mathbf{Y}}(\mathbf{y})$ is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{|y_2|^n} f_{\Gamma_0,A}\left(\frac{y_1}{(y_2)^n}, y_2\right). \quad (59)$$

Going back to the original RVs, that is, $Z = A^n\Gamma_0 = Y_1$ and $A = Y_2$, one gets

$$f_{Z,A}(z,a) = \frac{1}{|a|^n} f_{\Gamma_0,A}\left(\frac{z}{a^n}, a\right), \quad n = 0,1,2,\ldots. \quad (60)$$

Finally, considering the marginal density function of Z in (60), the 1-PDF of $\{Z_n;\ n \geq 0\}$ is given by

$$f_1(z,n) = \int_{a_1}^{a_2} \frac{1}{|a|^n} f_{\Gamma_0,A}\left(\frac{z}{a^n}, a\right) da, \qquad n = 0,1,2,\ldots,\ z \in \mathbb{R}. \quad (61)$$

Example 3. Let (Γ_0, A) be a random vector and let us assume that its PDF is given by

$$f_{\Gamma_0,A}(\gamma_0, a) = 4\gamma_0 a, \quad \text{if } 0 < \gamma_0, a < 1. \quad (62)$$

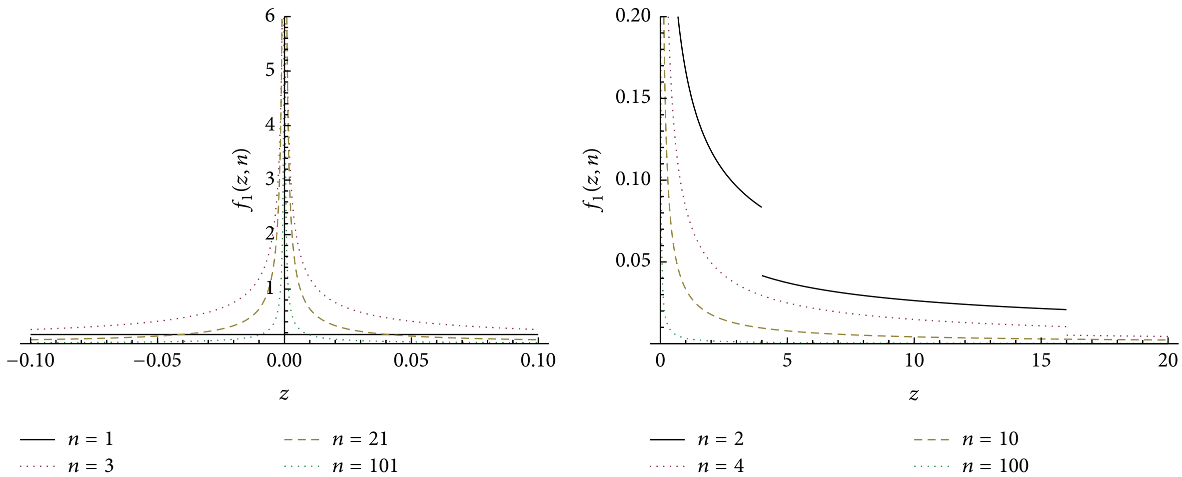


FIGURE 2: $f_1(z,n)$ in Example 2 at different values of n.

By (61)-(62), the following 1-PDF of $\{Z_n : n \geq 0\}$ is obtained:

$$f_1(z,n) = \begin{cases} 4z\int_0^1 a\,da = 2z, & n = 0,\ 0 < z < 1, \\ 4z\int_z^1 \frac{1}{a}da = -4z\ln z, & n = 1,\ 0 < z < 1, \\ 4z\int_{\sqrt[n]{z}}^1 \frac{1}{a^{2n-1}}da = \frac{2z\left(1-\sqrt[n]{z^{2(1-n)}}\right)}{1-n}, & n = 2,3,\ldots,\ 0 < z < 1. \end{cases} \tag{63}$$

Figure 3 shows $f_1(z,n)$ for $n \in \{0,1,2,\ldots,10\}$. From $n \geq 2$, we observe that the density of probability accumulates around $z = 0$, which is in agreement with the asymptotic behaviour of the solution which tends to zero as $n \to \infty$.

4. Case Study: Nonhomogeneous Discrete Initial Value Problem (II)

In this section, we deal with the computation of the 1-PDF $\{f_1(z,n) : n \geq 0\}$ of the solution discrete SP $\{Z_n : n \geq 0\}$ of the nonhomogeneous discrete initial value problem (II). This will be done for every one of the cases considered in Table 1. Now, the solution $\{Z_n : n \geq 0\}$ has the following form:

$$Z_n = \Gamma_0 + nB, \quad n = 0,1,2,\ldots. \tag{64}$$

As we did in Section 3 and for the sake of clarity in the presentation, we will recast the input parameters in (64) by lowercase letters when they indicate deterministic quantities.

4.1. Case II.1: Γ_0 *Is a Random Variable.* In this case, solution (64) takes the form

$$Z_n = \Gamma_0 + nb, \quad n = 0,1,2,\ldots. \tag{65}$$

Let $n \geq 0$ be an arbitrary and fixed integer and denote $Z = Z_n = \Gamma_0 + nb$. By applying [20, Proposition 2] to

$$Y = Z, \quad X = \Gamma_0, \quad \alpha = 1 \neq 0, \quad \beta = nb, \tag{66}$$

one obtains the 1-PDF

$$f_1(z,n) = f_{\Gamma_0}(z - nb), \quad n = 0,1,2,\ldots,\ z \in \mathbb{R}. \tag{67}$$

Example 4. Let Γ_0 be a gamma RV of parameters $\alpha, \beta > 0$, $\Gamma_0 \sim \text{Ga}(\alpha;\beta)$. Then, by (67), the 1-PDF of $\{Z_n : n \geq 0\}$ reads

$$f_1(z,n) = \frac{1}{\beta^\alpha\Gamma(\alpha)}(z-nb)^{\alpha-1}e^{-(z-nb)/\beta}, \quad n = 0,1,2,\ldots,\ nb \leq z < +\infty, \tag{68}$$

where $\Gamma(\alpha)$ means the classical gamma function. Notice that, for each $n = 0,1,2,\ldots$, the domain of z follows from the corresponding domain of a gamma distribution. In Figure 4, the 1-PDF $f_1(z,n)$ is plotted at different values of n considering $\Gamma_0 \sim \text{Ga}(2;1)$ and $b = 1/2$. In this case, from

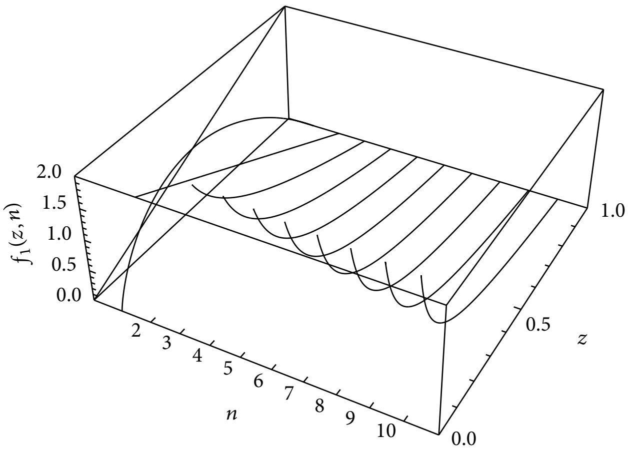


FIGURE 3: $f_1(z,n)$, $n \in \{0,1,2,\dots,10\}$, in Example 3, where (Γ_0, A) is a random vector, whose PDF is given by (62).

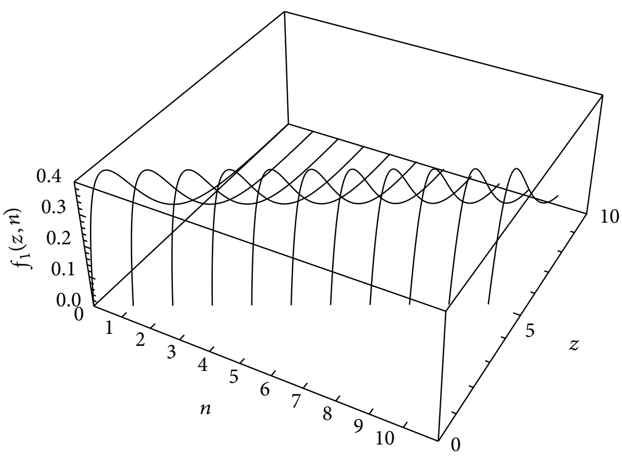


FIGURE 4: $f_1(z,n)$, $n \in \{0,1,2,\dots,10\}$, in Example 4, where $\Gamma_0 \sim \mathrm{Ga}(2;1)$ and $b = 1/2$.

the plot of $f_1(z,n)$, one observes that the expectation $\mathbb{E}[Z_n]$ increases as n does. It is straightforward to check that the expectation lies on the straight line $(1/2)n + 2$, whereas the variance takes the constant value 2.

4.2. Case II.2: B Is a Random Variable. In this case, the solution discrete stochastic process (64) takes the form

$$Z_n = \gamma_0 + nB, \quad n = 0,1,2,\dots. \tag{69}$$

For $n = 0$, $Z_0 = \gamma_0$ and the PDF is $\delta(z-\gamma_0)$. If $n \geq 1$ is an arbitrary and fixed integer, denoting $Z = Z_n = \gamma_0 + nB$, the PDF is obtained by applying [20, Proposition 2] to

$$\begin{aligned} Y &= Z, \\ X &= B, \\ \alpha &= n \neq 0, \\ \beta &= \gamma_0. \end{aligned} \tag{70}$$

This leads to the 1-PDF

$$f_1(z,n) = \begin{cases} \delta(z-\gamma_0), & n = 0 \\ \dfrac{1}{n} f_B\left(\dfrac{z-\gamma_0}{n}\right), & n = 1,2,\dots, \end{cases} \quad z \in \mathbb{R}. \tag{71}$$

Example 5. Let B be a RV with χ^2-distribution with ν degrees of freedom, $B \sim \chi^2(\nu)$, $\nu > 0$. Then, by (71), the 1-PDF of $\{Z_n : n \geq 0\}$ writes

$$f_1(z,n) = \begin{cases} \delta(z-\gamma_0), & n = 0,\ z \in \mathbb{R}, \\ \dfrac{2}{n^\nu}\dfrac{1}{2^{\nu/2}}\dfrac{1}{\Gamma(\nu/2)}(z-\gamma_0)^{\nu/2-1}e^{-(1/2)((z-\gamma_0)/n)^2}, & n = 1,2,\dots,\ \gamma_0 \leq z < \infty. \end{cases} \tag{72}$$

For each $n \geq 1$, the domain of z has been determined considering the domain of χ^2-distribution with ν degrees of freedom. For the sake of clarity, Figure 5 shows a 2D plot (a) and a 3D plot (b) of $f_1(z,n)$ at different values of n in the particular case that $B \sim \chi^2(3)$ and $\gamma_0 = 1$.

4.3. Case II.3: (Γ_0, B) Is a Random Vector. In accordance with the notation previously introduced, $f_{\Gamma_0,B}(\gamma_0, b)$ stands for the joint PDF of the random vector (Γ_0, B). Let $n \geq 0$ be an arbitrary and fixed integer and denote $Z = Z_n = \Gamma_0 + nB$. To compute the PDF of Z, first we will determine the joint PDF of the RVs Z and B by applying [20, Theorem 1] to the two-dimensional RV $\mathbf{Y} = \mathbf{r}(\mathbf{X})$ with

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} \Gamma_0 \\ B \end{bmatrix}, \\ \mathbf{Y} &= \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} r_1(\Gamma_0, B) \\ r_2(\Gamma_0, B) \end{bmatrix} = \begin{bmatrix} \Gamma_0 + nB \\ B \end{bmatrix}. \end{aligned} \tag{73}$$

From (73), the inverse transformation of $\mathbf{r}(\mathbf{X})$: $\mathbf{X} = \mathbf{r}^{-1}(\mathbf{Y}) = \mathbf{s}(\mathbf{Y})$ takes the form

$$\mathbf{X} = \begin{bmatrix} \Gamma_0 \\ B \end{bmatrix} = \begin{bmatrix} s_1(Y_1, Y_2) \\ s_2(Y_1, Y_2) \end{bmatrix} = \begin{bmatrix} Y_1 - nY_2 \\ Y_2 \end{bmatrix}. \tag{74}$$

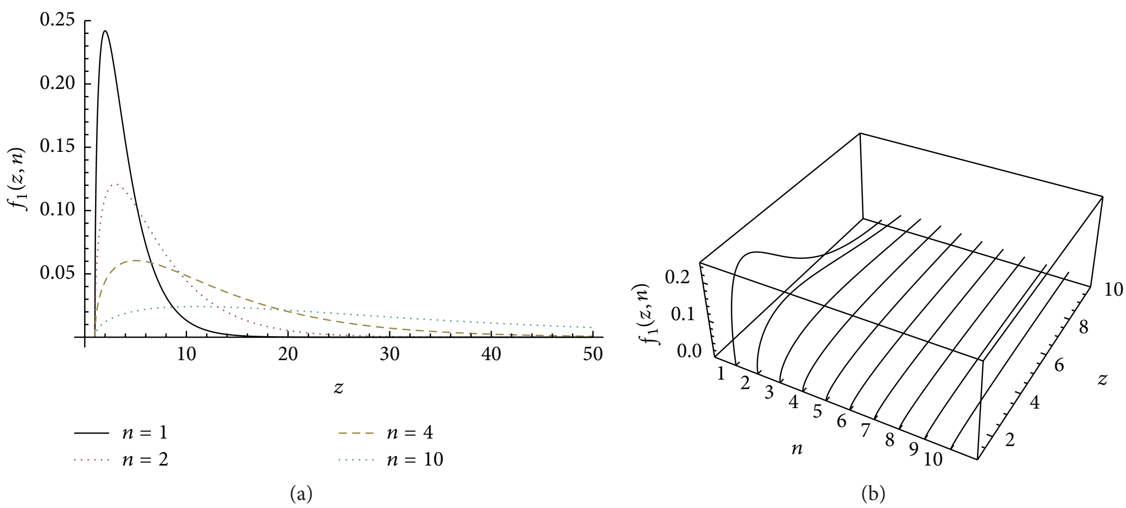


FIGURE 5: $f_1(z,n)$ in Example 5, where $B \sim \chi^2(3)$ and $\gamma_0 = 1$, at different values of n. (a) 2D plot for $n \in \{1,2,4,10\}$. (b) 3D plot for $n \in \{1,2,\ldots,10\}$.

By [20, Theorem 4] and taking into account the fact that $|J_2| = 1 \neq 0$, the joint PDF $f_{\mathbf{Y}}(\mathbf{y})$ is given by

$$f_{Y_1,Y_2}(y_1,y_2) = f_{\Gamma_0,B}(y_1 - ny_2, y_2). \quad (75)$$

Going back to the original RVs, that is, $Z = \Gamma_0 + nB = Y_1$ and $B = Y_2$, one gets

$$f_{Z,B}(z,b) = f_{\Gamma_0,B}(z - nb, b). \quad (76)$$

Finally, considering the marginal density function of Z, one gets the 1-PDF of $\{Z_n : n \geq 0\}$:

$$f_1(z,n) = \int_{b_1}^{b_2} f_{\Gamma_0,B}(z - nb, b)\, db, \qquad n = 0,1,2,\ldots,\ z \in \mathbb{R}. \quad (77)$$

Example 6. Let us consider the random vector (Γ_0, B) whose joint PDF is defined by

$$f_{\Gamma_0,B}(\gamma_0, b) = \frac{1}{4} + \frac{1}{4}(\gamma_0)^3 b - \frac{1}{4}\gamma_0 b^3, \qquad \text{if } -1 < \gamma_0, b < 1. \quad (78)$$

By (77), the 1-PDF of $\{Z_n : n \geq 0\}$ is

$$f_1(z,n) = \int_{\max\{-1,(z-1)/n\}}^{\min\{1,(z+1)/n\}} f_{\Gamma_0,B}(z - nb, b)\, db, \qquad -(n+1) \leq z \leq n+1,\ n = 0,1,2,\ldots. \quad (79)$$

Computing the above integral, one gets

$$f_1(z,n) = \begin{cases} f_{1a}(z,n), & \text{if } n = 0,1,2,\ldots,\ -(n+1) \leq z \leq -n+1, \\ f_{1b}(z,n), & \text{if } n = 0,1,2,\ldots,\ -n+1 \leq z \leq n-1, \\ f_{1c}(z,n), & \text{if } n = 0,1,2,\ldots,\ n-1 \leq z \leq n+1, \end{cases} \quad (80)$$

where

$$\begin{aligned} f_{1a}(z,n) &= \int_{-1}^{(z+1)/n} f_{\Gamma_0,B}(z - nb, b)\, db = \frac{1}{80n^4}\Big(-4n^7 - 15n^6 z + 20n^3(1+z) - (-4+z)(1+z)^4 \\ &\quad + n^5\left(4 - 20z^2\right) + 5n^4\left(4 + z - 2z^3\right) + n^2\left(-4 - 5z + z^5\right)\Big), \\ f_{1b}(z,n) &= \int_{(z-1)/n}^{(z+1)/n} f_{\Gamma_0,B}(z - nb, b)\, db = \frac{1 - n^2 + 5n^3 + 5z^2}{10n^4}, \\ f_{1c}(z,n) &= \int_{(z-1)/n}^{1} f_{\Gamma_0,B}(z - nb, b)\, db = \frac{1}{80n^4}\Big(-4n^7 - 20n^3(-1+z) + 15n^6 z + (-1+z)^4(4+z) \\ &\quad + n^5\left(4 - 20z^2\right) + 5n^4\left(4 - z + 2z^3\right) - n^2\left(4 - 5z + z^5\right)\Big). \end{aligned} \quad (81)$$

Figure 6 shows two equivalent plots of $f_1(z,n)$ given by (80)-(81). It is straightforward to check that the expectation and variance of the solution Z_n are given by

$$\mathbb{E}[Z_n] = 0, \qquad \mathbb{V}[Z_n] = \frac{1}{3}\left(1 + n^2\right), \qquad n = 0,1,2,\ldots, \quad (82)$$

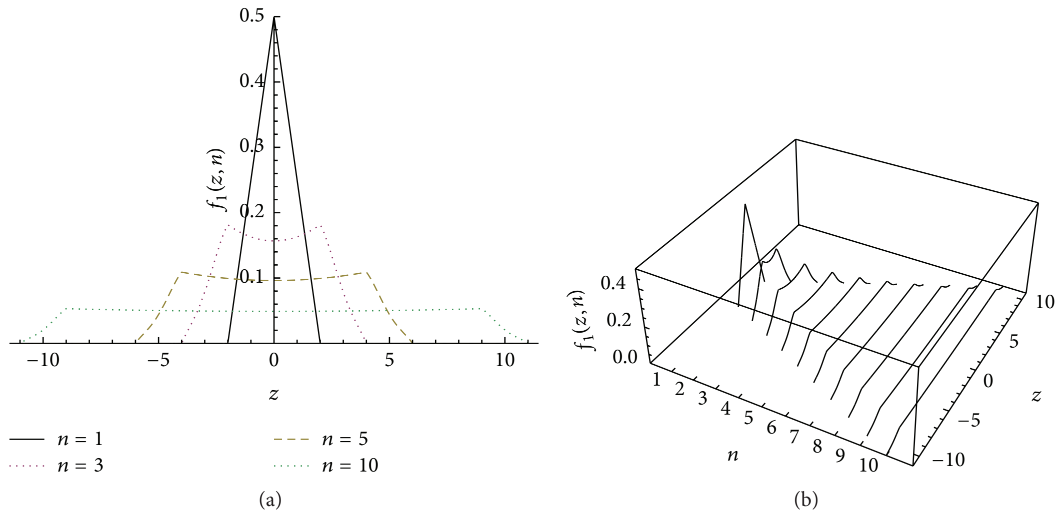


Figure 6: $f_1(z,n)$ in Example 6 at different values of n, where (Γ_0, B) is a random vector whose PDF is given by (78). (a) 2D plot for $n \in \{1,3,5,10\}$. (b) 3D plot for $n \in \{1,2,\dots,10\}$.

respectively. Notice that the values of the expectation and variance obtained from expressions (82) agree with the plots shown in Figure 6, where one observes that the 1-PDF $f_1(z,n)$ is, for each $n \geq 0$, symmetric about $z = 0$ and its support increases as n tends to infinity in such a way that the 1-PDF's shape becomes flattened. Then, the variability about zero, in this case the variance, increases as n does.

5. Case Study: Nonhomogeneous Discrete Initial Value Problem (III)

This section is addressed to determine the 1-PDFs $\{f_1(z,n) : n \geq 0\}$ of the solution SPs $\{Z_n : n \geq 0\}$ of problem (III) in Cases III.1–III.7 collected in Table 1. Now, the solution $\{Z_n : n \geq 0\}$ has the following form:

$$Z_0 = \Gamma_0, \qquad Z_n = \left(\Gamma_0 - \frac{B}{1-A}\right)A^n + \frac{B}{1-A}, \quad n = 1,2,\dots, \tag{83}$$

which is well defined due to the hypothesis $\mathbb{P}[\{\omega \in \Omega : A(\omega) = 1\}] = 0$.

Analogously to the previous sections, for the sake of clarity in the presentation, we will rewrite each one of the involved parameters in (83) by lowercase letters when it denotes a deterministic quantity.

5.1. Case III.1: Γ_0 Is a Random Variable. In this case, if $a \neq 0$, solution (83) takes the form

$$Z_n = a^n\Gamma_0 + \frac{b}{1-a}\left(1-a^n\right), \quad n = 0,1,2,\dots. \tag{84}$$

Let $n \geq 0$ be an arbitrary and fixed integer and denote $Z = Z_n = a^n\Gamma_0 + b(1-a^n)/(1-a)$. The application of [20, Proposition 2] to

$$Y = Z, \quad X = \Gamma_0, \quad \alpha = a^n \neq 0, \quad \beta = \frac{b}{1-a}\left(1-a^n\right) \tag{85}$$

permits computing the 1-PDF. This yields

$$f_1(z,n) = \frac{1}{|a|^n} f_{\Gamma_0}\left(\frac{z(1-a) - b(1-a^n)}{a^n(1-a)}\right), \quad n = 0,1,2,\dots, \; z \in \mathbb{R}. \tag{86}$$

For the trivial case $a = 0$, solution (83) takes the form

$$Z_0 = \Gamma_0, \qquad Z_n = b, \quad n = 1,2,\dots, \tag{87}$$

and hence the 1-PDF is given by

$$f_1(z,n) = \begin{cases} f_{\Gamma_0}(z), & n = 0, \\ \delta(z-b), & n = 1,2,\dots, \end{cases} \quad z \in \mathbb{R}. \tag{88}$$

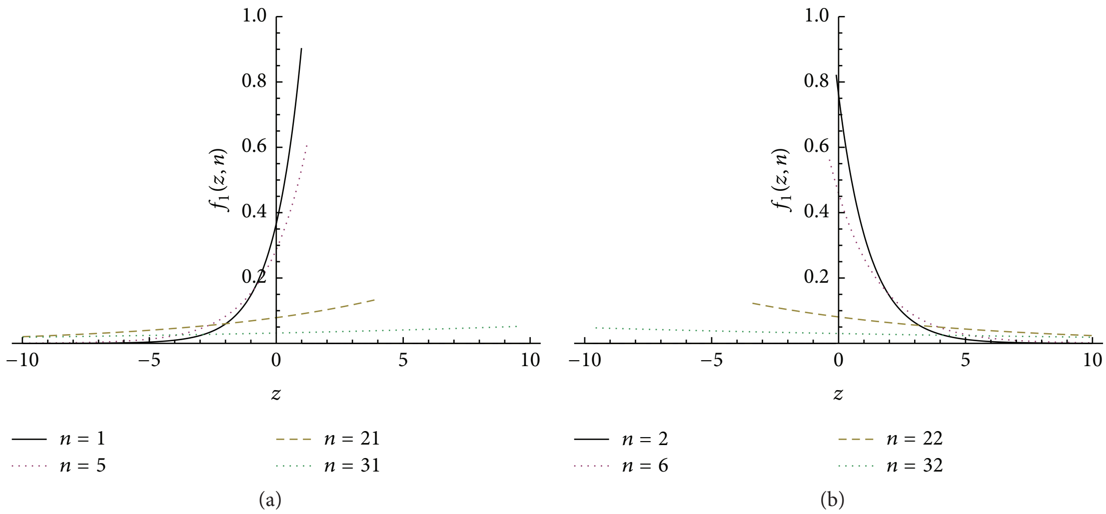


FIGURE 7: $f_1(z, n)$ in Example 7 at different values of n depending on whether n is odd (a) or even (b), where $\Gamma_0 \sim \text{Exp}(\lambda = 1)$, $a = -11/10$, and $b = 1$.

Remark 7. Notice that expression (51) obtained in Case I.1 is a particular case of (86) taking $b = 0$. Similarly, if the parameter a tends to 1 in (86), one gets formula (67) of Case II.1.

Example 7. Let Γ_0 be an exponential RV of parameter $\lambda > 0$, $\Gamma_0 \sim \text{Exp}(\lambda)$. Then, by (86), the 1-PDF of $\{Z_n : n \geq 0\}$ writes

$$f_1(z,n) = \begin{cases} -\dfrac{\lambda}{a^n} e^{-\lambda(z/a^n - b(1-a^n)/(a^n(1-a)))} & \text{if } n \text{ odd}, \; z < \dfrac{b(1-a^n)}{1-a}, \\ \dfrac{\lambda}{a^n} e^{-\lambda(z/a^n - b(1-a^n)/(a^n(1-a)))} & \text{if } n \text{ even}, \; z > \dfrac{b(1-a^n)}{1-a}, \end{cases} \quad a < 0, \tag{89}$$

$$f_1(z,n) = \frac{\lambda}{a^n} e^{-\lambda(z/a^n - b(1-a^n)/(a^n(1-a)))}, \quad z > \frac{b(1-a^n)}{1-a}, \; a > 0, \tag{90}$$

where the domain has been determined taking into account the domain of an exponential RV. Figure 7 shows $f_1(z, n)$ at different values of n depending on whether n is odd (a) or even (b) for $\lambda = 1$, $a = -11/10$, and $b = 1$.

Below, we show an example with the aim of illustrating that once the 1-PDF has been computed, important statistical moments of the solution SP, such as the expectation and variance, can be determined straightforwardly.

Example 8. Within the context of Example 7 and assuming for illustrative purposes that, for instance, $a < 0$, the statistical moment of order n of Z_n can be determined directly using expression (89) of $f_1(z, n)$ in the following way:

$$m_Z(n,k) = \mathbb{E}\left[(Z_n)^k\right] = \begin{cases} \displaystyle\int_{-\infty}^{b(1-a^n)/(1-a)} z^k f_1(z,n)\,dz, & \text{if } n \text{ is odd}, \\ \displaystyle\int_{b(1-a^n)/(1-a)}^{+\infty} z^k f_1(z,n)\,dz, & \text{if } n \text{ is even}, \end{cases} \quad k = 0, 1, 2, \ldots. \tag{91}$$

For example, taking $\lambda = 1$, $a = -11/10$, and $b = 1$, the mean and the variance of Z_n are given by

$$\mathbb{E}[Z_n] = m_Z(n,1) = \begin{cases} \dfrac{1}{21}\left(10 - 10^{-n} 11^{n+1}\right), & \text{if } n \text{ is odd}, \\ \dfrac{1}{21}\left(10 + 10^{-n} 11^{n+1}\right), & \text{if } n \text{ is even}, \end{cases}$$
$$\mathbb{V}[Z_n] = m_Z(n,2) - (m_Z(n,1))^2 = \left(\frac{121}{100}\right)^n, \quad n = 0, 1, 2, \ldots. \tag{92}$$

Note that the expression of the variance does not depend on whether n is even or odd.

In Figure 8, the expectation and variance of Z_n have been plotted using expressions (92) to carry out computations.

In addition to computing the mean and the variance, further significant information related to the solution SP can be computed from the 1-PDF, such as the probability of specific sets in which we could be interested. For example,

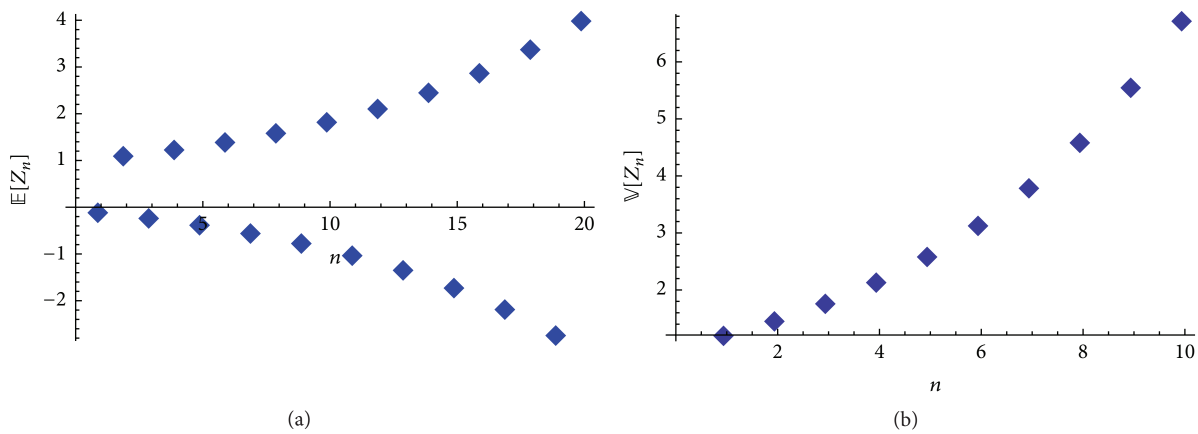


FIGURE 8: Mean (a) and variance (b) of Z_n in Example 8, where $\Gamma_0 \sim \mathrm{Exp}(\lambda = 1)$, $a = -11/10$, and $b = 1$.

the probability that the solution varies between the values $v_1 = 2$ and $v_2 = 3$ is given by

$$\mathbb{P}[2 \le Z_n \le 3] = \int_2^3 f_1(z,n)\,dz = \begin{cases} 0 & \text{if } n \text{ is odd}, \ n \le 11, \\ e^{(1/21)(-10)((-1)^n+1)}\left(1 - e^{(2/21)(2^{n+4}(5/11)^n+5(-1)^n)}\right) & \text{if } n \text{ is odd}, \ 13 \le n \le 17, \\ e^{(1/21)(-10)11^{-n}(10^n+11^n)}\left(e^{3(10/11)^n} - e^{(5/11)^n 2^{n+1}}\right) & \text{if } n \text{ is odd}, \ n \ge 19, \\ e^{(1/21)(-5)(19(10/11)^n+2)}\left(e^{3(10/11)^n} - e^{(5/11)^n 2^{n+1}}\right) & \text{if } n \text{ is even}. \end{cases} \quad (93)$$

5.2. Case III.2: B Is a Random Variable. Let us assume $a \ne -1$. In this case, the discrete solution SP (83) takes the form

$$Z_0 = \gamma_0, \qquad Z_n = \frac{1-a^n}{1-a}B + \gamma_0 a^n, \quad n = 1,2,\ldots. \quad (94)$$

Let $n \ge 1$ be an arbitrary and fixed integer and denote $Z = Z_n$. By applying [20, Proposition 2] to

$$Z = Y, \qquad X = B, \qquad \alpha = \frac{1-a^n}{1-a} \ne 0, \qquad \beta = \gamma_0 a^n, \quad (95)$$

and taking into account the fact that $Z_0 = \gamma_0$, one obtains the 1-PDF

$$f_1(z,n) = \begin{cases} \delta(z-\gamma_0), & n = 0, \\ \left|\dfrac{1-a}{1-a^n}\right| f_B\left(\dfrac{1-a}{1-a^n}(z-\gamma_0 a^n)\right), & n = 1,2,\ldots, \end{cases} \qquad z \in \mathbb{R}. \quad (96)$$

If $a = -1$, then the discrete solution SP (83) is

$$Z_n = \frac{1-(-1)^n}{2}B + \gamma_0(-1)^n = \begin{cases} \gamma_0 & \text{if } n \text{ is even}, \\ B - \gamma_0 & \text{if } n \text{ is odd}. \end{cases} \quad (97)$$

Then, for $n \ge 0$, an arbitrary but fixed integer, applying [20, Proposition 2] to (97), one obtains the 1-PDF of Z_n defined by (97):

$$f_1(z,n) = \begin{cases} \delta(z-\gamma_0), & n \text{ even}, \\ f_B(z+\gamma_0), & n \text{ odd}, \end{cases} \qquad z \in \mathbb{R}. \quad (98)$$

Remark 9. Notice that expression (71) in Case II.2 can be obtained assuming that parameter a tends to 1 in (96).

Example 9. Let B be a RV having a gamma distribution of parameters $\alpha, \beta > 0$; that is, $B \sim \mathrm{Ga}(\alpha;\beta)$. Let us fix $a \ne -1$. Then, by (96), the 1-PDF of $\{Z_n : n \ge 0\}$ reads

$$f_1(z,n) = \begin{cases} \delta(z-\gamma_0) & \text{if } n = 0, \ z \in \mathbb{R}, \\ f_{1a}(z,n), & \text{if } \dfrac{1-a}{1-a^n} > 0, \ n = 1,2,\ldots, \ z \ge \gamma_0 a^n, \\ f_{1b}(z,n), & \text{if } \dfrac{1-a}{1-a^n} < 0, \ n = 1,2,\ldots, \ z \le \gamma_0 a^n, \end{cases} \quad (99)$$

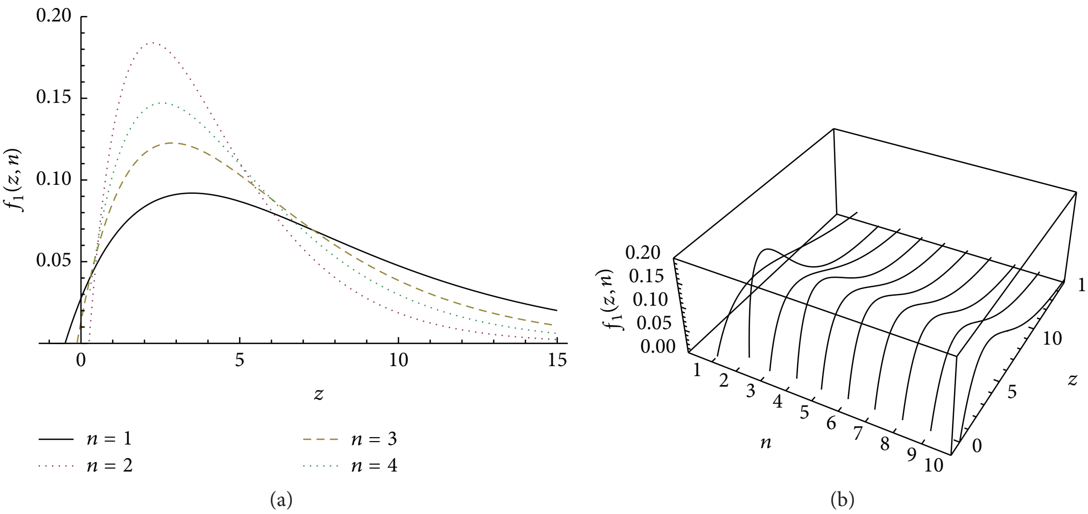


FIGURE 9: 2D and 3D plots of $f_1(z,n)$ in Example 9, where $B \sim \mathrm{Ga}(2;4)$, $a=-1/2$, and $\gamma_0=1$. (a) $n \in \{1,2,3,4\}$. (b) $n \in \{1,2,\ldots,10\}$.

where

$$f_{1a}(z,n) = \frac{1-a}{1-a^n}\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\left(\frac{1-a}{1-a^n}(z-\gamma_0 a^n)\right)^{\alpha-1} \cdot e^{(1/\beta)((a-1)/(1-a^n))(z-\gamma_0 a^n)},$$
$$f_{1b}(z,n) = \frac{a-1}{1-a^n}\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\left(\frac{1-a}{1-a^n}(z-\gamma_0 a^n)\right)^{\alpha-1} \cdot e^{(1/\beta)((a-1)/(1-a^n))(z-\gamma_0 a^n)}. \quad (100)$$

Taking into account the fact that the domain of a gamma RV is $(0,\infty)$, one deduces the domain of z specified in (99) for each $n \geq 1$. Figure 9 shows, $f_1(z,n)$ at different values of n assuming that $B \sim \mathrm{Ga}(2;4)$, $a=-1/2$, and $\gamma_0=1$. Note that, in accordance with (99), $f_1(z,0)=\delta(z-1)$, $-\infty<z<\infty$.

5.3. Case III.3: A Is a Random Variable. So far, we have taken advantage of RVT method to compute the 1-PDF of the solution of problem (2). The success of this approach has relied on the capability to find out an exact expression for the inverse transformation of the mapping that determines the solution in terms of the random inputs. However, in many practical cases, such expression can just be found in an approximate form rather than in an exact form. Under these circumstances, the RVT method can still be very useful. In fact, as it was shown in the analysis of Case III.3 of [20], the application of the Lagrange-Bürmann theorem [27] together with the RVT technique permits determining reliable approximations of the 1-PDF of the solution SP of the continuous counterpart of problem (2). Below, we provide an illustrative example dealing with a particular case of (83) where just the parameter A is assumed to be random. To avoid repetitions, we omit the theoretical development which can be found in [20].

Example 10. Throughout this example, we will use the notation introduced in [20]. Let A be a beta RV of parameters $\alpha=2$, $\beta=3$: $A \sim \mathrm{Be}(2;3)$ and $\gamma_0=1$ and $b=1$. In Figure 10, the approximation of $f_1(z,n)$ is shown at different values of $n \geq 1$. This plot has been made by considering expressions [20, eqs. (78)–(80)] with $k=1$, being $\mathscr{A}_1=[0,1]$ because of monotony of $r(A)$. To carry out computations, $\mathscr{A}_1$ has been divided into 7 subintervals in agreement with the process described in [20]. In each subinterval, an approximation of degree $N_j=2$ has been considered. If $n=0$, $f_1(z,0)=\delta(z-\gamma_0)$.

5.4. Case III.4: (Γ_0, B) Is a Random Vector. Let us denote by $f_{\Gamma_0,B}(\gamma_0,b)$ the joint PDF of the random vector (Γ_0,B).

If $a \neq 0$, the discrete solution SP (83) takes the form

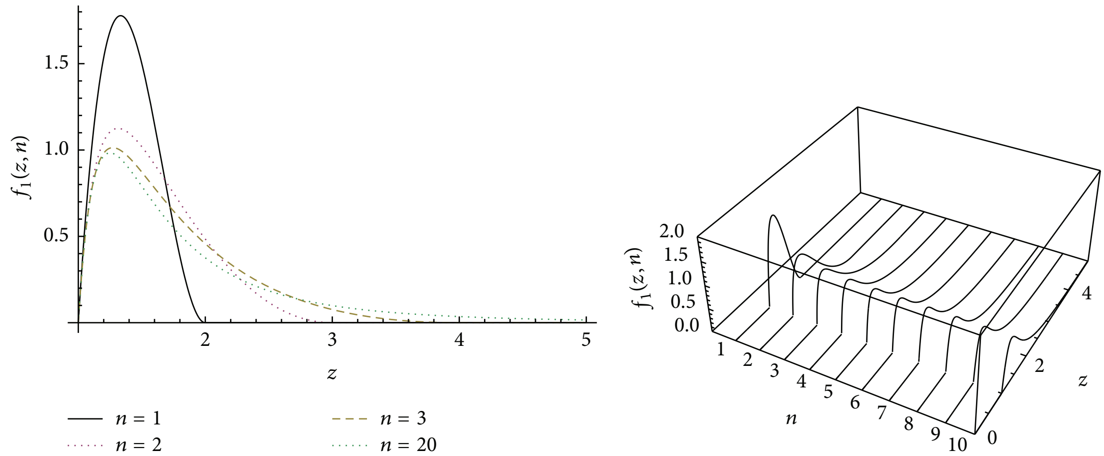
$$Z_n = \frac{1-a^n}{1-a}B + a^n\Gamma_0, \quad n=0,1,2,\ldots. \quad (101)$$

Let us fix $n: n \geq 0$ and denote $Z=Z_n=(1-a^n)/(1-a)B+a^n\Gamma_0$. To compute the 1-PDF of Z, first we will determine the joint PDF of the RVs Z and B by applying [20, Theorem 4] to the two-dimensional RV $\mathbf{Y}=\mathbf{r}(\mathbf{X})$ being

$$\mathbf{X} = \begin{bmatrix}\Gamma_0 \\ B\end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix}Y_1 \\ Y_2\end{bmatrix} = \begin{bmatrix}r_1(\Gamma_0,B) \\ r_2(\Gamma_0,B)\end{bmatrix} = \begin{bmatrix}\dfrac{1-a^n}{1-a}B + a^n\Gamma_0 \\ B\end{bmatrix}. \quad (102)$$

From (102), the inverse transformation of $\mathbf{r}(\mathbf{X})$, $\mathbf{X}=\mathbf{r}^{-1}(\mathbf{Y})=\mathbf{s}(\mathbf{Y})$, takes the form

$$\mathbf{X} = \begin{bmatrix}\Gamma_0 \\ B\end{bmatrix} = \begin{bmatrix}s_1(Y_1,Y_2) \\ s_2(Y_1,Y_2)\end{bmatrix} = \begin{bmatrix}\left(Y_1 - \dfrac{1-a^n}{1-a}Y_2\right)\dfrac{1}{a^n} \\ Y_2\end{bmatrix}. \quad (103)$$



(a) (b)

FIGURE 10: 2D and 3D plots of $f_1(z,n)$ in Example 10, where $A \sim \mathrm{Be}(2;3)$, $\gamma_0 = 1$, and $b = 1$. (a) $n \in \{0, 1, 2, 3\}$. (b) $n \in \{0, 1, 2, \ldots, 10\}$.

By [20, Theorem 4] and taking into account the fact that $|J_2| = 1/|a|^n \neq 0$, the joint PDF $f_{\mathbf{Y}}(\mathbf{y})$ is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{|a|^n} f_{\Gamma_0,B}\left(\left(y_1 - \frac{1-a^n}{1-a} y_2\right)\frac{1}{a^n}, y_2\right). \quad (104)$$

Going back to the original RVs, that is, $Z = ((1-a^n)/(1-a))B + a^n\Gamma_0 = Y_1$ and $B = Y_2$, one gets

$$f_{Z,B}(z,b) = \frac{1}{|a|^n} f_{\Gamma_0,B}\left(\left(z - \frac{1-a^n}{1-a} b\right)\frac{1}{a^n}, b\right). \quad (105)$$

Finally, considering the marginal PDF of Z in (105), one obtains the 1-PDF of Z for $a \neq 0$:

$$f_1(z,n) = \frac{1}{|a|^n}\int_{b_1}^{b_2} f_{\Gamma_0,B}\left(\left(z - \frac{1-a^n}{1-a} b\right)\frac{1}{a^n}, b\right) db, \quad n = 0, 1, 2, \ldots, \quad z \in \mathbb{R}. \quad (106)$$

For the trivial case $a = 0$, solution (101) takes the form

$$Z_0 = \Gamma_0$$
$$Z_n = B, \quad n = 1, 2, \ldots, \quad (107)$$

and hence the 1-PDF of Z is given by the marginal PDFs of Γ_0 and B:

$$f_1(z,n) = \begin{cases} f_{\Gamma_0}(\gamma_0) = \displaystyle\int_{b_1}^{b_2} f_{\Gamma_0,B}(\gamma_0, b)\, db, & n = 0, \\ f_B(b) = \displaystyle\int_{\gamma_{0,1}}^{\gamma_{0,2}} f_{\Gamma_0,B}(\gamma_0, b)\, d\gamma_0, & n = 1, 2, \ldots, \end{cases} \quad z \in \mathbb{R}. \quad (108)$$

Remark 11. Notice that expression (77) in Case II.3 can be obtained assuming that parameter a tends to 1 in (106).

Example 11. Let us assume that the random vector $\boldsymbol{\eta} = (\Gamma_0, B)^\top \sim N(\boldsymbol{\mu}_{\boldsymbol{\eta}}, \Sigma_{\boldsymbol{\eta}})$ has a bidimensional Gaussian distribution whose mean vector and covariance matrix are given by

$$\boldsymbol{\mu}_{\boldsymbol{\eta}} = \begin{pmatrix} \mu_{\Gamma_0} \\ \mu_B \end{pmatrix},$$
$$\Sigma_{\boldsymbol{\eta}} = \begin{pmatrix} (\sigma_{\Gamma_0})^2 & \rho_{\Gamma_0,B}\sigma_{\Gamma_0}\sigma_B \\ \rho_{\Gamma_0,B}\sigma_{Z_0}\sigma_B & (\sigma_B)^2 \end{pmatrix}, \quad (109)$$

respectively. In (109), $\rho_{\Gamma_0,B}$ denotes the correlation coefficient of RVs Γ_0 and B. Let us assume $a \neq 0$. By (106), the 1-PDF of $\{Z_n : n \geq 0\}$ reads as follows:

$$f_1(z,n) = \frac{1}{|a|^n}\frac{1}{2\pi\sqrt{\det(\Sigma_{\boldsymbol{\eta}})}}\int_{-\infty}^{+\infty} e^{-(1/2)(\boldsymbol{\zeta}-\boldsymbol{\mu}_{\boldsymbol{\eta}})^\top(\Sigma_{\boldsymbol{\eta}})^{-1}(\boldsymbol{\zeta}-\boldsymbol{\mu}_{\boldsymbol{\eta}})} db, \quad n = 0, 1, 2, \ldots, \quad z \in \mathbb{R}, \quad (110)$$

being

$$\boldsymbol{\zeta} = \begin{pmatrix} \left(z - \dfrac{1-a^n}{1-a} b\right)\dfrac{1}{a^n} \\ b \end{pmatrix}. \quad (111)$$

To determine the domain of integration in (110), we have taken into account the fact that B is also a Gaussian RV. Figure 11 shows a graphical representation of $f_1(z,n)$ at different values of n assuming that $\mu_{\Gamma_0} = 1$, $\mu_B = 0$, $\sigma_{\Gamma_0} = 0.1$, $\sigma_B = 0.1$, $\rho_{\Gamma_0,B} = 0.5$, and $a = -11/10$.

5.5. Case III.5: (Γ_0, A) Is a Random Vector. In this case, solution (83) takes the form

$$Z_n = A^n\Gamma_0 + \frac{b}{1-A}(1 - A^n), \quad n = 0, 1, 2, \ldots. \quad (112)$$

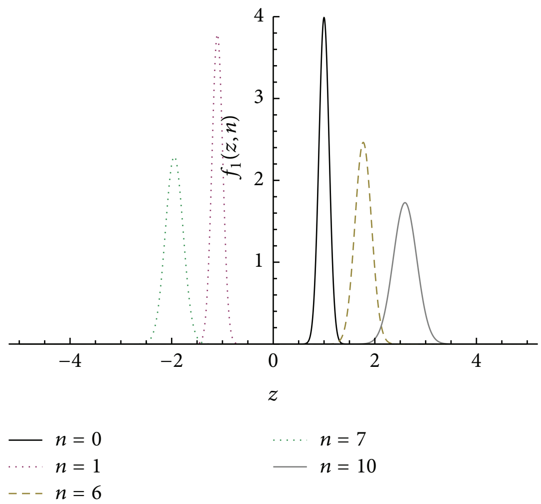


FIGURE 11: $f_1(z,n)$ in Example 11 at different values of n. The input parameters are assumed to be $\boldsymbol{\eta} = (\Gamma_0, B)^{\top} \sim N(\boldsymbol{\mu}_{\boldsymbol{\eta}}, \Sigma_{\boldsymbol{\eta}})$, where the mean vector and the variance-covariance matrix are defined by (109), with the values $\mu_{\Gamma_0} = 1$, $\mu_B = 0$, $\sigma_{\Gamma_0} = 0.1$, $\sigma_B = 0.1$, $\rho_{\Gamma_0,B} = 0.5$, and $a = -11/10$.

Let us denote by $f_{\Gamma_0,A}(\gamma_0, a)$ the joint PDF of the random vector (Γ_0, A). Let us fix $n : n \geq 0$ and denote $Z = Z_n = A^n\Gamma_0 + b/(1-A)(1-A^n)$. To compute the 1-PDF of Z, first we will determine the joint PDF of the RVs Z and A by applying [20, Theorem 4] to the two-dimensional RV $\mathbf{Y} = \mathbf{r}(\mathbf{X})$ with

$$\mathbf{X} = \begin{bmatrix} \Gamma_0 \\ A \end{bmatrix},$$

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} r_1(\Gamma_0, A) \\ r_2(\Gamma_0, A) \end{bmatrix} = \begin{bmatrix} A^n\Gamma_0 + \dfrac{b}{1-A}(1-A^n) \\ A \end{bmatrix}. \tag{113}$$

From (113), the inverse transformation of $\mathbf{r}(\mathbf{X})$, $\mathbf{X} = \mathbf{r}^{-1}(\mathbf{Y}) = \mathbf{s}(\mathbf{Y})$, takes the form

$$\mathbf{X} = \begin{bmatrix} \Gamma_0 \\ A \end{bmatrix} = \begin{bmatrix} s_1(Y_1, Y_2) \\ s_2(Y_1, Y_2) \end{bmatrix} = \begin{bmatrix} \left(Y_1 - \dfrac{b}{1-Y_2}\left(1-(Y_2)^n\right)\right)\dfrac{1}{(Y_2)^n} \\ Y_2 \end{bmatrix}. \tag{114}$$

Notice that $Y_2 \neq 0$ w.p. 1. By [20, Theorem 4] and taking into account the fact that $|J_2| = 1/|y_2|^n \neq 0$, the joint PDF $f_{\mathbf{Y}}(\mathbf{y})$ is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{|y_2|^n} \cdot f_{\Gamma_0,A}\left(\left(y_1 - \frac{b}{1-y_2}\left(1-(y_2)^n\right)\right)\frac{1}{(y_2)^n}, y_2\right). \tag{115}$$

Going back to the original RVs, that is, $Z = A^n\Gamma_0 + b/(1-A)(1-A^n) = Y_1$ and $A = Y_2$, one gets

$$f_{Z,A}(z, a) = \frac{1}{|a|^n} f_{\Gamma_0,A}\left(\left(z - \frac{b}{1-a}(1-a^n)\right)\frac{1}{a^n}, a\right). \tag{116}$$

Finally, considering the marginal PDF of Z in (116), the 1-PDF of $\{Z_n : n \geq 0\}$ is given by

$$f_1(z, n) = \int_{a_1}^{a_2} \frac{1}{|a|^n} f_{\Gamma_0,A}\left(\left(z - \frac{b}{1-a}(1-a^n)\right)\frac{1}{a^n}, a\right) da, \quad n = 0, 1, 2, \ldots, \; z \in \mathbb{R}. \tag{117}$$

Example 12. Let us consider the two-dimensional Gaussian vector $\boldsymbol{\eta} = (\Gamma_0, A)^{\top} \sim N(\boldsymbol{\mu}_{\boldsymbol{\eta}}, \Sigma_{\boldsymbol{\eta}})$, whose mean and covariance matrix are given by

$$\boldsymbol{\mu}_{\boldsymbol{\eta}} = \begin{pmatrix} \mu_{\Gamma_0} \\ \mu_A \end{pmatrix}, \qquad \Sigma_{\boldsymbol{\eta}} = \begin{pmatrix} (\sigma_{\Gamma_0})^2 & \rho_{\Gamma_0,A}\sigma_{\Gamma_0}\sigma_A \\ \rho_{\Gamma_0,A}\sigma_{\Gamma_0}\sigma_A & (\sigma_A)^2 \end{pmatrix}, \tag{118}$$

respectively. In (118), $\rho_{\Gamma_0,A}$ denotes the correlation coefficient between RVs Γ_0 and A. Notice that A has a Gaussian distribution; hence, $a_1 = -\infty$ and $a_2 = +\infty$. Then, according to (117), the 1-PDF of $\{Z_n : n \geq 0\}$ is given by

$$f_1(z, n) = \frac{1}{2\pi\sqrt{\det(\Sigma_{\boldsymbol{\eta}})}} \int_{-\infty}^{+\infty} \frac{1}{|a|^n} e^{-(1/2)(\boldsymbol{\zeta}-\boldsymbol{\mu}_{\boldsymbol{\eta}})^{\top}(\Sigma_{\boldsymbol{\eta}})^{-1}(\boldsymbol{\zeta}-\boldsymbol{\mu}_{\boldsymbol{\eta}})} da, \quad n = 0, 1, 2, \ldots, \; z \in \mathbb{R}, \tag{119}$$

where

$$\boldsymbol{\zeta} = \begin{pmatrix} \left(z - \dfrac{b}{1-a}(1-a^n)\right)\dfrac{1}{a^n} \\ a \end{pmatrix}. \tag{120}$$

Figure 12 shows a graphical representation of the 1-PDF $f_1(z,n)$ given by (117) at different values of n assuming that $\mu_{\Gamma_0} = 1$, $\mu_A = 1.5$, $\sigma_{\Gamma_0} = 0.1$, $\sigma_A = 0.1$, $\rho_{\Gamma_0,A} = 0.5$, and $b = 1$. For the sake of clarity, we have split the representation of $f_1(z,n)$ into two plots due to the significant differences in the vertical scales required depending on the values of n.

5.6. Case III.6: (B, A) Is a Random Vector.

Solution (83) takes the form

$$Z_n = A^n\gamma_0 + \frac{1-A^n}{1-A}B, \quad n = 0, 1, 2, \ldots. \tag{121}$$

Let us denote by $f_{B,A}(b, a)$ the joint PDF of the random vector (B, A). Let us fix $n : n \geq 1$ and denote $Z = Z_n =$

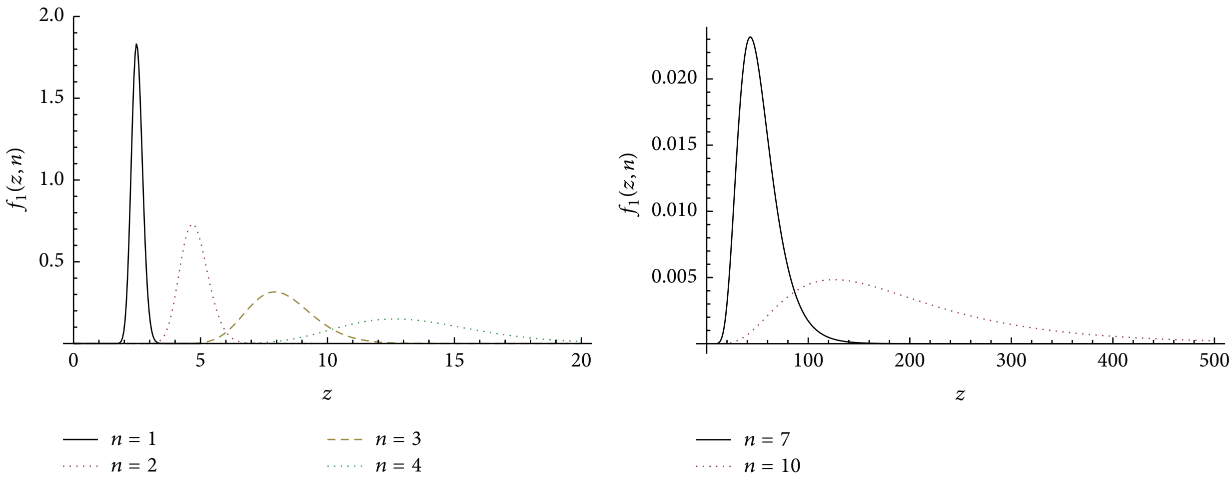


FIGURE 12: $f_1(z,n)$ in Example 12 at different values of n. The input parameters are assumed to be $\boldsymbol{\eta} = (\Gamma_0, A)^\top \sim N(\boldsymbol{\mu}_{\boldsymbol{\eta}}, \Sigma_{\boldsymbol{\eta}})$, where the mean vector and the variance-covariance matrix are defined by (118), with the values $\mu_{\Gamma_0} = 1$, $\mu_A = 1.5$, $\sigma_{\Gamma_0} = 0.1$, $\sigma_A = 0.1$, $\rho_{\Gamma_0,A} = 0.5$, and $b = 1$.

$A^n\gamma_0 + (1 - A^n)/(1 - A)B$. In order to compute the 1-PDF of Z, first we will determine the joint PDF of the RVs Z and A by applying [20, Theorem 4] to the two-dimensional RV $\mathbf{Y} = \mathbf{r}(\mathbf{X})$ with

$$\mathbf{X} = \begin{bmatrix} B \\ A \end{bmatrix},$$
$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} r_1(B,A) \\ r_2(B,A) \end{bmatrix} = \begin{bmatrix} A^n\gamma_0 + \dfrac{1-A^n}{1-A}B \\ A \end{bmatrix}. \tag{122}$$

From (122), the inverse transformation of $\mathbf{r}(\mathbf{X})$, $\mathbf{X} = \mathbf{r}^{-1}(\mathbf{Y}) = \mathbf{s}(\mathbf{Y})$, takes the form

$$\mathbf{X} = \begin{bmatrix} B \\ A \end{bmatrix} = \begin{bmatrix} s_1(Y_1,Y_2) \\ s_2(Y_1,Y_2) \end{bmatrix} = \begin{bmatrix} \left(Y_1 - (Y_2)^n\gamma_0\right)\dfrac{1-Y_2}{1-(Y_2)^n} \\ Y_2 \end{bmatrix}. \tag{123}$$

By [20, Theorem 4] and taking into account the fact that $|J_2| = |(1-y_2)/(1-(y_2)^n)| \neq 0$, the joint PDF $f_{\mathbf{Y}}(\mathbf{y})$ is given by

$$f_{Y_1,Y_2}(y_1,y_2) = \left|\frac{1-y_2}{1-(y_2)^n}\right| f_{B,A}\left(\left(y_1 - (y_2)^n\gamma_0\right)\frac{1-y_2}{1-(y_2)^n}, y_2\right). \tag{124}$$

Going back to the original RVs, that is, $Z = A^n\gamma_0 + (1 - A^n)/(1-A)B = Y_1$ and $A = Y_2$, one gets

$$f_{Z,A}(z,a) = \left|\frac{1-a}{1-a^n}\right| f_{B,A}\left(\left(z - a^n\gamma_0\right)\frac{1-a}{1-a^n}, a\right), \quad n = 1,2,\ldots. \tag{125}$$

Finally, considering the marginal PDF of Z in (125) and the case where $n = 0$, which gives $Z_0 = \gamma_0$, one gets the 1-PDF of $\{Z_n : n \geq 0\}$:

$$f_1(z,n) = \begin{cases} \delta(z-\gamma_0), & n = 0, \\ \displaystyle\int_{a_1}^{a_2} \left|\frac{1-a}{1-a^n}\right| f_{B,A}\left(\left(z - a^n\gamma_0\right)\frac{1-a}{1-a^n}, a\right) da, & n = 1,2,\ldots, \end{cases} \quad z \in \mathbb{R}. \tag{126}$$

Example 13. So far, we have considered standard distributions in one or more dimensions to illustrate the obtained theoretical results. Now, we will assume that the joint PDF of the input parameters B and A is constructed by means of a copula transformation. Let us assume that B and A are uniform RVs defined on the interval $]0,1[$; that is, $B, A \sim \text{Un}(]0,1[)$. We transform these RVs by the Farlie-Gordon-Morgenstern copula [28], so that a two-dimensional RV (B, A) with joint PDF

$$f_{B,A}(b,a) = \frac{2}{3}(2 - b - a + 2ba), \quad \text{if } 0 < b, a < 1, \tag{127}$$

is defined. This random vector satisfies the notion that the marginal distributions of $f_{B,A}(b,a)$ keep the one-dimensional distributions of A and B. Hereinafter, let us take $\gamma_0 = 0$. Taking into account the fact that $0 < a, b < 1$, by (126), for z and n previously fixed, one must calculate a such that

$$0 < z\frac{1-a}{1-a^n} < 1. \tag{128}$$

In general, the variable a cannot be determined in a closed form. Using Mathematica software to carry out computations numerically, in Figure 13, several representations of $f_1(z,n)$ have been plotted.

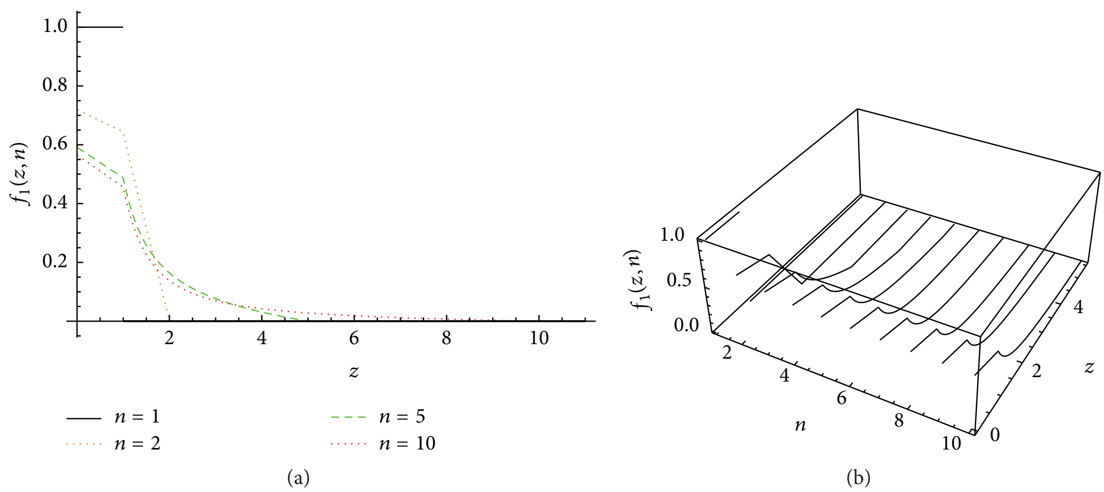


FIGURE 13: 2D and 3D plots of $f_1(z,n)$ in Example 13 at different values of n, where $\gamma_0 = 0$ and the PDF of the two-dimensional RV (B, A) is given by (127). (a) $n \in \{1, 2, 5, 10\}$. (b) $n \in \{1, 2, \dots, 10\}$.

5.7. Case III.7: (Γ_0, B, A) *Is a Random Vector.* In this last case, solution (83) takes the form

$$Z_n = A^n \Gamma_0 + \frac{B}{1-A}\left(1 - A^n\right), \quad n = 0, 1, 2, \dots. \tag{129}$$

Let us denote by $f_{\Gamma_0,B,A}(\gamma_0, b, a)$ the joint PDF of the random vector (Γ_0, B, A). Let us fix $n : n \geq 0$ and denote $Z = Z_n = A^n\Gamma_0 + B/(1-A)(1-A^n)$. To determine the 1-PDF of Z, first we will calculate the joint PDF of the RVs Z, B, and A by applying [20, Theorem 4] to the three-dimensional RV $\mathbf{Y} = \mathbf{r}(\mathbf{X})$ with

$$\mathbf{X} = \begin{bmatrix} \Gamma_0 \\ B \\ A \end{bmatrix},$$

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} r_1\left(\Gamma_0, B, A\right) \\ r_2\left(\Gamma_0, B, A\right) \\ r_3\left(\Gamma_0, B, A\right) \end{bmatrix} = \begin{bmatrix} A^n\Gamma_0 + \dfrac{B}{1-A}\left(1 - A^n\right) \\ B \\ A \end{bmatrix}. \tag{130}$$

From (130), the inverse transformation of $\mathbf{r}(\mathbf{X})$, $\mathbf{X} = \mathbf{r}^{-1}(\mathbf{Y}) = \mathbf{s}(\mathbf{Y})$, takes the form

$$\mathbf{X} = \begin{bmatrix} \Gamma_0 \\ B \\ A \end{bmatrix} = \begin{bmatrix} s_1\left(Y_1, Y_2, Y_3\right) \\ s_2\left(Y_1, Y_2, Y_3\right) \\ s_3\left(Y_1, Y_2, Y_3\right) \end{bmatrix} = \begin{bmatrix} \left(Y_1 - \dfrac{Y_2}{1-Y_3}\left(1 - \left(Y_3\right)^n\right)\right)\dfrac{1}{\left(Y_3\right)^n} \\ Y_2 \\ Y_3 \end{bmatrix}, \tag{131}$$

where $Y_3 \neq 0$ w.p. 1. By [20, Theorem 4] and taking into account the fact that $|J_3| = 1/|y_3|^n \neq 0$, the joint PDF $f_{\mathbf{Y}}(\mathbf{y})$ is given by

$$f_{Y_1,Y_2,Y_3}\left(y_1, y_2, y_3\right) = \frac{1}{\left|y_3\right|^n} \cdot f_{\Gamma_0,B,A}\left(\left(y_1 - \frac{y_2}{1-y_3}\left(1 - \left(y_3\right)^n\right)\right)\frac{1}{\left(y_3\right)^n}, y_2, y_3\right). \tag{132}$$

Going back to the original RVs, that is, $Z = A^n\Gamma_0 + (B/(1 - A))(1 - A^n) = Y_1$, $B = Y_2$, and $A = Y_3$, one gets

$$f_{Z,B,A}(z, b, a) = \frac{1}{|a|^n} f_{\Gamma_0,B,A}\left(z - \frac{b}{1-a}\left(1 - a^n\right)\frac{1}{a^n}, b, a\right). \tag{133}$$

Finally, considering the marginal PDF of Z in (133), the 1-PDF of $\{Z_n : n \geq 0\}$ is given by

$$f_1(z, n) = \int_{a_1}^{a_2}\int_{b_1}^{b_2} \frac{1}{|a|^n} f_{\Gamma_0,B,A}\left(z - \frac{b}{1-a}\left(1 - a^n\right)\frac{1}{a^n}, b, a\right) db\, da, \quad n = 0, 1, 2, \dots,\ z \in \mathbb{R}. \tag{134}$$

Example 14. Let us assume that $\boldsymbol{\eta} = (Z_0, B, A)^\top \sim N(\boldsymbol{\mu}_{\boldsymbol{\eta}}, \Sigma_{\boldsymbol{\eta}})$; that is, $\boldsymbol{\eta}$ is a Gaussian random vector with mean $\boldsymbol{\mu}_{\boldsymbol{\eta}} = (\mu_1, \mu_2, \mu_3)^\top \in \mathbb{R}^3$ and variance-covariance matrix $\Sigma_{\boldsymbol{\eta}} \in \mathbb{R}^{3\times 3}$. By (134), the 1-PDF of Z_n writes

$$f_1(z, n) = \frac{1}{2\pi\sqrt{2\pi}\sqrt{\det\left(\Sigma_{\boldsymbol{\eta}}\right)}} \cdot \iint_{-\infty}^{\infty} \frac{1}{|a|^n} e^{-(1/2)(\boldsymbol{\zeta}-\boldsymbol{\mu}_{\boldsymbol{\eta}})^\top(\Sigma_{\boldsymbol{\eta}})^{-1}(\boldsymbol{\zeta}-\boldsymbol{\mu}_{\boldsymbol{\eta}})} db\, da, \tag{135}$$

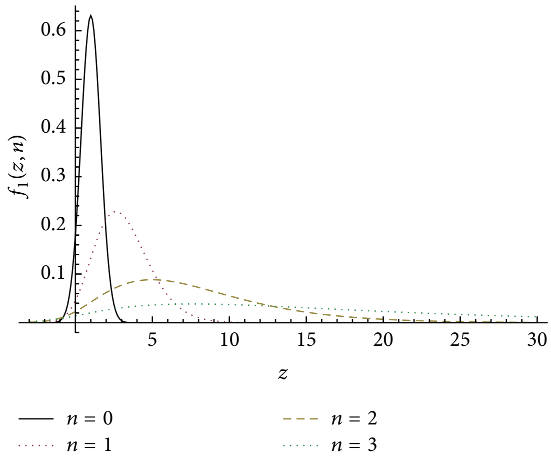


FIGURE 14: $f_1(z,n)$, $n \in \{0,1,2,3\}$, in Example 14. The input parameters ($\Gamma_0, B,$ and A) are assumed to be a Gaussian distribution with mean and variance-covariance matrix defined by (137).

where

$$\boldsymbol{\zeta} = \begin{pmatrix} \left(z - \dfrac{b}{1-a}\left(1-a^n\right)\right)\dfrac{1}{a^n} \\ b \\ a \end{pmatrix}. \tag{136}$$

In order to illustrate the theoretical results previously established, let us fix the mean vector and the variance-covariance matrix as follows:

$$\boldsymbol{\mu}_{\boldsymbol{\eta}} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \Sigma_{\boldsymbol{\eta}} = \frac{1}{10}\begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 2 \end{pmatrix}. \tag{137}$$

In this example, we do not make the 1-PDF, $f_1(z,n)$, explicit, since its expression is cumbersome. In Figure 14, we have plotted $f_1(z,n)$ at different values of n.

6. Conclusions

In this paper, we have provided general explicit formulae to compute the first probability density function (1-PDF) of the solution stochastic process to random first-order linear difference equations. It has been done in the general case where the involved random inputs are statistically dependent. The study has been based on the Random Variable Transformation technique. When solving random difference equations, most of the available studies focus on the computation of the solution stochastic process and its expectation and variance functions. However, the computation of explicit formulae to determine the 1-PDF is more advisable since it permits the computation of other higher-order moments and the probability of certain sets of interest as well. We have shown, through the theoretical development, that the study here presented generalizes its deterministic counterpart. In addition, all the theoretical results have been illustrated by a comprehensive list of examples. Finally, note that our analysis can be extended to determine the 1-PDF of the solution to random nonlinear first-order difference equations in future studies.

Appendix

In order to facilitate the handling of all the results obtained throughout the paper in practice, in the following cases, Cases I–III, we sum up all the expressions of the 1-PDF of the solution stochastic process of problem (2) according to the cases listed in Table 1.

Notice that the domains are defined in expression (3).

Case I. Expression of the 1-PDF of the solution SP of problem $Z_{n+1} = AZ_n$, $n = 0, 1, 2, \ldots$ and $Z_0 = \Gamma_0$, is listed below from expressions (A.1)–(A.14).

Case I.1. $a \neq 0$

$$f_1(z,n) = \frac{1}{|a|^n} f_{\Gamma_0}\left(\frac{z}{a^n}\right), \quad n = 0, 1, 2, \ldots, \quad z \in \mathbb{R}. \tag{A.1}$$

$a = 0$

$$f_1(z,n) = \begin{cases} f_{\Gamma_0}(z), & n = 0, \\ \delta(z), & n = 1, 2, \ldots, \end{cases} \quad z \in \mathbb{R}. \tag{A.2}$$

Case I.2. $\gamma_0 = 0$

$$f_1(z,n) = \delta(z), \quad n = 0, 1, 2, \ldots, \quad z \in \mathbb{R}. \tag{A.3}$$

$\gamma_0 \neq 0$

$n = 0$

$$f_1(z,n) = \delta(z - \gamma_0), \quad z \in \mathbb{R}. \tag{A.4}$$

$n = 1$

$$f_1(z,n) = \frac{1}{|\gamma_0|} f_A\left(\frac{z}{\gamma_0}\right), \quad \begin{cases} \gamma_0 a_1 \le z \le \gamma_0 a_2, & \text{if } \gamma_0 > 0, \\ \gamma_0 a_2 \le z \le \gamma_0 a_1, & \text{if } \gamma_0 < 0, \end{cases} \quad z \in \mathbb{R}. \tag{A.5}$$

$n \ge 3$ and odd

(i) $a_1 > 0$ or $a_2 < 0$

$$f_1(z,n) = \frac{1}{|\gamma_0|\, n}\left(\frac{z}{\gamma_0}\right)^{(1-n)/n} f_A\left(\sqrt[n]{\frac{z}{\gamma_0}}\right), \quad \begin{cases} \gamma_0 (a_1)^n \le z \le \gamma_0 (a_2)^n, & \text{if } \gamma_0 > 0, \\ \gamma_0 (a_2)^n \le z \le \gamma_0 (a_1)^n, & \text{if } \gamma_0 < 0. \end{cases} \tag{A.6}$$

(ii) $a_1 a_2 \leq 0$

$$f_1(z,n) = \frac{1}{|\gamma_0| n}\left(\frac{z}{\gamma_0}\right)^{(1-n)/n} f_A\left(\sqrt[n]{\frac{z}{\gamma_0}}\right),$$
$$\begin{cases} \gamma_0 (a_1)^n \leq z < 0,\ 0 < z \leq \gamma_0 (a_2)^n, & \text{if } \gamma_0 > 0, \\ \gamma_0 (a_2)^n \leq z < 0,\ 0 < z \leq \gamma_0 (a_1)^n, & \text{if } \gamma_0 < 0. \end{cases} \quad \text{(A.7)}$$

n even

(i) $a_1 \geq 0$

$$f_1(z,n) = \frac{1}{|\gamma_0| n}\left|\left(\frac{z}{\gamma_0}\right)^{(1-n)/n}\right| f_A\left(+\sqrt[n]{\frac{z}{\gamma_0}}\right),$$
$$\begin{cases} \gamma_0 (a_1)^n \leq z \leq \gamma_0 (a_2)^n, & \text{if } \gamma_0 > 0, \\ \gamma_0 (a_2)^n \leq z \leq \gamma_0 (a_1)^n, & \text{if } \gamma_0 < 0. \end{cases} \quad \text{(A.8)}$$

(ii) $a_2 \leq 0$

$$f_1(z,n) = \frac{1}{|\gamma_0| n}\left|\left(\frac{z}{\gamma_0}\right)^{(1-n)/n}\right| f_A\left(-\sqrt[n]{\frac{z}{\gamma_0}}\right),$$
$$\begin{cases} \gamma_0 (a_2)^n \leq z \leq \gamma_0 (a_1)^n, & \text{if } \gamma_0 > 0, \\ \gamma_0 (a_1)^n \leq z \leq \gamma_0 (a_2)^n, & \text{if } \gamma_0 < 0. \end{cases} \quad \text{(A.9)}$$

(iii) $a_1 a_2 < 0$

(a) $a_2 \geq |a_1|$

$$f_1(z,n) = f_Z^1(z,n) + f_Z^2(z,n), \quad \text{(A.10)}$$

where

$$f_Z^1(z,n) = \frac{1}{|\gamma_0| n}\left|\left(\frac{z}{\gamma_0}\right)^{(1-n)/n}\right| \cdot \left\{f_A\left(-\sqrt[n]{\frac{z}{\gamma_0}}\right) + f_A\left(+\sqrt[n]{\frac{z}{\gamma_0}}\right)\right\},$$
$$\begin{cases} 0 < z \leq \gamma_0 (a_1)^n, & \text{if } \gamma_0 > 0, \\ \gamma_0 (a_1)^n \leq z < 0, & \text{if } \gamma_0 < 0, \end{cases} \quad \text{(A.11)}$$

$$f_Z^2(z,n) = \frac{1}{|\gamma_0| n}\left|\left(\frac{z}{\gamma_0}\right)^{(1-n)/n}\right| f_A\left(+\sqrt[n]{\frac{z}{\gamma_0}}\right),$$
$$\begin{cases} \gamma_0 (a_1)^n < z \leq \gamma_0 (a_2)^n, & \text{if } \gamma_0 > 0, \\ \gamma_0 (a_2)^n \leq z < \gamma_0 (a_1)^n, & \text{if } \gamma_0 < 0. \end{cases}$$

(b) $a_2 < |a_1|$

$$f_1(z,n) = f_Z^1(z,n) + f_Z^2(z,n), \quad \text{(A.12)}$$

where

$$f_Z^1(z,n) = \frac{1}{|\gamma_0| n}\left|\left(\frac{z}{\gamma_0}\right)^{(1-n)/n}\right| \cdot \left\{f_A\left(-\sqrt[n]{\frac{z}{\gamma_0}}\right) + f_A\left(+\sqrt[n]{\frac{z}{\gamma_0}}\right)\right\},$$
$$\begin{cases} 0 < z \leq \gamma_0 (a_2)^n, & \text{if } \gamma_0 > 0, \\ \gamma_0 (a_2)^n \leq z < 0, & \text{if } \gamma_0 < 0, \end{cases} \quad \text{(A.13)}$$

$$f_Z^2(z,n) = \frac{1}{|\gamma_0| n}\left|\left(\frac{z}{\gamma_0}\right)^{(1-n)/n}\right| f_A\left(-\sqrt[n]{\frac{z}{\gamma_0}}\right),$$
$$\begin{cases} \gamma_0 (a_2)^n < z \leq \gamma_0 (a_1)^n, & \text{if } \gamma_0 > 0, \\ \gamma_0 (a_1)^n \leq z < \gamma_0 (a_2)^n, & \text{if } \gamma_0 < 0. \end{cases}$$

Case I.3. Consider

$$f_1(z,n) = \int_{a_1}^{a_2} \frac{1}{|a|^n} f_{\Gamma_0,A}\left(\frac{z}{a^n}, a\right) da, \quad n = 0,1,2,\ldots,\ z \in \mathbb{R}. \quad \text{(A.14)}$$

Case II. Expression of the 1-PDF of the solution SP of problem $Z_{n+1} = Z_n + B$, $n = 0,1,2,\ldots$ and $Z_0 = \Gamma_0$, is listed below from expressions (A.15)–(A.17).

Case II.1. Consider

$$f_1(z,n) = f_{\Gamma_0}(z - nb), \quad n = 0,1,2,\ldots,\ z \in \mathbb{R}. \quad \text{(A.15)}$$

Case II.2. Consider

$$f_1(z,n) = \begin{cases} \delta(z - \gamma_0), & n = 0, \\ \dfrac{1}{n} f_B\left(\dfrac{z - \gamma_0}{n}\right), & n = 1,2,\ldots, \end{cases} \quad z \in \mathbb{R}. \quad \text{(A.16)}$$

Case II.3. Consider

$$f_1(z,n) = \int_{b_1}^{b_2} f_{\Gamma_0,B}(z - nb, b)\, db, \quad n = 0,1,2,\ldots,\ z \in \mathbb{R}. \quad \text{(A.17)}$$

Case III. Expression of the 1-PDF of the solution SP of problem $Z_{n+1} = AZ_n + B$, $n = 0,1,2,\ldots$ and $Z_0 = \Gamma_0$, is listed below from expressions (A.18)–(A.27).

Case III.1. $a \neq 0$

$$f_1(z,n) = \frac{1}{|a|^n} f_{\Gamma_0}\left(\frac{z(1-a) - b(1-a^n)}{a^n(1-a)}\right), \quad n = 0,1,2,\ldots,\ z \in \mathbb{R}. \quad \text{(A.18)}$$

$a = 0$

$$f_1(z,n) = \begin{cases} f_{\Gamma_0}(z), & n = 0, \\ \delta(z-b), & n = 1,2,\ldots, \end{cases} \quad z \in \mathbb{R}. \tag{A.19}$$

Case III.2. $a \neq -1$

$$f_1(z,n) = \begin{cases} \delta(z-\gamma_0), & n = 0, \\ \left|\dfrac{1-a}{1-a^n}\right| f_B\left(\dfrac{1-a}{1-a^n}(z-\gamma_0 a^n)\right), & n = 1,2,\ldots, \end{cases} \quad z \in \mathbb{R}. \tag{A.20}$$

$a = -1$

$$f_1(z,n) = \begin{cases} \delta(z-\gamma_0), & n \text{ even}, \\ f_B(z+\gamma_0), & n \text{ odd}, \end{cases} \quad z \in \mathbb{R}. \tag{A.21}$$

Case III.3. Consider

$$f_1(z,n) = \sum_{j=1}^{k} f_A\left(s_{j,N_j}(z)\right)\left|\frac{ds_{j,N_j}(z)}{dz}\right| \quad (\text{using Lagrange-Bürmann formula}). \tag{A.22}$$

Case III.4. $a \neq 0$

$$f_1(z,n) = \frac{1}{|a|^n}\int_{b_1}^{b_2} f_{\Gamma_0,B}\left(\left(z-\frac{1-a^n}{1-a}b\right)\frac{1}{a^n}, b\right) db, \quad n = 0,1,2,\ldots, \; z \in \mathbb{R}. \tag{A.23}$$

$a = 0$

$$f_1(z,n) = \begin{cases} f_{\Gamma_0}(\gamma_0) = \displaystyle\int_{b_1}^{b_2} f_{\Gamma_0,B}(\gamma_0,b)\,db, & n = 0, \\ f_B(b) = \displaystyle\int_{\gamma_{0,1}}^{\gamma_{0,2}} f_{\Gamma_0,B}(\gamma_0,b)\,d\gamma_0, & n = 1,2,\ldots, \end{cases} \quad z \in \mathbb{R}. \tag{A.24}$$

Case III.5. Consider

$$f_1(z,n) = \int_{a_1}^{a_2} \frac{1}{|a|^n} f_{\Gamma_0,A}\left(\left(z-\frac{b}{1-a}(1-a^n)\right)\frac{1}{a^n}, a\right) da, \quad n = 0,1,2,\ldots, \; z \in \mathbb{R}. \tag{A.25}$$

Case III.6. Consider

$$f_1(z,n) = \begin{cases} \delta(z-\gamma_0), & n = 0, \\ \displaystyle\int_{a_1}^{a_2}\left|\frac{1-a}{1-a^n}\right| f_{B,A}\left((z-a^n\gamma_0)\frac{1-a}{1-a^n}, a\right) da, & n = 1,2,\ldots, \end{cases} \quad z \in \mathbb{R}. \tag{A.26}$$

Case III.7. Consider

$$f_1(z,n) = \int_{a_1}^{a_2}\int_{b_1}^{b_2} \frac{1}{|a|^n} f_{\Gamma_0,B,A}\left(z - \frac{b}{1-a}(1-a^n)\frac{1}{a^n}, b, a\right) db\,da, \quad n = 0,1,2,\ldots, \; z \in \mathbb{R}. \tag{A.27}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, Springer, New York, NY, USA, 6th edition, 2010.

[2] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, vol. 23 of *Applications of Mathematics: Stochastic Modelling and Applied Probability*, Springer, New York, NY, USA, 3rd edition, 1992.

[3] E. Allen, *Modeling with Itô Stochastic Differential Equations*, Springer, New York, NY, USA, 2007.

[4] W. Enders, *Applied Econometric Times Series*, John Wiley & Sons, New York, NY, USA, 1995.

[5] M. A. El-Beltagy and N. A. Al-Mulla, "Solution of the stochastic heat equation with nonlinear losses using Wiener-Hermite expansion," *Journal of Applied Mathematics*, vol. 2014, Article ID 843714, 9 pages, 2014.

[6] C. A. Braumann, "Growth and extinction of populations in randomly varying environments," *Computers & Mathematics with Applications*, vol. 56, no. 3, pp. 631–644, 2008.

[7] M. A. El-Beltagy and A. S. Al-Johani, "Numerical approximation of higher-order solutions of the quadratic nonlinear

stochastic oscillatory equation using WHEP technique," *Journal of Applied Mathematics*, vol. 2013, Article ID 685137, 21 pages, 2013.

[8] T. T. Soong, *Random Differential Equations in Science and Engineering*, Academic Press, New York, NY, USA, 1973.

[9] T. C. Gard, *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York, NY, USA, 1988.

[10] R. G. Ghanem and P. D. Spanos, *Stochastic Finite Elements: A Spectral Approach*, Courier Dover, New York, NY, USA, 2003.

[11] B. He, "Building structures vibration differential equations under random excitation," in *Computational Structural Engineering*, pp. 787–793, Springer, The Hague, the Netherlands, 2009.

[12] A. K. Golmankhaneh, N. A. Porghoveh, and D. Baleanu, "Mean square solutions of second-order random differential equations by using homotopy analysis method," *Romanian Reports in Physics*, vol. 65, no. 2, pp. 350–361, 2013.

[13] K. Nouri and H. Ranjbar, "Mean square convergence of the numerical solution of random differential equations," *Mediterranean Journal of Mathematics*, vol. 12, no. 3, pp. 1123–1140, 2015.

[14] J.-C. Cortés, L. Jódar, M.-D. Roselló, and L. Villafuerte, "Solving initial and two-point boundary value linear random differential equations: a mean square approach," *Applied Mathematics and Computation*, vol. 219, no. 4, pp. 2204–2211, 2012.

[15] A. Hussein and M. M. Selim, "Solution of the stochastic transport equation of neutral particles with anisotropic scattering using RVT technique," *Applied Mathematics and Computation*, vol. 213, no. 1, pp. 250–261, 2009.

[16] A. Hussein and M. M. Selim, "A developed solution of the stochastic Milne problem using probabilistic transformations," *Applied Mathematics and Computation*, vol. 216, no. 10, pp. 2910–2919, 2010.

[17] A. Hussein and M. M. Selim, "Solution of the stochastic radiative transfer equation with Rayleigh scattering using RVT technique," *Applied Mathematics and Computation*, vol. 218, no. 13, pp. 7193–7203, 2012.

[18] M. A. El-Tawil, W. El-Tahan, and A. Hussein, "A proposed technique of SFEM on solving ordinary random differential equation," *Applied Mathematics and Computation*, vol. 161, no. 1, pp. 35–47, 2005.

[19] M. El-Tawil, W. El-Tahan, and A. Hussein, "Using FEM–RVT technique for solving a randomly excited ordinary differential equation with a random operator," *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 856–867, 2007.

[20] M.-C. Casabán, J.-C. Cortés, J.-V. Romero, and M.-D. Roselló, "Determining the first probability density function of linear random initial value problems by the random variable transformation (RVT) technique: a comprehensive study," *Abstract and Applied Analysis*, vol. 2014, Article ID 248512, 25 pages, 2014.

[21] R. E. Walpole, R. H. Myers, S. L. Myers, and K. Ye, *Probability and Statistics for Engineers and Scientists*, Prentice Hall, New Jersey, NJ, USA, 8th edition, 2011.

[22] A. Papoulis and S. U. Pillai, *Probability, Random Variables and Stochastic Processes*, McGraw-Hill, New York, NY, USA, 4th edition, 2002.

[23] G. Casella and R. L. Berger, *Statistical Inference*, Brooks Cole, Pacific Grove, Calif, USA, 2002.

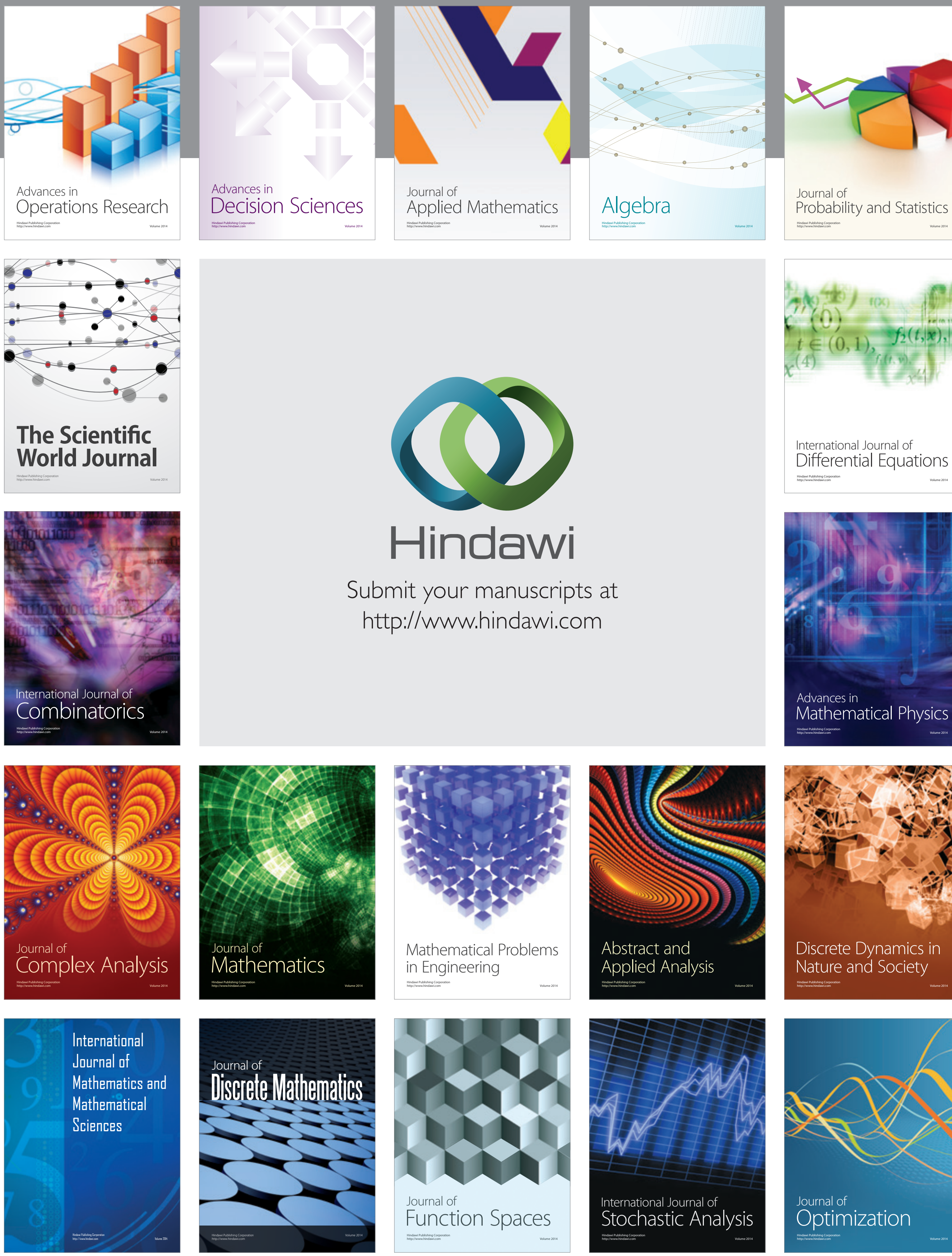
[24] M. A. El-Tawil, "The approximate solutions of some stochastic differential equations using transformations," *Applied Mathematics and Computation*, vol. 164, no. 1, pp. 167–178, 2005.

[25] Z. Yang and D. Xu, "Mean square exponential stability of impulsive stochastic difference equations," *Applied Mathematics Letters*, vol. 20, no. 8, pp. 938–945, 2007.

[26] G. Calbo, J.-C. Cortés, and L. Jódar, "Random matrix difference models arising in long-term medical drug strategies," *Applied Mathematics and Computation*, vol. 217, no. 5, pp. 2149–2161, 2010.

[27] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, Dover, New York, NY, USA, 1972.

[28] R. B. Nelsen, *An Introduction to Copulas*, Springer, New York, NY, USA, 1999.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