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Introduction to generalized topological spaces

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Abstract

We introduce the notion of generalized topological space (gt-space). Generalized topology of gt-space has the structure of frame and is closed under arbitrary unions and finite intersections modulo small subsets. The family of small subsets of a gt-space forms an ideal that is compatible with the generalized topology. To support the definition of gt-space we prove the frame embedding modulo compatible ideal theorem. We provide some examples of gt-spaces and study key topological notions (continuity, separation axioms, cardinal invariants) in terms of generalized spaces.

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1. Introduction

The notion of i-topological space was presented in [9]. That was our first try to develop the concept of topological space modulo small sets. Generalized topological space presented in this paper also acts modulo small sets that are encapsulated in an ideal, but has completely other form that makes things easier. The new form let us easily profit from compatibility of ideal with generalized topology, since now it is the basic property of generalized space (we had to assume compatibility in the preceding work).

We start with developing the frame-theoretical framework for generalized spaces and prove frame embedding modulo compatible ideal theorem that let us to make the definitions of generalized topological notions more transparent.

2. Frame embedding modulo compatible ideal

In what follows, we assume that a frame $(T, \leq, \vee, \wedge)$ and a complete Boolean lattice $(F, \leq, \cup, \cap, ^c)$ such that $T \subseteq F$ are fixed. We also assume that the inclusion from T into F is an order embedding preserving zero. For all $a, b \in F$ we define the operation $\setminus$ as follows: $a \setminus b = a \cap b^c$.

With respect to the fixed element $z \in F$, we say that $a \in F$ is *z-empty* iff $a \leq z$, and *z-nonempty* in other case [8]. If the converse is not stated, we always assume that some $z \in F$ is fixed. We use the notion of z-emptiness in order to use the following equivalent form of infinite distributivity.

Proposition 2.1 ([8]). *The following are equivalent:*

(ID1) *T is infinitely distributive, i.e. for every b in T and every subset $A \subseteq T$,*

$$b \wedge \bigvee A = \bigvee_{a \in A} (b \wedge a),$$

(ID2) *for every $d, z \in T$ and every $A \subseteq T$, if d is z-nonempty and $d \leq \bigvee A$ then there exists $a_0 \in A$ such that $d \wedge a_0$ is z-nonempty.*

The notion of *compatible ideal* in topological spaces was studied in numerous papers of T.R. Hamlett and D. Janković [3, 4, 5, 6, 7]. We generalize this notion in the natural way.

Definition 2.2. We say that an element $a \in F$ is *small* with respect to a subfamily $H \subseteq F$ iff there exists a family $U \subseteq T$ such that

$$a \leq \bigvee U \text{ and } a \cap u \in H \text{ for all } u \in U.$$

Denote the family of all small elements by $sm(H)$. We do not specify T in the notation $sm(H)$ since T is always fixed. The family H is said to be *compatible with* T, denote by $H \sim T$, iff it contains all small elements from $sm(H)$.

An element $a \in F$ is said to be *join-generated* by the family $U \subseteq T$ iff $a \leq (\bigvee U) \setminus (\bigcup U)$. An element $b \in F$ is said to be *meet-generated* by the family $V \subseteq T$ iff V is finite and $b \leq (\bigcap V) \setminus (\bigwedge V)$. In what follows, we are going to consider the following families:

1. G – the family of all join- and meet-generated elements,
2. $i(G)$ – the family of finite joins of elements of G,
3. $c(G)$ – the family of finite joins of elements of $sm(G)$.

Lemma 2.3. *The families $i(G)$ and $c(G)$ are ideals.*

Proof. Both $i(G)$ and $c(G)$ are closed under finite joins. Thus, we have to show that both families are lower sets.

Consider $b \in F$ and $a \in i(G)$ with $b \leq a$. Then there exists finite $A \subseteq G$ such that $a = \bigcup A$. Then $b = b \cap a = b \cap \bigcup A = \bigcup_{a \in A} (b \cap a)$. Clearly, $b \cap a \leq a$ and, hence, $b \cap a \in G$ holds for every $a \in A$, since $a \in G$ and G is a lower set. Hence, b lies in $i(G)$ as a finite join of elements of G.

Assume that $b \in F$ and $a \in sm(G)$ with $b \leq a$. Then there exists a family $U \subseteq T$ such that $a \leq \bigvee U$ and $a \cap u \in G$ for all $u \in U$. Clearly, it holds that $b \leq \bigvee U$ and $b \cap u \in G$ for all $u \in U$, since G is a lower set. We conclude that $sm(G)$ is a lower set.

Consider $b \in F$ and $a \in c(G)$ with $b \leq a$. Then there exists finite $A \subseteq sm(G)$ such that $a = \bigcup A$. Then $b = b \cap a = b \cap \bigcup A = \bigcup\limits_{a \in A} (b \cap a)$. Clearly, $b \cap a \leq a$ and, hence, $b \cap a \in sm(G)$ holds for every $a \in A$, since $a \in sm(G)$ and $sm(G)$ is a lower set. Hence, b lies in $c(G)$ as a finite join of elements of $sm(G)$. □

Lemma 2.4. *The following inclusions hold: $G \subseteq sm(G)$, $i(G) \subseteq c(G)$, $i(G) \subseteq sm(i(G))$ and $c(G) \subseteq sm(c(G))$.*

Proof. Assume that $H \subseteq F$ is a lower set, $a \in H$ and there exists a family $U \subseteq T$ such that $a \leq \bigvee U$. Then $a \in sm(H)$. Since for every element $a \in G$ there is $U \in T$ such that $a \leq \bigvee U$, we conclude that $G \subseteq sm(G)$. The other inclusions are the corollary from the later one. □

Lemma 2.5. *Given $U = \{ u_1, u_2, ..., u_k\} \subseteq T$ and a z-nonempty $v \in T$ such that $v \leq u_1$, then there exists a z-nonempty $w \in T$ that satisfies $w \leq v$ and one of the following:*

(i) $w \leq \bigwedge U$;
(ii) $w \wedge u_j$ *is z-empty for some* $u_j \in U$.

Proof. Define the decreasing chain $V = \{ v_1, v_2, ..., v_k \}$ as follows:

$$v_1 = v \text{ and } v_i = v_{i-1} \wedge u_i \text{ for } i \in \{ 2, 3, ..., k \}.$$

At least one element of this chain is z-nonempty since $v_1 = v$. Put $w = v_j$ where v_j is the last z-nonempty element of the chain. Clearly, it holds that $w \leq v$. There are two possibilities: either $j < k$ and then $w \wedge u_{j+1}$ is z-empty, or $j = k$ and then $w \leq \bigwedge U$. The lemma is proved. □

Lemma 2.6. *Given $z \in T$, a z-nonempty $y \in T$ satisfying $y \backslash z \leq \bigcup A$ for some $A = \{ a_1, a_2, ..., a_n \} \subseteq G$, then there exist a z-nonempty $w \in T$ and $a_j \in A$ such that $w \leq y$ and $w \cap a_j = 0$.*

Proof. For every $a_i \in A$ we denote by U_i the corresponding family of elements from T by that a_i is join- or meet-generated. We define the covering family as follows:

$$C_1 = \{ z \} \cup U_1 \cup U_2 \cup ... \cup U_n.$$

It holds that $y \leq \bigvee C_1$. By Proposition 2.1, we conclude that $v_1 = y \wedge u$ is z-nonempty for some $u \in C_1$. Without loss of generality, we assume that $v_1 \leq u \in U_1$. There are two possibilities for a_1: either a_1 is join-generated and then $v_1 \cap a_1 = 0$, we put $w = v_1$ and the proof is complete, or a_1 is meet-generated and then, applying Lemma 2.5 for U_1 and v_1, we obtain a z-nonempty element $w_1 \leq v_1$. If $w_1 \leq \bigwedge U_1$ then $w_1 \cap a_1 = 0$; we put $w = w_1$ and the proof is complete. If the other case, we possess the element $u_1 \in U_1$

such that $w_1 \wedge u_1$ is z-empty. We repeat the whole process from the beginning. Define the covering family C_2 as follows:

$$C_2 = \{ z \} \cup \{ u_1 \} \cup U_2 \cup U_3 \cup ... \cup U_n.$$

It holds that $w_1 \leq \bigvee C_2$. Again, by Proposition 2.1, $v_2 = w_1 \wedge u$ is z-nonempty for some $u \in C_2$. Without loss of generality, we assume that $v_2 \leq u \in U_2$. Again, there are two possibilities for a_2: either a_2 is join-generated and then $w = v_2$ satisfies the conditions of the theorem, or a_2 is meet-generated and we apply Lemma 2.5 for U_2 and v_2 and obtain a z-nonempty element $w_2 \leq v_2$. If it holds that $w_2 \leq \bigwedge U_2$ then $w = w_2$ is the one we need. In the other case, we possess the element $u_2 \in U_2$ such that $w_2 \wedge u_2$ is z-empty. We continue in the same way as above defining the covering family C_3.

Through the process, we obtain the decreasing chain of z-nonempty elements $y \geq v_1 \geq w_1 \geq v_2 \geq \ldots$. If the process stops at some $w = v_j$ or $w = w_j$, where $j \in \{ 1, \ldots, n \}$, it means that the proof is complete. Let us consider the other case. That is, we assume that the process did not stop and we possess the chain $y \geq w_1 \geq w_2 \geq \cdots \geq w_n$. As above, we define the covering family C_{n+1}:

$$C_{n+1} = \{ z \} \cup \{ u_1 \} \cup \{ u_2 \} \cup \cdots \cup \{ u_{n-1} \} \cup \{ u_n \}.$$

It holds that w_n is z-nonempty and $w_n \leq \bigvee C_{n+1}$. Hence, by Proposition 2.1, there exists $u \in C_{n+1}$ such that $w_n \wedge u$ is z-nonempty. But such u does not exist. Means the process stopped at some previous step. The proof is complete. □

Lemma 2.7. *For every $y, z \in T$, it holds that $y \setminus z \in i(G)$ iff $y \leq z$.*

Proof. If $y \leq z$ then $y \setminus z = 0 \in i(G)$. Let us proof the other implication.

Assume that $y \setminus z \neq 0$. Then there exists the family $A_1 = \{ a_1, a_2, \ldots, a_n \} \subseteq G$ such that $y \setminus z \leq \bigcup A_1$. Applying Lemma 2.6, we obtain a z-nonempty element $w_1 \leq y$ such that $w_1 \cap a = 0$ for some $a \in A_1$. Without loss of generality, assume that $w_1 \cap a_1 = 0$. Then from $w_1 \setminus z \leq \bigcup A_1$ and infinite distributivity of F we imply that

$$w_1 \setminus z = (w_1 \setminus z) \cap \bigcup A_1 = \bigcup_{a \in A_1} ((w_1 \setminus z) \cap a) = \bigcup_{a \in A_2} ((w_1 \setminus z) \cap a) = (w_1 \setminus z) \cap \bigcup A_2,$$

that is $w_1 \setminus z \leq \bigcup A_2$ where $A_2 = \{ a_2, \ldots, a_n \}$. We continue this process and obtain the decreasing chain of $y \geq w_1 \geq \cdots \geq w_n$ of z-nonempty elements. For the last element w_n it holds that $w_n \setminus z \leq 0$. Means w_n is z-empty. But an element cannot be both z-nonempty and z-empty. Hence, our assumption that $y \setminus z \neq 0$ was false, and we conclude that $y \setminus z \in i(G)$ implies $y \leq z$. □

Lemma 2.8. *For every $y, z \in T$, it holds that $y \setminus z \in c(G)$ iff $y \leq z$.*

Proof. If $y \leq z$ then $y \setminus z = 0 \in c(G)$. Let us proof the other implication.

Assume that $y \setminus z \neq 0$. Then there exists the family $A = \{ a_1, a_2, \ldots, a_n \} \subseteq sm(G)$ such that $y \setminus z \leq \bigcup A$. For every $a_i \in A$ we denote by U_i the corresponding family of elements from T (Def. 2.2). We define the covering family as follows:

$$C_1 = \{ z \} \cup U_1 \cup U_2 \cup ... \cup U_n.$$

It holds that $y \leq \bigvee C_1$. By Proposition 2.1, we conclude that $v_1 = y \wedge u_1$ is z-nonempty for some $u_1 \in C_1$.Without loss of generality, we assume that $u_1 \in U_1$. Consider the following auxiliary families:

$$D_1 = U_2 \cup ... \cup U_n \text{ and } A_1 = \{ a_2, \ldots, a_n \}.$$

Assume that $v_1 \wedge u$ is z-empty for all $u \in D_1$. Then it follows from infinite distributivity that $v_1 \cap \bigcup A_1 \leq (v_1 \cap \bigvee D_1) \setminus z \in G$. On the other hand, $v_1 \cap a_1 \leq u_1 \cap a_1 \in G$ since a_1 is a small element. Hence, $v_1 \setminus z = v_1 \cap \bigcup A \in i(G)$. Applying Lemma 2.7, we conclude that $v_1 \leq z$ – a contradiction! Means, $v_2 = v_1 \wedge u_2 = y \wedge u_1 \wedge u_2$ is z-nonempty for some $u_2 \in D_1$. Without loss of generality assume that $u_2 \in U_2$.

As above, the assumption that $v_2 \wedge u$ is z-empty for all $u \in U_3 \cup \cdots \cup U_n$ will bring us to the contradiction. Hence, $v_3 = v_2 \wedge u_3 = y \wedge u_1 \wedge u_2 \wedge u_3$ is z-nonempty for, without loss of generality, some $u_3 \in U_3$.

We continue this process till we obtain a z-nonempty $v_n = u_1 \wedge \cdots \wedge u_n$. Since all $a_i \in A$ are small elements, we imply that $u_i \cap a_i \in G$, for all u_i that form v_n. Hence, it holds that $v_n \setminus z \in i(G)$. This is a contradiction with Lemma 2.7, and we conclude that the assumption that $y \setminus z \neq 0$ is false. The proof is complete. □

Lemma 2.9. *Consider $a \in F$ and $U \subseteq T$ such that $a \leq \bigvee U$ and $a \cap u \in c(G)$ for all $u \in U$. Then there exist $b, c \in F$ such that $a = b \cup c$, $b \in sm(G)$ and $c \cap u \in sm(G)$ for all $u \in U$.*

Proof. Fix $u \in U$. Since $a \cap u \in c(G)$, there exist $A = \{ a_1, \ldots, a_n \} \subseteq sm(G)$ such that $a \cap u = \bigcup A$. For every $a_i \in A$ there is $V_i \subseteq T$ satisfying $a_i \leq \bigvee V_i$ and $a_i \cap v \in G$ for all $v \in V_i$. Write $V = \bigcup\limits_{i=1}^{n} V_i$. Then $a \cap u \subseteq u$ and $a \cap u \subseteq \bigvee V$. Hence, there exists $b^u \in G$ such that

$$(a \cap u) \setminus b^u \leq u \wedge \bigvee V = \bigvee_{v \in V} (u \wedge v).$$

Write $c_i = a_i \setminus b^u$ and $W_i = \{ u \wedge v \mid v \in V_i \}$.

Consider c_1. It holds that $c_1 \cap w \in G$ for all $w \in W_1$. On the other hand, $\bigvee\limits_{i=1}^{n} W_i \leq u$. Hence, $w \setminus \bigvee\limits_{i=2}^{n} W_i \in G$ holds for all $w \in W_1$. Applying Lemma 2.7, we conclude that $w \leq \bigvee\limits_{i=2}^{n} W_i$ for all $w \in W_i$. Thus, $c_1 = 0$.

We continue this process for all c_i where $i = 2, \ldots, n-1$. At the end, we will imply that $c_1 = \cdots = c_{n-1} = 0$ and $c_n = (a \cap u) \setminus b^u$. It is clear that $c_n \in sm(G)$. Write $c^u = c_n$.

We do the same steps for each $u \in U$. Write $b = \bigcup \{ b^u \mid u \in U \}$ and $c = \bigcup \{ c^u \mid u \in U \}$. Then b and c satisfy the necessary properties. The proof is complete. □

Lemma 2.10. *It holds that $c(G)$ is an ideal, $sm(c(G)) = c(G)$ and, hence, $c(G) \sim T$.*

Proof. We already proved that $c(G)$ is an ideal and that $c(G) \subseteq sm(c(G))$ (Lemmas 2.3 and 2.4). Let us prove that $sm(c(G)) \subseteq c(G)$.

Consider $a \in F$ and $U \subseteq T$ such that $a \leq \bigvee U$ and $a \cap u \in c(G)$ for all $u \in U$. Applying Lemma 2.9 and since $c(G)$ contains finite joins of small elements, without loss of generality, we can assume that $a \cap u \in sm(G)$ for all $u \in U$.

Fix $u \in U$. Then there exists a family $V^u \subseteq T$ such that $a \cap u \leq \bigvee V^u$ and $a \cap u \cap v \in G$ for all $v \in V^u$. Consider the family $W^u = \{v \wedge u \mid v \in V^u\}$. Clearly, $\bigvee W^u \leq u$ and there exists $c^u \in G$ such that $(a \cap u) \setminus c^u \leq \bigvee W^u$ and $c^u \leq u$.

Denote $b = \bigcup\{(a \cap u) \setminus c^u \mid u \in U\}$ and $c = \{c^u \mid u \in U\}$. Then $a = b \cup c$ and both b and c are small elements. We conclude that $a \in c(G)$. $\square$

Proposition 2.11. *Given an ideal $I \subseteq F$, then the relation $\preceq$ defined as follows is a preorder on F:*

$$a \preceq b \text{ iff } a \setminus b \in I.$$

The relation $\approx$ defined as follows is an equivalence on F:

$$a \approx b \text{ iff } a \setminus b \in I \text{ and } b \setminus a \in I.$$

Proof. The reflexivity holds for $\preceq$ and $\approx$ since $0 \in I$. The transitivity for $\preceq$ and $\approx$ holds since I is closed under finite joins. The symmetry follows from the definition of $\approx$. $\square$

The relations $\preceq$ and $\approx$ considered in the previous proposition are called *the preorder generated by the ideal I* and *the equivalence generated by the ideal I*, respectively.

Now we are ready to prove the frame embedding modulo compatible ideal theorem. Briefly speaking, this theorem says that an order embedding of a frame into a complete Boolean lattice preserving zero is always a frame embedding modulo compatible ideal. The assumption $T \subseteq F$ simplifies the notations but is not essential. We could speak as well about an order embedding $\varphi\colon T \to F$ preserving zero and consider the image $\varphi T \subseteq F$ instead of T.

Theorem 2.12. *Let $(T, \vee, \wedge)$ be a frame and $(F, \cup, \cap, {}^*)$ be a complete Boolean lattice such that $T \subseteq F$ and the inclusion from T into F is an order embedding preserving zero. Then $I = c(G)$ is the least ideal satisfying the following:*

(i) *for every $U \subseteq T$ there is $a \in I$ such that $\bigvee U = (\bigcup U) \cup a$;*
(ii) *for every $v, w \in T$ there is $b \in I$ such that $v \wedge w = (v \cap w) \setminus b$;*
(iii) *$I \cap T = \{0\}$;*
(iv) *for every $u, v \in T$, it holds that $u \preceq v$ iff $u \leq v$;*
(v) *$I \sim T$.*

Proof. It follows from the construction of I that (i) and (ii) hold. The statement (iv) is proved in Lemma 2.8 and applying Lemma 2.8 for all $y \in T$ and $z = 0$ we imply (iii). The statement (v) is proved in Lemma 2.10.

The least ideal satisfying (i),(ii) and (v) should contain all elements of $sm(G)$. On the other hand, the least ideal that contains all elements of $sm(G)$ is $c(G)$. Hence, $c(G)$ is the least ideal satisfying (i), (ii) and (v). $\square$

Remark 2.13. In Theorem 2.12, if we assume that T is not only a frame but is a complete completely distributive lattice is it then possible to construct the ideal in such a way that (i), (iii)-(v) hold and (ii) holds for arbitrary subfamilies of T? The counterexample for it was introduced in [8].

3. Definition of gt-space. Examples

We start with the immediate corollary from Theorem 2.12 that is the motivation for the following definition of generalized topological space.

Corollary 3.1. *Let X be a nonempty set. Assume that $T \subseteq 2^X$ forms a frame with respect to $\subseteq$ and $\varnothing, X \in T$. Then there exists the least ideal $I \subseteq 2^X$ such that:*

(i) *for every $\mathcal{U} \subseteq T$ holds $\bigvee \mathcal{U} \setminus \bigcup \mathcal{U} \in I$;*
(ii) *for every $V, W \in T$ holds $(V \cap W) \setminus (V \wedge W) \in I$;*
(iii) *$T \cap I = \{\varnothing\}$;*
(iv) *$U \preceq V$ (Prop. 2.11), i.e. $U \setminus V \in I$, implies $U \subseteq V$ for every $U, V \in T$;*
(v) *$U \approx V$ (Prop. 2.11), i.e. $U \Delta V \in I$, implies $U = V$ for every $U, V \in T$;*
(vi) *the ideal I is compatible with T, write $I \sim T$ (Def. 2.2), i.e. $A \subseteq X$ and $\mathcal{U} \subseteq T$ with $A \subseteq \bigvee \mathcal{U}$ and $A \cap U \in I$, for all $U \in \mathcal{U}$, imply that $A \in I$.*

Definition 3.2. Let X be a nonempty set. A family $T \subseteq 2^X$ is called a *generalized topology* (or *topology modulo ideal*) and the pair (X, T) is called a *generalized topological space* (*gt-space* for short, or *topological space modulo ideal*) provided that:

(GT1) $\varnothing, X \in T$;
(GT2) $(T, \subseteq)$ is a frame.

The elements of X are called *points*, and the elements of T are called *open sets*. We say that $Y \subseteq X$ is a *neighborhood of a point x* iff there is $U \in T$ such that $x \in U \subseteq Y$. We use the notation $T(x)$ for the family of all open neighborhoods of a point x.

An ideal $J \subseteq 2^X$ satisfying (i)-(v) of Corollary 3.1 is called *suitable*. In there is no chance for confusion, we keep the notation I, sometimes with the appropriate index, to denote *the least suitable ideal* (the existence of it is proved in Corollary 3.1). If the ideal is not specified in a definition or construction then it is always the least suitable ideal.

If there is no specification or index, we use the symbols $\preceq$ and $\approx$ to denote the preoder and equivalence, respectively, generated by the least suitable ideal (Prop. 2.11). We keep the notations $\vee$ and $\wedge$ for the frame operations of generalized topology.

A topological space is a trivial example of gt-space where the least suitable ideal consists only of the empty set. In order to distinguish between topological spaces and gt-spaces that are not topological spaces, we provide the following classification.

Definition 3.3. A gt-space is called

(1) *crisp gt-space* (*crisp space* for short) iff its least suitable ideal is $\{\varnothing\}$;
(2) *proper gt-space* iff it is not crisp.

Example 3.4 (Right arrow gt-space)**.** *Consider the real line* $\mathbb{R}$ *and the family of "right arrows"*

$$\mathcal{A} = \{ [a, b) \mid a, b \in \mathbb{R} \cup \{-\infty, +\infty\} \text{ and } a < b \}.$$

We say that a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ *is well separated iff for all* $[a, b), [c, d) \in \mathcal{A}'$ *it holds that* $b < c$ *or* $d < a$*. Construct the family* T_{ra} *as follows:*

$$T_{ra} = \{ \varnothing, \mathbb{R} \} \cup \left\{ \bigcup \mathcal{A}' \mid \mathcal{A}' \subseteq \mathcal{A} \text{ and } \mathcal{A}' \text{ is well separated} \right\}.$$

Clearly T_{ra} *is a complete lattice and it is also easy to see that* T_{ra} *is infinitely distributive (e.g. by Proposition 2.1). Then the pair* $(\mathbb{R}, T_{ra})$ *forms a gt-space. The respective least ideal for this gt-space is the family* D *of nowhere dense subsets of the real line. Let us prove that.*

Consider a nowhere dense subset $A \subseteq \mathbb{R}$ *and the open subset* $U = \bigvee \{ V \in T_{ra} \mid V \cap A = \varnothing \}$*. Fix a point* $a \in A$ *and assume that* $a \notin U$*. Assume that we could find the closest "right arrow" on the right from the point* a*, that is there exists* $b = \min \{ r \in U \mid r > a \}$*. Then there is an interval* $(c, d) \subseteq (a, b)$ *such that* $(c, d) \cap A = \varnothing$*, since* A *is nowhere dense, and, hence, there is a non-empty open* $V \subseteq (c, d)$ *with* $V \cap A = \varnothing$*. The latter means that* $V \subseteq U$ *but this is a contradiction and we conclude that such minimal point* b *does not exist.*

Then the part of U *lying on the right from the point* a *is a union of a well separated subfamily of "right arrows". This and the fact that* A *is nowhere dense imply that there exists an interval* (c, d) *such that* $a < c$*,* $(c, d) \cap U = \varnothing$ *and* $(c, d) \cap A = \varnothing$*. Hence, there is a non-empty open* $V \subseteq (c, d)$ *with* $V \cap U = \varnothing$ *and* $V \cap A = \varnothing$*. The latter is a contradiction and we conclude that our assumption that* $a \notin U$ *is false.*

Thus, we proved that $A \subseteq U$*. This means that* $U = \mathbb{R}$ *and, hence,* A *belongs to the least suitable ideal as a join-generated subset of* $\mathbb{R}$*.*

Example 3.5. *Consider the family* $T_{ra}^{\mathbb{Q}} = \{ U \cap \mathbb{Q} \mid U \in T_{ra} \}$ *where* T_{ra} *is a generalized topology from the previous example. Then* $(\mathbb{Q}, T_{ra}^{\mathbb{Q}})$ *is also a gt-space.*

Example 3.6. *Let* (X, T) *be a topological space. The family of all regular open subsets of* X *is denoted by* $R(T)$*. It is known [2] that* $R(T)$ *forms a frame with respect to* $\subseteq$*. Clearly,* $\varnothing, X \in R(T)$*. Hence,* $(X, R(T))$ *is a gt-space.*

Example 3.7. *Let T be a usual topology on $\mathbb{R}^2$. Consider the following family*

$$\tau = \left\{ \mathbb{R}^2 \setminus U \mid U \in R(T) \right\}.$$

Then (X, τ) is a gt-space. Briefly speaking, this is a generalized topology consisting of "2-dimensional figures without 1-dimensional protuberances and cracks".

Example 3.8. *Let X be a nonempty set X and $S \subseteq 2^X$. Assume that S separates the elements of X, is a complete, completely distributive lattice with respect to $\subseteq$, contains $\varnothing$ and X, arbitrary meets coincide with intersections, and finite joins with unions. Then (X, S) is called a* texture [1].

A ditopology [1] *on a texture (X, S) is a pair (τ, κ) of subsets of S, where the set of* open sets τ *and the set of* closed sets κ *satisfy the following:*

(1) $S, \varnothing \in \tau$,
(2) $G_1, G_2 \in \tau$ implies $G_1 \cap G_2 \in \tau$,
(3) $\mathcal{G} \subseteq \tau$ implies $\bigvee \mathcal{G} \in \tau$,
(4) $S, \varnothing \in \kappa$,
(5) $K_1, K_2 \in \kappa$ implies $K_1 \cup K_2 \in \kappa$,
(6) $\mathcal{K} \subseteq \kappa$ implies $\bigcap \mathcal{K} \in \kappa$.

Let (τ, κ) be a ditopology on a texture (X, S). Then (X, τ) forms a gt-space.

4. Closed sets. Interior and closure operators

The concept of gt-space makes it possible to preserve the classical definition for closed sets. Nevertheless, it would be interesting to consider the notion of "closed subset modulo ideal" in the further research.

Definition 4.1. Let (X, T) be a gt-space. A subset $Y \subseteq X$ is called *closed* iff $Y = X \setminus U$ for some $U \in T$.

Proposition 4.2. *Let (X, T) be a gt-space. The family of all closed subsets of X is a complete lattice where the join and meet operations are the following:*

$$\bigvee \mathcal{A} = X \setminus \bigwedge \{X \setminus A \mid A \in \mathcal{A}\}$$

and

$$\bigwedge \mathcal{A} = X \setminus \bigvee \{X \setminus A \mid A \in \mathcal{A}\}$$

where $\mathcal{A}$ is a family of closed subsets of X.

Proof. The proof is an easy exercise since T is a complete lattice. □

Proposition 4.3. *Let (X, T) be a gt-space. For all closed subsets $A, B \subseteq X$, it holds that $A \preceq B$ iff $A \subseteq B$.*

Proof. Consider closed subsets $A, B \subseteq X$ such that $A \preceq B$. Then $X \setminus A$ and $X \setminus B$ are open, and it holds that $X \setminus B \preceq X \setminus A$. The latter implies that $X \setminus B \subseteq X \setminus A$, and, hence, $A \subseteq B$. □

We use the set operator ψ [3] and the local function * [5] as interior and closure operators in gt-spaces. Note that these operators would not make so much topological sense in our framework if the ideal would not be compatible with a gt-topology (Def. 3.1). Indeed, to prove that $\psi(A) \preceq A \preceq A^*$ holds for all $A \subseteq X$ in a gt-space (X, T) we need the ideal I to be compatible with T.

Definition 4.4. Let (X, T) be a gt-space. The operators ${}^*\colon 2^X \to 2^X$ and $\psi\colon 2^X \to 2^X$ are defined as follows, for all $A \subseteq X$:

$$\psi(A) = \{x \in X \mid \text{ exists } U \in T(x) \text{ such that } U \preceq A\},$$
$$A^* = \{x \in X \mid \text{ for all } U \in T(x) \text{ it holds that } A \cap U \notin I\}.$$

The operator ψ is called *the interior operator* (*interior operator modulo ideal*) and * is called *the closure operator* (*closure operator modulo ideal*).

Theorem 4.5. *In a gt-space (X, T), the following hold for every $A, B \subseteq X$:*

(i) $\psi(A) = \bigvee\{U \subseteq X \mid U \preceq A,\ U \text{ is open}\}$, *and*
$A^* = \bigwedge\{B \subseteq X \mid A \preceq B,\ B \text{ is closed}\}$ *(Def. 4.1);*
(ii) $\psi(A) = X \setminus (X \setminus A)^*$*;*
(iii) *A is open iff $A = \psi(A)$, and A is closed iff $A = A^*$;*
(iv) $\psi(X) = X$ *and* $\varnothing^* = \varnothing$*;*
(v) $\psi(A) \preceq A \preceq A^*$*;*
(vi) $\psi(\psi(A)) = \psi(A)$ *and* $(B^*)^* = B^*$*;*
(vii) $\psi(A \cap B) = \psi(A) \wedge \psi(B)$ *and* $(A \cup B)^* = A^* \vee B^*$.

Proof. The statement (i) is a straight corollary from the definition of the interior and closure operators. Let us prove (ii). Consider a subset $A \subseteq X$. A point $x \in X$ belongs to $\psi(A)$ if there exists $U \in T(x)$ such that $U \preceq A$, that is $U \cap (X \setminus A) \in I$. The latter means that $x \notin (X \setminus A)^*$ and, hence, $x \in X \setminus (X \setminus A)^*$. Then it follows that $\psi(A) \subseteq X \setminus (X \setminus A)^*$.

Assume that $x \in X \setminus (X \setminus A)^*$, that is $x \notin (X \setminus A)^*$. Then there exists $V \in T(x)$ such that $V \cap (X \setminus A) \in I$. Hence, $V \preceq A$ and we conclude that $x \in \psi(A)$. The proof is complete.

For (iii)-(vii), we provide the proofs only for the interior operator. The proofs for the closure operator could be done in the similar way.

Let us prove (iii). If A is open then $A \preceq A$, and $U \preceq A$ implies $U \subseteq A$ for all open U. Hence, $A = \psi(A)$. On the other hand, if $A = \psi(A)$ then A is the join of all such open U that $U \preceq A$, that is A is open.

The statement (iv) is obvious. To prove (v), it is enough to remember that $I \sim T$. The property (vi) is a straight corollary from (iii).

To prove (vii), let us consider the following chain of implications: $U \subseteq \psi(A \cap B)$ iff $U \preceq A \cap B$ iff $U \preceq A$ and $A \preceq B$ iff $U \subseteq \psi(A) \cap \psi(B)$ iff $U \subseteq \psi(A) \wedge \psi(B)$. The proof is complete. □

5. Generalized continuous (g-continuous) mappings

We generalize the notion of continuous mapping in a natural way. Like in the previous section, note that the proof of the essential Theorem 5.2 is not possible without the ideal being compatible with a gt-topology (Def. 3.1).

Definition 5.1. Let (X, T_X) and (Y, T_Y) be gt-spaces. A mapping $f\colon X \to Y$ is called a *generalized continuous mapping* (or *g-continuous mapping* for short) provided that there exists a frame homomorphism $h\colon T_Y \to T_X$ such that $h(U) \approx f^{-1}(U)$ holds for every $U \in T_Y$.

The g-continuous mapping f is called a *generalized homeomorphism* (or *g-homeomorphism* for short) iff f is a bijection and f^{-1} is g-continuous.

Theorem 5.2. *Given gt-spaces* (X, T_X) *and* (Y, T_Y)*, and a g-continuous mapping* $f\colon X \to Y$*, then the following hold:*

(i) *the corresponding frame homomorphism* $h\colon T_Y \to T_X$ *is unique;*
(ii) $f^{-1}(B) \in I_X$ *holds for all* $B \in I_Y$*.*

Proof. Assume that there exists a frame homomorphism $g\colon T_Y \to T_X$ such that $g(U) \approx f^{-1}(U)$ for every $U \in T_Y$. Then $h(U) \approx g(U)$ for every $U \in T_Y$. Since $h(U) \in T_X$ and $g(U) \in T_X$, it follows from Corollary 3.1 that $h(U) = g(U)$ for all $U \in T_Y$. Hence, $h = g$, and we proved (i).

Assume that $B \subseteq Y$ is join-generated. Then there exists a family $\mathcal{V} \subseteq T_Y$ such that $B \subseteq (\bigvee \mathcal{V}) \setminus (\bigcup \mathcal{V})$. Denote by A the preimage of B:

$$A = f^{-1}(B) \subseteq f^{-1}\left(\bigvee \mathcal{V}\right) \setminus \bigcup_{V \in \mathcal{V}} f^{-1}(V).$$

Divide A in three subsets and prove that they belong to the ideal I_X:

(1) $A \setminus h(\bigvee \mathcal{V}) \in I_X$ holds since $h(\bigvee \mathcal{V}) \approx f^{-1}(\bigvee \mathcal{V})$;
(2) $(A \cap h(\bigvee \mathcal{V})) \setminus \bigcup_{V \in \mathcal{V}} h(V) \in I_X$ holds
since $h(\bigvee \mathcal{V}) = \bigvee_{V \in \mathcal{V}} h(V)$ and $h(\bigvee \mathcal{V}) \setminus \bigcup_{V \in \mathcal{V}} h(V) \in I_X$;
(3) $A \cap \bigcup_{V \in \mathcal{V}} h(V) \in I_X$ holds
since $A \cap h(V) \subseteq h(V) \setminus f^{-1}(V) \in I_X$, for all $V \in \mathcal{V}$, and $I_X \sim T_X$.

In a similar way, we prove that the preimage of every meet-generated subset of Y lies in the ideal I_X.

Now, consider a subset $B \subseteq Y$ and a family $\mathcal{V} \subseteq T_Y$ satisfying the following: for all $V \in \mathcal{V}$, it holds that $B \cap V$ is a join- or meet-generated subset of Y. Denote by $A = f^{-1}(B)$. As above, divide A in three subsets and prove that they belong to the ideal I_X:

(4) $A \setminus h(\bigvee \mathcal{V}) \in I_X$ and $(A \cap h(\bigvee \mathcal{V})) \setminus \bigcup_{V \in \mathcal{V}} h(V) \in I_X$ as above;
(5) $A \cap \bigcup_{V \in \mathcal{V}} h(V) \in I_X$ holds
since $A \cap h(V) \approx A \cap f^{-1}(V) = f^{-1}(B \cap V) \in I_X$, for all $V \in \mathcal{V}$, and $I_X \sim T_X$.

The rest of the proof is obvious. □

Proposition 5.3. *Given gt-spaces* (X, T_X)*,* (Y, T_Y) *and* (Z, T_Z) *and g-continuous mappings* $f\colon X \to Y$ *and* $g\colon Y \to Z$*, then the composition* $g \circ f\colon X \to Z$ *is also a g-continuous mapping.*

Proof. Denote by h_f and h_g the corresponding frame homomorphisms for f and g, respectively. It is known that the composition of frame homomorphisms is also a frame homomorphism. Hence, $h_f \circ h_g$ is a frame homomorphism. We have to show that $f^{-1}\left(g^{-1}(V)\right) \approx h_f\left(h_g(V)\right)$ holds for all $V \in T_Z$. The latter holds, since $f^{-1}(B) \in I_X$ for all $B \in I_Y$. □

The next proposition makes it easier to check if a mapping is g-continuous in some particular cases. We will use it when proving Theorem 6.5 (Urysohn Lemma for gt-spaces).

Definition 5.4. Let (X, T) be a gt-space. A family $\mathcal{B} \subseteq T$ is called a *base for* T provided that for every open set U there exists a subfamily $\mathcal{B}_0 \subseteq \mathcal{B}$ such that $U = \bigvee \mathcal{B}_0$.

Proposition 5.5. *Let (X, T_X) and (Y, T_Y) be gt-spaces, $\mathcal{B} \subseteq T_Y$ be a base, and $f\colon X \to Y$ be a mapping. Assume that*

(1) *$f^{-1}(B) \in I_X$ holds for all $B \in I_Y$,*
(2) *for every $V \in \mathcal{B}$ there is $V' \in T_X$ such that $V' \approx f^{-1}(V) \subseteq V'$.*

Then f is a g-continuous mapping.

Proof. Define a mapping $h_0\colon \mathcal{B} \to T_X$ such that $h_0(V) \approx f^{-1}(V) \subseteq h_0(V)$, for all $V \in \mathcal{B}$. Such a mapping exists under our assumption, and is unique since (X, T_X) is a gt-space. Take an open subset $U \subseteq Y$, and consider a family $\mathcal{V} \subseteq \mathcal{B}$ such that $U = \bigvee \mathcal{V}$. Then

$$f^{-1}(U) = f^{-1}\left(\bigvee \mathcal{V}\right) \approx f^{-1}\left(\bigcup \mathcal{V}\right) = \bigcup_{V \in \mathcal{V}} f^{-1}(V) \subseteq \bigcup_{V \in \mathcal{V}} h_0(V) \approx \bigvee_{V \in \mathcal{V}} h_0(V).$$

Denote $A = \bigcup_{V \in \mathcal{V}} h_0(V) \setminus \bigcup_{V \in \mathcal{V}} f^{-1}(V)$. Then $A \subseteq \bigvee_{V \in \mathcal{V}} h_0(V)$ and $A \cap h_0(V) \in I_X$, for all $V \in \mathcal{V}$. Since $I_X \sim T_X$, we conclude that $A \in I_X$. Hence,

$$f^{-1}(U) \approx \bigvee_{V \in \mathcal{V}} h_0(V).$$

Consider $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{B}$ such that $U = \bigvee \mathcal{V}_1 = \bigvee \mathcal{V}_2$. Then, according to the latter observation, it holds that

$$f^{-1}(U) \approx \bigvee_{V \in \mathcal{V}} h_0(V_1) \approx \bigvee_{V \in \mathcal{V}} h_0(V_2).$$

Since both joins $\bigvee_{V \in \mathcal{V}} h_0(V_1)$ and $\bigvee_{V \in \mathcal{V}} h_0(V_2)$ are open in T_X, we conclude that they are equal. Define the mapping $h\colon T_Y \to T_X$ as follows, $U \in T_Y$:

$$h(U) = h_0(U), \text{ if } U \in \mathcal{B};$$
$$h(U) = \bigvee_{V \in \mathcal{V}} h_0(V), \text{ if } \bigvee \mathcal{V} = U \notin \mathcal{B} \text{ and } \mathcal{V} \subseteq \mathcal{B}.$$

Then $h(U) \approx f^{-1}(U)$ holds for all $U \in T_Y$. Note that h preserves arbitrary joins of the elements of $\mathcal{V}$. Let us prove that h preserves arbitrary joins of arbitrary open subsets of Y. Consider $\mathcal{U} \subseteq T_Y$ and a family $\mathcal{V} \subseteq \mathcal{B}$ satisfying $\bigvee \mathcal{U} = \bigvee \mathcal{V}$. We have to show that

$$\bigvee_{U \in \mathcal{U}} h(U) = h\left(\bigvee \mathcal{U}\right).$$

It holds that $\bigvee_{U\in\mathcal{U}} h(U) \approx \bigcup_{U\in\mathcal{U}} h(U)$, and it follows from the assumption (1) and the construction of h that

$$\bigcup_{U\in\mathcal{U}} f^{-1}(U) = f^{-1}\left(\bigcup \mathcal{U}\right) \approx f^{-1}\left(\bigvee \mathcal{U}\right) \approx h\left(\bigvee \mathcal{U}\right) =$$

$$= h\left(\bigvee \mathcal{V}\right) = f^{-1}\left(\bigvee \mathcal{V}\right) \approx f^{-1}\left(\bigcup \mathcal{V}\right) \approx \bigcup_{V\in\mathcal{V}} f^{-1}(V).$$

Hence, to complete the proof, it is enough to show that the subsets $A, B \subseteq X$ defined as follows are elements of the ideal I_X:

$$A = \bigcup_{U\in\mathcal{U}} h(U) \setminus \bigcup_{U\in\mathcal{U}} f^{-1}(U) \text{ and } B = \bigcup_{V\in\mathcal{V}} f^{-1}(V) \setminus \bigcup_{U\in\mathcal{U}} h(U).$$

The subset A lies in the ideal, since $I_X \sim T_X$. Under our assumption, for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subseteq U$. Then, for every such V and U, it holds that $f^{-1}(V) \subseteq h(V) \subseteq h(U)$. Thus, $\bigcup_{V\in\mathcal{V}} f^{-1}(V) \subseteq \bigcup_{U\in\mathcal{U}} h(U)$, that is $B = \varnothing \in I_X$. The proof is complete. □

In the following example we show that for given gt-spaces (X, T_X) and (Y, T_Y) and a frame homomorphism $h\colon T_Y \to T_X$ it is possible that there does not exist a g-continuous mapping $f\colon X \to Y$ such that h is its corresponding frame homomorphism.

Example 5.6. *Consider the gt-spaces $(\mathbb{R}, T_{ra})$ and $(\mathbb{Q}, T_{ra})$ (Examples 3.4 and 3.5). Assume that there exists a g-continuous mapping of the given gt-spaces $f\colon \mathbb{R} \to \mathbb{Q}$ such that the corresponding frame homomorphism is the identity mapping* $\mathrm{id}\colon T_{ra} \to T_{ra}$.

We mentioned already that the least suitable ideal of the gt-space $(\mathbb{R}, T_{ra})$ is the family D of nowhere dense subsets of $\mathbb{R}$. Since f is a g-continuous mapping, it follows that $U\Delta f^{-1}(U) \in D$ for every $U \in T_{ra}$. Hence, $|U\Delta f^{-1}(U)| \leq \aleph_0$ holds for every $U \in T_{ra}$.

Assume that $|f^{-1}(q)| \leq \aleph_0$ for every $q \in \mathbb{Q}$. Then $\aleph_0 \geq |f^{-1}\mathbb{Q}| \neq |\mathbb{R}|$. The later inequality is a contradiction, and we conclude that there exists $q \in \mathbb{Q}$ such that $|f^{-1}(q)| > \aleph_0$. Denote $U_q = [q, +\infty)$.

Since $|U_q \Delta f^{-1}(U)_q| \leq \aleph_0$, we imply that $|U_q \cap f^{-1}(U)_q| > \aleph_0$. Then there exists $p \in \mathbb{Q}$ such that $|[p, +\infty) \cap f^{-1}(q)| > \aleph_0$. Then it follows that $|[p, +\infty)\Delta f^{-1}[p, +\infty)| > \aleph_0$, and we conclude that f is not a g-continuous mapping.

6. Separation axioms

In this section, we consider T_1 separation axiom and normal spaces. We use the notation $\mathbf{I}$ for the real number interval $[0, 1]$ with the usual topology of open intervals.

Definition 6.1. The gt-space (X, T) is said to be T_1 iff for every $x, y \in X$ there is $U \in T$ such that $x \in U$ and $y \notin U$.

Proposition 6.2. *Given a T_1 gt-space (X, T) and $x \in T$, then one and only one of the following holds: $\{ x \} \in I$ or $\{ x \}$ is closed.*

Proof. Consider the set $A = \{ \bigcup U \in T \mid x \notin U \} = X \setminus \{ x \}$. There are two possibilities: A is open or A is not open. Assume that A is open. Then $\{ x \}$ is closed and it cannot be that $\{ x \} \in I$, since it is impossible that X and A are both open, $X \neq A$ and $X \approx A$.

Assume that $\{ x \} \in I$. Then it is impossible that A is open, since it cannot be that X and A are both open, $X \neq A$ and $X \approx A$. Hence, $\{ x \}$ is not closed. □

In the following example we show that the property of T_1 gt-space from the previous proposition is necessary but not sufficient for T_1.

Example 6.3. *Let $X = \{ 0 \} \cup [1, 2)$. Consider a family*

$$\mathcal{A} = \{ \{ 0 \} \cup [1, b) \mid b \in (1, 2) \} \cup \{ (a, b) \mid a, b \in (1, 2) \text{ and } a \leq b \} .$$

Define T as a family consisting of empty set, the elements of $\mathcal{A}$ and such disjoint units of elements of $\mathcal{A}$ that, for every $U \in T$ and $b \in (1, 2)$, it holds that $(1, b) \subseteq U$ implies $\{ 0, 1 \} \subseteq U$. Then (X, T) is a gt-space and the least suitable ideal is $I = \{ \{ \varnothing \}, \{ 0 \}, \{ 1 \}, \{ 0, 1 \} \}$.

For every point $x \in (1, 2)$, it holds that $\{ x \}$ is a closed set, since $\{ 0 \} \cup [1, x) \cup (x, 2) \in T$. On the other hand, $\{ x \}$ is not an element of the ideal. The sets $\{ 0 \}$ and $\{ 1 \}$ are elements of the ideal and are not closed sets.

This gt-space is not T_1 since we cannot separate 0 and 1.

Definition 6.4. A gt-space (X, T) is called *normal* iff for every disjoint nonempty closed $A, B \subseteq X$ there exist $U, V \in T$ such that $A \subseteq U$, $B \subseteq V$ and $U \wedge V = \varnothing$.

Theorem 6.5 (Urysohn's Lemma for gt-spaces)**.** *Let (X, T) be a normal gt-space and $A, B \subseteq X$ be disjoint nonempty closed subsets. Assume that finite meets of open subsets coincide with intersections. Then there exists a g-continuous mapping $f \colon X \to \mathbf{I}$ such that $f(A) = 0$ and $f(B) = 1$.*

Proof. Let us organize the rational numbers from $\mathbf{I}$ into a sequence $r_0, r_1, r_2, \ldots$ where $r_0 = 0$ and $r_1 = 1$. For every rational number r, we are going to define an open subset W_r such that the following is satisfied for every index k:

$$W^*_{r_i} \subseteq W_{r_j} \text{ for } r_i < r_j \text{ and } i, j \leq k. \tag{6.1}$$

Consider disjoint nonempty closed subsets $A, B \subseteq X$. Since X is normal, there exist $U, V \in T$ such that $A \subseteq U$, $B \subseteq V$ and $U \wedge V = \varnothing$. Denote

$$W_0 = U \text{ and } W_1 = X \setminus B.$$

Then $W_0 \preceq X \setminus V = (X \setminus V)^* \subseteq W_1$. By Theorem 4.5, we conclude that $W^*_0 \subseteq W_1$, and, hence, (6.1) is satisfied for $k = 1$.

Assume that the open subsets W_{r_i} satisfying (6.1) are already constructed for all $i \leq n \geq 1$. Denote by

$$r_l = \max\{r_i \mid i \leq n \text{ and } r_i < r_{n+1}\} \text{ and} r_m = \min\{r_i \mid i \leq n \text{ and } r_{n+1} < r_i\}.$$

It holds that $r_l \leq r_m$, and, hence, $W^*_{r_l} \subseteq W_{r_m}$. Since X is normal, there exist open U, V such that $W^*_{r_l} \subseteq U$, $X \setminus W_{r_m} \subseteq V$, and $U \wedge V = \varnothing$. Then $U \preceq X \setminus V = (X \setminus V)^* \subseteq W_{r_m}$, and $U^* \subseteq W_{r_m}$. Denote $W_{n+1} = U$. By the finite induction, we obtained the sequence $W_{r_0}, W_{r_1}, W_{r_2}, \dots$ satisfying (6.1) for all indexes k, and

$$A \subseteq W_0 \text{ and } B \subseteq X \setminus W_1. \tag{6.2}$$

We define the function $f\colon X \to \mathbf{I}$ as follows:

$$f(x) = \begin{cases} \inf\{r \mid x \in W_r\}, & \text{if } x \in W_1, \\ 1, & \text{if } x \in X \setminus W_1. \end{cases}$$

Consider $x \in X$ and $a, b \in \mathbf{I}$. It holds that $f(x) < b$ iff there is a rational number $r < b$ such that $x \in W_r$. Hence, $f^{-1}(\,[0,b)\,) = \bigcup\{W_r \mid r < b\}$. It holds that $f(x) > a$ iff there are rational numbers r and r' such that $a < r < r'$ and $x \notin W_{r'}$. Then it follows from (6.1) that $x \notin W^*_r \subseteq W_{r'}$ and $f^{-1}(\,(a,1]\,) = \bigcup\{X \setminus W^*_r \mid r > a\}$. For every $a < b$, define the corresponding open subset of X as follows:

$$U_{a,b} = \Big(\bigvee\{X \setminus W^*_r \mid r > a\}\Big) \wedge \Big(\bigvee\{W_r \mid r < b\}\Big).$$

Then, under the assumption of the current theorem, the following holds for all intervals (a, b), which form a base for the topology of $\mathbf{I}$:

$$\begin{aligned} f^{-1}(\,(a,b)\,) &= \Big(\bigcup\{X \setminus W^*_r \mid r > a\}\Big) \cap \Big(\bigcup\{W_r \mid r < b\}\Big) \\ &\subseteq \Big(\bigvee\{X \setminus W^*_r \mid r > a\}\Big) \cap \Big(\bigvee\{W_r \mid r < b\}\Big) = U_{a,b} \in T \end{aligned}$$

and

$$U_{a,b} \setminus f^{-1}(\,(a,b)\,) \in I_X.$$

Since $\mathbf{I}$ is a crisp space (Def. 3.3), it holds that $f^{-1} I_{\mathbf{I}} = f^{-1}\{\varnothing\} \subseteq I_X$. Then, by Proposition 5.5, the mapping f is g-continuous. Finally, it follows from (6.2) that $f(A) = 0$ and $f(B) = 1$. ☐

7. Normalized spaces. Cardinal invariants

We introduce the notion of *normalized gt-space*. We exploit it as an auxiliary tool to prove some results in this section.

Definition 7.1. Let (X, T) be a gt-space. The operator ${}^N\colon T \to 2^X$ is called the *normalization operator* provided that, for every $U \in T$:

$$U^N = \{x \in U \mid U \wedge V \neq \varnothing \text{ for all } V \in T(x)\}.$$

The family $T^N = \{U^N \mid U \in T\}$ is called the *normalization of* T.

Proposition 7.2. *Given a gt-space* (X, T)*, then the following hold:*

(i) T^N *is a frame, isomorphic to* T*;*
(ii) (X, T^N) *is a gt-space.*

And the following conditions are equivalent:

(iii) $T = T^N$;
(iv) $U \wedge V = \varnothing$ *iff* $U \cap V = \varnothing$ *for all* $U, V \in T$.

Definition 7.3. A gt-space (X, T) is called *normalized* iff $T = T^N$.

Cardinal invariant is a function associating a cardinal number to each space and taking the same value on homeomorphic spaces.

Definition 7.4. Let (X, T) be a gt-space and $\mathcal{B} \subseteq T$ be a base (Def. 5.4). The smallest cardinal number of the form $|\mathcal{B}|$, where $\mathcal{B}$ is a base, is called the *weight* of the given space and is denoted by $w(X, T)$, $w(X)$, or $w(T)$.

Definition 7.5. Let (X, T) be a gt-space. A family $\mathcal{N} \subseteq 2^X$ is called a *network for* T provided that for every open set U there exists a subfamily $\mathcal{N}_0 \subseteq \mathcal{N}$ and $A \in I$ such that $U = A \cup (\bigcup \mathcal{N}_0)$. The smallest cardinal number of the form $|\mathcal{N}|$, where $\mathcal{N}$ is a network, is called the *network weight* of the given space and is denoted by $nw(X, T)$, $nw(X)$, or $nw(T)$.

The statement of the following theorem is the immediate corollary from the fact that every base is a network.

Theorem 7.6. *In a gt-space* (X, T)*, it holds that* $nw(T) \leq w(T)$.

Theorem 7.7. *Let* (X, T) *be a gt-space. Assume that* $nw(T) \leq \mathfrak{m}$. *Then for every family* $\mathcal{U}$ *of of open sets there exists a subfamily* $\mathcal{U}_0 \subseteq \mathcal{U}$ *such that* $|\mathcal{U}_0| \leq \mathfrak{m}$ *and* $\bigvee \mathcal{U}_0 = \bigvee \mathcal{U}$.

Proof. Fix a network $\mathcal{N}$ with $|\mathcal{N}| \leq \mathfrak{m}$. Consider a subfamily $\mathcal{N}_0 \subseteq \mathcal{N}$ with the following property: $M \in \mathcal{N}_0$ iff there exists $U \in \mathcal{U}$ satisfying $M \subseteq U$. Clearly, it holds that $|\mathcal{N}_0| \leq \mathfrak{m}$ and for every $U \in \mathcal{U}$ holds $U \setminus (\bigcup \mathcal{N}_0) \in I$.

Consider a mapping $\varphi\colon \mathcal{N}_0 \to \mathcal{U}$ such that $M \subseteq \varphi(M)$, for every $M \in \mathcal{N}_0$. Denote $\mathcal{U}_0 = \varphi\mathcal{N}_0$. Then it holds that $|\mathcal{U}_0| \leq |\mathcal{N}_0| \leq \mathfrak{m}$ and $\bigcup \mathcal{N}_0 \subseteq \bigvee \mathcal{U}_0$.

Take $U \in \mathcal{U}$. Since $U \setminus (\bigcup \mathcal{N}_0)$ lies in the ideal, it holds that $U \setminus (\bigvee \mathcal{U}_0)$ also lies in the ideal, and, hence, $U \subseteq \bigvee \mathcal{U}_0$. Thus, we proved the inclusion $\bigvee \mathcal{U} \subseteq \bigvee \mathcal{U}_0$. The converse inclusion is obvious. The proof is complete. □

Definition 7.8. Let (T, X) be a gt-space. A family $\mathcal{C} \subseteq T$ is called a *cover of* X iff $\bigvee \mathcal{C} = X$. A subfamily $\mathcal{C}_0 \subseteq \mathcal{C}$ is called a *subcover* iff it is a cover.

The smallest cardinal number $\mathfrak{m}$ such that, for every cover $\mathcal{C}$ there exists a subcover $\mathcal{C}_0$ satisfying $|\mathcal{C}_0| \leq \mathfrak{m}$ is called the *Lindelöf number* of the given space and is denoted as $l(X, T)$, $l(X)$, or $l(T)$.

The following result is the immediate corollary from Theorem 7.7.

Theorem 7.9. *In a gt-spaced* (X, T)*, it holds that* $l(T) \leq nw(T)$.

Lemma 7.10. *In a normalized gt-space* (X, T)*, for every* $\mathcal{U} \subseteq T$ *and every* $V \in T$ *that satisfies* $V \cap (\bigvee \mathcal{U}) \neq \varnothing$ *there exists a nonempty open* $W \subseteq V$ *such that* $W \subseteq \bigcup \mathcal{U}$.

Proof. Under the assumption, $V \cap (\bigvee \mathcal{U}) \neq \varnothing$ implies that $W_0 = V \wedge (\bigvee \mathcal{U})$ is nonempty. Then there exists $U \in \mathcal{U}$ such that $W_0 \wedge U \neq \varnothing$, since $W_0 \subseteq \bigvee \mathcal{U}$. Put $W = W_0 \wedge U$. It holds that $W \subseteq U \subseteq \bigcup \mathcal{U}$. The proof is complete. □

Lemma 7.11. *In a normalized gt-space* (X, T)*, for every finite family* $\mathcal{U} \subseteq T$ *and every* $V \in T$ *that satisfies* $V \cap U_1 \neq \varnothing$ *for some* U_1 *from* $\mathcal{U}$ *there exists a nonempty open* $W \subseteq V$ *such that the intersection of* W *and* $(\bigcap \mathcal{U}) \setminus (\bigwedge \mathcal{U})$ *is empty.*

Proof. Put $\mathcal{U} = \{U_1, U_2, \ldots, U_k\}$, and $A = (\bigcap \mathcal{U}) \setminus (\bigwedge \mathcal{U})$.

(1) $V \cap U_1 \neq \varnothing$ implies $W_1 = V \wedge U_1$ is nonempty.
If $W_1 \cap A = \varnothing$ then put $W = W_1$.

(2) If $W_1 \cap A \neq \varnothing$ then $W_2 = W_1 \wedge U_2$ is nonempty.
If $W_2 \cap A = \varnothing$ then put $W = W_2$.
...

(k) If $W_{k-1} \cap A \neq \varnothing$ then $W_k = W_{k-1} \wedge U_k$ is nonempty.

If the process stops at some $W = W_i$, where $1 \leq i \leq k-1$, then the proof is complete. Otherwise, we obtain a nonempty $W = W_k$ and then $W \subseteq \bigwedge \mathcal{U}$. Hence, $W \cap A = \varnothing$. The proof is complete. □

Proposition 7.12. *In a normalized gt-space* (X, T)*, for every nonempty* $V \in T$ *and every* $A \in I$ *the inclusion* $A \subseteq V$ *implies that there exists a nonempty open* $W \subseteq V$ *such that* $W \cap A = \varnothing$*.*

Proof. Apply Lemma 7.10 and Lemma 7.11 to verify that for every $V \in T$ and $A = A_1 \cup A_2 \cup \cdots \cup A_n \in I$, where every A_i is join- or meet-generated, the inclusion $A \subseteq V$ implies that there exists a nonempty open $W \subseteq V$ such that $W \cap A_1 = \varnothing$. The remainder of the proof is obvious. □

Definition 7.13. Let (X, T) a gt-spaces. A subset $A \subseteq X$ is called *dense* if $A^* = X$ (Def. 4.4). The smallest cardinal number of the form $|A|$, where A is a dense subset of X, is called the *density* of the given gt-space and is denoted by $d(X, T)$, $d(X)$, or $d(T)$.

Proposition 7.14. *In a normalized gt-space* (X, T)*, it holds* $d(T) \leq nw(T)$*.*

Proof. Let $\mathcal{N}$ be a network for X satisfying $nw(T) = |\mathcal{N}|$. Fixing an arbitrary point in every set from $\mathcal{N}$, we construct the set A. Then it holds that $|A| \leq |\mathcal{N}| = nw(T)$. On the other hand, for every $x \in X$ and every $V \in T(x)$, if $V \cap A \in I$ then, by Proposition 7.12, there exists an open subset W such that $W \cap A = \varnothing$. But the latter equality is impossible, since A has a nonempty intersection with every member of the network. Therefore, we conclude that A is dense, and hence $d(X) \leq |A| \leq |\mathcal{N}| = nw(T)$. The proof is complete. □

Definition 7.15. In a gt-space (X, T), the least cardinal number $\mathfrak{m}$ such that $|\mathcal{S}| \leq \mathfrak{m}$ holds for every family of nonempty open subsets $\mathcal{S}$ with the property that $U \wedge V = \varnothing$ holds for all $U, V \in \mathcal{S}$ is called the *Suslin number* and is denoted by $c(X, T)$, $c(X)$, or $c(T)$.

Proposition 7.16. *In a normalized gt-pace* (X, T)*, it holds that* $c(T) \leq d(T)$*.*

Proof. Consider $A \subseteq X$ such that $|A| = d(T)$, and a family of nonempty open subsets $\mathcal{S}$ with the property that $U \wedge V = \varnothing$ holds for all $U, V \in \mathcal{S}$. Assume that $|A| < |\mathcal{S}|$. Then there exists a nonempty $U \in \mathcal{S}$ such that $A \cap U = \varnothing$. The latter statement is a contradiction, since A is dense. Hence, we conclude that $|\mathcal{S}| \leq |A|$, and this holds for all $\mathcal{S} \subseteq T$ with the mentioned property. Then $c(T) \leq d(T)$. □

Proposition 7.17. *Let* (X, T) *be a gt-space and* $(X, \tilde{T})$ *be its normalization (Def. 7.1). Then the following hold:*

(i) $c(T) = c(\tilde{T})$;
(ii) $d(T) = d(\tilde{T})$;
(iii) $nw(T) = nw(\tilde{T})$;
(iv) $w(T) = w(\tilde{T})$;
(v) $l(T) = l(\tilde{T})$.

Proof. We omit the proof, since it is a long but rather easy exercise. □

The following result is the natural corollary from Propositions 7.14, 7.16, and 7.17.

Theorem 7.18. *In a gt-space* (X, T)*, it holds* $c(T) \leq d(T) \leq nw(T)$.

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