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Some results and examples concerning Whyburn spaces

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Abstract

We prove some cardinal inequalities valid in the classes of Whyburn and hereditarily weakly Whyburn spaces and we construct examples of non-Whyburn and non-weakly Whyburn spaces to illustrate that some previously known results cannot be generalized.

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1. Introduction

A Hausdorff space X is said to be *Whyburn* if whenever $A \subseteq X$ is not closed and $x \in \text{cl}(A) \setminus A$, there is $B \subseteq A$ such that $\text{cl}(B) \setminus A = \{x\}$. The space is *weakly Whyburn* if whenever $A \subseteq X$ is not closed, there is $B \subseteq A$ such that $|\text{cl}(B) \setminus A| = 1$. These classes of spaces have been studied previously in [1], [4] and also earlier in [7] and elsewhere under the names AP-spaces, and WAP-spaces. If $A \subseteq X$, then the *Whyburn closure* of A, denoted by $\text{wcl}(A)$ is defined as

$$A \cup \bigcup \{\text{cl}(C) : C \subseteq A,\ |\text{cl}(C) \setminus A| = 1\}.$$

It follows immediately that a space is weakly Whyburn if and only if every Whyburn closed set is closed. Undefined terminology can be found in [2] or [5] and all spaces are assumed to be (at least) Hausdorff.

2. Whyburn and weakly Whyburn spaces

In [4], a pseudocompact Whyburn space which is not Fréchet was constructed and Proposition 2.1 of [7] states that a weakly Whyburn compact Hausdorff space must have a non-trivial convergent sequence. It is easy to see that this latter result generalizes to countably compact Hausdorff spaces and also to feebly compact spaces with an infinite set of isolated points (recall that a space is *feebly compact* if every locally finite family of non-empty open sets is finite). The question then arises whether this result is true for all pseudocompact or feebly compact spaces. To answer this question we need the following terminology.

If Y is a non-empty scattered space, then we set

$Y_0 = \{x : \{x\} \text{ is open}\}$ and for each ordinal α,

$Y_\alpha = \{x : \{x\} \text{ is open in } Y \setminus \bigcup\{Y_\beta : \beta < \alpha\}\}$.

The *dispersion order of* Y is then the least ordinal for which $Y_\alpha = \varnothing$. For the sequel, we note that for each $n \in \omega$, the dispersion order of the countable ordinal $\omega^n + 1$ is $n + 1$.

We also need two lemmas, the simple proof of the first of which we omit.

Lemma 2.1. *For $n \in \omega$, an infinite scattered subset A of a T_1-space X has dispersion order at most n if and only if it is the union of n discrete subspaces.*

Lemma 2.2. *If Y is a scattered metric space of finite dispersion order $n+1$, where $n \geq 1$, and $x \in Y_n$, then for any $\epsilon > 0$, there is an embedding $h : \omega^n + 1 \to Y$ such that $h(\omega^n) = x$ and $\operatorname{diam}(h[\omega^n + 1]) \leq \epsilon$.*

Proof. The proof is by induction on the dispersion order of Y. If $n = 1$, then each point $x \in Y_1$ is the limit of a sequence S in Y_0; S can be taken to have arbitrarily small diameter and $S \cup \{x\}$ is homeomorphic to $\omega + 1$.

Suppose now that the result is true for each $n < k$ and let Y be a scattered space of dispersion order $k + 1$. Suppose that $x \in Y_k$ and $\epsilon > 0$; pick a sequence $\langle x_m \rangle = S \subseteq Y_{k-1}$ converging to x such that $\operatorname{diam}(S) < \epsilon/2$. Since Y is hereditarily collectionwise normal, we may find mutually disjoint open sets U_m such that $x_m \in U_m$; each set U_m is scattered and has dispersion order k. Applying the inductive hypothesis, for each $m \in \omega$, we may find an embedding $h_m : \omega^{k-1} + 1 \to U_m$ such that $h_m(\omega^{k-1}) = x_m$ and such that $\operatorname{diam}(T_m) < \epsilon/4^m$, where $T_m = h[\omega^{k-1} + 1]$. Let $T = \bigcup\{T_m : m \in \omega\} \cup \{x\}$; it is straightforward to check that T is homeomorphic to $\omega^k + 1$ since each neighbourhood of x contains all but finitely many of the sets T_m; furthermore, $\operatorname{diam}(T) < \epsilon/2 + \epsilon/4 + \epsilon/16 < \epsilon$. □

Example 2.3. There is a Whyburn H-closed (hence feebly compact) Hausdorff space with no non-trivial convergent sequences.

Proof. We consider the space $X = [0, 1]$ with the usual metric topology μ. Let τ be the topology on X generated by

$$\mu \cup \{X \setminus D : D \subseteq X \text{ is } \mu\text{−discrete}\}.$$

Since $\{X \setminus D : D \subseteq X \text{ is } \mu\text{−discrete}\}$ is a filter of dense subsets of (X, μ) it follows that (X, τ) is H-closed. Furthermore, it is clear that (X, τ) is Hausdorff and has no convergent non-trivial sequences. Even more is true: It follows from Lemma 2.1 that every scattered subspace of (X, μ) of finite dispersion order is closed in the topology τ. We will show that (X, τ) is a Whyburn space. To this end, suppose that $A \subseteq X$ is not closed and let $x \in \mathrm{cl}_\tau(A) \setminus A$. Now in (X, μ), A is the union of a scattered subset $C \subseteq A$ and a dense-in-itself subset $B \subseteq A$, hence either (i) $x \in \mathrm{cl}_\tau(B)$ or (ii) $x \in \mathrm{cl}_\tau(C)$. We consider the cases separately.

(i) Since B is dense-in-itself, every non-empty open subset of B contains a dense subset homeomorphic to the rationals, $\mathbb{Q}$. Choose a nested local base at x of μ-closed sets $\mathcal{V} = \{V_n : n \in \omega\}$; we may assume that $V_{n+1} \subseteq \mathrm{int}(V_n)$ and $B \cap (\mathrm{int}(V_n) \setminus V_{n+1}) \neq \varnothing$ for each $n \in \omega$. Since $\mathbb{Q}$ is universal for countable metric spaces, for each $n \in \omega$, in the open subset $B \cap (\mathrm{int}(V_n) \setminus V_{n+1})$ of B we may find a subspace D_n homeomorphic to the compact ordinal $\omega^n + 1$ which has dispersion order $n + 1$; let $D = \bigcup\{D_n : n \in \omega\}$. It is easy to see that D is scattered and has dispersion order ω and since $x \in \mathrm{cl}_\mu(D)$ a straightforward argument shows that $\mathrm{cl}_\tau(D) \setminus D = \{x\}$.

(ii) Choose a nested local base at x of μ-closed sets $\mathcal{V} = \{V_n : n \in \omega\}$. If $x \in \mathrm{cl}_\tau(C)$, since each scattered subspace of (X, μ) of finite dispersion order is τ-closed, it follows that for each $n \in \omega$, $C \cap V_n$ has (countably) infinite dispersion order κ and since every countable limit ordinal has cofinality ω, we may assume without loss of generality that $\kappa = \omega$. Then, for each $n \in \omega$, using the previous lemma we may find embeddings $h_n : \omega^n + 1 \to V_n \cap C$ and it is not hard to see that the maps h_n may be chosen so that if $m \neq n$, then $T_n \cap T_m = \varnothing$, where $T_k = h_k[\omega^k + 1]$. Each of the sets T_k is μ-compact and τ-discrete but $T = \bigcup\{T_k : k \in \omega\}$ has infinite dispersion order and so $x \in \mathrm{cl}_\mu(T)$. Furthermore, since for each μ-neighbourhood V of x, the set $T \setminus V$ is τ-closed, it follows that $\mathrm{cl}(T) \setminus C = \{x\}$. □

In the sequel $d(X)$, $L(X)$, $t(X)$ and $\psi(X)$ will denote respectively the density, tightness, Lindelöf number and pseudocharacter of a space (X, τ) and $\psi(x, X)$ will denote the pseudocharacter of x in X. If (X, τ) is a Hausdorff space and $x \in X$, then let $\psi_c(x, X) = \min\{|\mathcal{U}| : \{x\} = \bigcap\{\mathrm{cl}(U) : x \in U \in \mathcal{U} \subseteq \tau\}\}$.

Theorem 2.4. *A k-space is weakly Whyburn if and only if for each non-closed set $A \subseteq X$, there is some compact set $K \subseteq X$ and $x \notin A$ such that* $\mathrm{cl}(K \cap A) = (K \cap A) \cup \{x\} = K$.

Proof. The sufficiency is clear, since $K \cap A$ is not closed in K. For the necessity, suppose that (X, τ) is a Hausdorff weakly Whyburn k-space and that $A \subseteq X$ is not closed in X. Then there is some compact set $C \subseteq X$ such that $C \cap A$ is not closed in C. Since C is a closed subset of X, C is weakly Whyburn and hence there is some $x \in C \setminus A$ and a set $B \subseteq C \cap A$ such that $\mathrm{cl}(B) \setminus A = \{x\}$. Clearly $\mathrm{cl}(B)$ is the required compact subset of X. □

Corollary 2.5. *A weakly Whyburn k-space is pseudoradial.*

Proof. This is an immediate consequence of the previous lemma and the fact that a compact weakly Whyburn space is pseudoradial (see [7]). □

The next result extends Theorem 3 of [1] to the class of Hausdorff spaces.

Theorem 2.6. *If X is a weakly Whyburn Lindelöf P-space and for each $x \in X$, $\psi(x, X) < \aleph_\omega$, then X is pseudoradial.*

Proof. For any Hausdorff space $\psi_c(x, X) \leq L(X)\psi(x, X)$ (see 2.8(c) of [3]) and hence $\psi_c(x, X) < \aleph_\omega$ for each $x \in X$. Let $A \subseteq X$ be a non-closed set and $B \subseteq A$ such that $\mathrm{cl}(B) \setminus A = \{x\}$ for some $x \in X$. Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be a family of minimal cardinality κ of open sets in $\mathrm{cl}(B)$ such that $\bigcap\{\mathrm{cl}(U_\alpha) : \alpha < \kappa\} = \{x\}$. Since X is a P-space and x is not isolated, κ is a regular uncountable cardinal. Since κ is minimal, for each $\alpha \in \kappa$ we may choose $x_\alpha \in \bigcap\{\mathrm{cl}(U_\beta) : \beta < \alpha\} \setminus \{x\} \subseteq A$. Since $\mathrm{cl}(B)$ is Lindelöf, the set so constructed $\{x_\alpha : \alpha < \kappa\}$ must have a complete accumulation point $z \in \mathrm{cl}(B)$. Since $\bigcap\{\mathrm{cl}(U_\alpha) : \alpha < \kappa\} = \{x\}$ and the well-ordered net $S = \langle x_\alpha \rangle_{\alpha \in \kappa}$ is finally in each set $\mathrm{cl}(U_\beta)$ it follows that $z = x$ and x is the unique complete accumulation point of S. Furthermore, $S = \langle x_\alpha \rangle_{\alpha \in \kappa}$ must converge to x, for otherwise there would exist a subset of S of size κ with no complete accumulation point. □

Theorem 2.7. *The product of two Whyburn spaces, one of which is a k-space and the other is locally compact is weakly Whyburn.*

Proof. Suppose that X is a Whyburn k-space and Y is a Whyburn locally compact space. It is known (see [7]) that a compact Whyburn Hausdorff space is Fréchet-Urysohn and it is easy to see that the same is true of a Whyburn Hausdorff k-space. It then follows from 3.3.J of [2] that $X \times Y$ is sequential and hence weakly Whyburn. □

Question 2.8. Is the product of two Whyburn k-spaces, weakly Whyburn?

Theorem 2.9. *If X is weakly Whyburn, then $|X| \leq d(X)^{t(X)}$.*

Proof. If X is finite, the result is trivial; thus we assume that X is infinite. Suppose that $d(X) = \delta$, $t(X) = \kappa$ and $D \subsetneq X$ is a dense (proper) subset of cardinality δ. Let $D = D_0$ and define recursively an ascending chain of subspaces $\{D_\alpha : \alpha \leq \kappa^+\}$ as follows:

Since X is weakly Whyburn, there is some $x \in X \setminus D$ and $B_x \subseteq D$ such that $\mathrm{cl}(B_x) \setminus D = \{x\}$; clearly, we have $|\mathrm{cl}(B_x)| \leq \delta \leq \delta^\kappa$ and we may assume that $|B_x| \leq \kappa$.

We then define

$$D_1 = \bigcup\{\mathrm{cl}(B) : B \subseteq D_0, |B| \leq \kappa, |\mathrm{cl}(B) \setminus D_0| = 1\}.$$

Clearly $D_1 \supsetneq D_0$ and since there are at most δ^κ such sets B it follows that $|D_1| \leq \delta^\kappa$.

Suppose now that for each $\beta < \alpha \leq \kappa^+$ we have defined dense sets D_β such that $|D_\beta| \leq \delta^\kappa$ and $D_\gamma \subseteq D_\lambda$ whenever $\gamma < \lambda < \alpha$. If α is a limit ordinal, then define $D_\alpha = \bigcup\{D_\beta : \beta < \alpha\}$ and then $|D_\alpha| \leq |\alpha|.\delta^\kappa \leq \kappa^+.\delta^\kappa = \delta^\kappa$. If on the other hand $\alpha = \beta + 1$, and $D_\beta \subsetneq X$, then since X is weakly Whyburn there is some $x \in X \setminus D_\beta$ and $B_x \subseteq D_\beta$ such that $\mathrm{cl}(B_x) \setminus D_\beta = \{x\}$. Again we have that $|\mathrm{cl}(B_x)| \le \delta^\kappa$ and we may assume that $|B_x| \le \kappa$. Now we may define

$$D_\alpha = \bigcup\{\mathrm{cl}(B) : B \subseteq D_\beta, |B| \le \kappa, |\mathrm{cl}(B) \setminus D_\beta| = 1\}.$$

Clearly $D_\alpha \supsetneq D_\beta$ and since there are at most $(\delta^\kappa)^\kappa$ such sets B it follows that $|D_\alpha| \le \delta^\kappa$.

To complete the proof it suffices to show that for some $\alpha \le \kappa^+$, we have that $D_\alpha = X$. Suppose to the contrary that $\Delta = \bigcup\{D_\alpha : \alpha < \kappa^+\} \neq X$; $|\Delta| \le \kappa^+.\delta^\kappa = \delta^\kappa$. Then, since X is weakly Whyburn and has tightness κ, there is some $z \in X \setminus \Delta$ and some set $B \subseteq \Delta$ of cardinality at most κ, such that $\mathrm{cl}(B) \setminus \Delta = \{z\}$. Since the sets $\{D_\alpha : \alpha < \kappa^+\}$ form an ascending chain and $\mathrm{cf}(\kappa^+) > \kappa$, it follows that for some $\gamma < \kappa^+$, $B \subseteq \bigcup\{D_\alpha : \alpha < \gamma\}$ and hence $z \in D_{\gamma+1}$, a contradiction. □

Lemma 2.10. *If X is hereditarily weakly Whyburn, then $|X| \le 2^{d(X)}$.*

Proof. Suppose to the contrary that $|X| > 2^{d(X)}$. Let Δ be a dense subset of X of minimal cardinality, $\mathcal{A} = \{A \subseteq \Delta : |\mathrm{cl}_X(A)| \le 2^{d(X)}\}$ and $Y = \bigcup\{\mathrm{cl}_X(A) : A \in \mathcal{A}\}$. Since $|\mathcal{P}(\Delta)| = 2^{d(X)}$, it follows that $|Y| \le 2^{d(X)}$ and hence if we put $Z = \Delta \cup (X \setminus Y)$, then $|Z| > 2^{d(X)}$. Now if $B \in \mathcal{P}(\Delta) \setminus \mathcal{A}$, then $|\mathrm{cl}_X(B) \cap Z| > 2^{d(X)}$ thus showing that Δ is Whyburn closed in Z but not closed. Thus Z is not weakly Whyburn and hence X is not hereditarily weakly Whyburn. □

3. The Whyburn property in scattered and submaximal spaces

We recall our convention that all spaces are Hausdorff. A space is said to be *submaximal* if every dense subset is open. A standard procedure for constructing submaximal topologies is as follows. Suppose that (X, τ) is a (Hausdorff) space and $\mathcal{D}$ is a maximal filter in the family of dense subsets of X. Then the topology σ generated by the subbase $\tau \cup \mathcal{D}$ is submaximal and is called a *submaximalization* of τ. Note that σ is semiregular if and only if τ is semiregular and submaximal (then $\sigma = \tau$). Obviously, a scattered space is submaximal if and only if it has dispersion order 2.

As we mentioned earlier, every regular scattered space is weakly Whyburn and the Katětov extension of ω (see 4.8(n) of [5]) shows that this is not true in the class of Urysohn spaces. Thus it is natural to ask the following two questions.

(1) Must a dense-in-itself submaximal Whyburn space be regular?, and
(2) Is every scattered semiregular space Whyburn?

We give a partial answer to the first question by showing that a submaximalization of a resolvable space is never Whyburn and answer the second by constructing a semiregular scattered space of dispersion order 2 which is not weakly Whyburn.

Recall that a space is *resolvable* if it possesses two mutually disjoint dense subsets.

Theorem 3.1. *A submaximalization of a resolvable Hausdorff space is not weakly Whyburn.*

Proof. Suppose that (X, τ) is a resolvable T_2-space and $\mathcal{F}$ is a maximal filter of dense sets in X. We first show that there is $F \in \mathcal{F}$ such that $X \setminus F$ is somewhere dense in X. To this end, suppose to the contrary that no such F exists, then for each $F \in \mathcal{F}$, $U_F = X \setminus F$ is nowhere dense. Now let D and D' be complementary dense subsets of X; clearly $D, D' \notin \mathcal{F}$. For each $F \in \mathcal{F}$, since $\text{int}(F) = X \setminus \text{cl}(X \setminus F)$, it follows that $\text{int}(F)$ is dense in X and so too are $D \cap \text{int}(F) \subseteq D \cap F$ and $D' \cap \text{int}(F)) \subseteq D' \cap F$. Since $\mathcal{F}$ is maximal, any dense set which meets each element of $\mathcal{F}$ in a dense set is an element of $\mathcal{F}$ and so it follows that $D \in \mathcal{F}$ and $D' \in \mathcal{F}$ contradicting the fact that $\mathcal{F}$ is a filter.

Now let σ be the topology generated by $\tau \cup \mathcal{F}$ and $F \in \mathcal{F}$ be such that $X \setminus F$ is somewhere dense; thus $\text{int}_\sigma(\text{cl}_\sigma(X \setminus F)) = U \neq \varnothing$. Let $V = U \cap F$, $x \in \text{cl}_\sigma(V) \setminus F$ and note that V is infinite. Then if $B \subseteq V$ is such that $x \in \text{cl}_\sigma(B)$, it follows that $W = \text{int}_\sigma(B) \neq \varnothing$. But then, $\text{cl}_\sigma(W) \cap (X \setminus F) = \text{cl}_\tau(W) \cap (X \setminus F) = \text{cl}_\tau(\text{cl}_\tau(W) \cap (X \setminus F)) \cap (X \setminus F)$ which is infinite. □

An example of a scattered submaximal Whyburn (even first countable) space which is not regular (nor even semiregular) is easy to construct. Let $\mathbb{Q}$ denote the rational numbers and $X = \mathbb{Q} \times \{0, 1\}$ with the following topology:

Each point of $\mathbb{Q} \times \{0\}$ is isolated and a basic open neighbourhood of $(q, 1)$ is of the form $\{(q, 1)\} \cup [(U_q \setminus \{q\}) \times \{0\}]$ where U_q is a Euclidean neighbourhood of $q \in \mathbb{Q}$.

A space X is said to be *ω-resolvable* if X possesses infinitely many mutually disjoint dense subsets. The construction of the next example depends on the existence of a countable ω-resolvable Hausdorff space which is not weakly Whyburn. Before constructing such a space, the following lemma is needed

Lemma 3.2. *A space X, $\omega \subseteq X \subseteq \beta\omega$ is hereditarily weakly Whyburn if and only if X is scattered.*

Proof. The sufficiency is clear since a subspace of a scattered space is scattered and it was proved in [4] that a regular scattered space is weakly Whyburn. Furthermore, it is easy to see that if the dispersion order of X is 2, then it is Whyburn also.

For the inverse implication, suppose that $D \subseteq X \setminus \omega$ is dense in itself and let $Y = \omega \cup D$; if Y were weakly Whyburn, then we could find $p \in D$ and $B \subseteq \omega$ such that $\text{cl}_Y(B) \setminus \omega = \{p\}$, in other words, $\text{cl}_Y(B) = B \cup \{p\}$. However, $\text{cl}_Y(B) = \text{cl}_{\beta\omega}(B) \cap Y$ and so $\text{cl}_Y(B) \cap D$ is an open subset of D to which p belongs; since p is not isolated, this set must be infinite. □

By way of contrast to the last result we note that under CH the subspace of P-points of $\beta\omega \setminus \omega$ has character ω_1 and it then follows from Proposition 2.7 of [4] that this space is Whyburn.

Consider a countable dense-in-itself subset $D \subseteq \mathrm{cl}_{\beta\mathbb{Q}}(\mathbb{N}) \setminus \mathbb{N} \subseteq \beta\mathbb{Q} \setminus \mathbb{Q}$ (where once again, $\mathbb{Q}$ denotes the set of rational numbers with the Euclidean topology). Let $X = \mathbb{Q} \cup D$; X is a countable Tychonoff space which, is clearly ω-resolvable. That $\mathbb{N}$ is Whyburn closed in X follows from the previous lemma and the fact that $\mathrm{cl}_{\beta\mathbb{Q}}(\mathbb{N})$ is homeomorphic to $\beta\omega$.

Example 3.3. There is a semiregular scattered space (of dispersion order 2) which is not weakly Whyburn.

Proof. Let (Z, σ) be an ω-resolvable (dense-in-itself) countable Tychonoff space which is not weakly Whyburn and let $\mathcal{F}$ be an infinite family of mutually disjoint dense subsets of (Z, σ) and $\phi : Z \to \mathcal{F}$ a bijection. Let $X = Z \times \{0, 1\}$ and for each $z \in Z$, let $\mathcal{V}_z$ be an open neighbourhood base at z. We define a topology τ on $X = Z \times \{0, 1\}$ as follows:

Each point of $Z \times \{0\}$ is isolated and an open neighbourhood of $(z, 1)$ is of the form

$$W_{V,z} = \{(z, 1)\} \cup (V \times \{0\}) \setminus (\{(z, 0)\} \cup \phi(z)), \text{ where } V \in \mathcal{V}_z.$$

The space (X, τ) is a scattered space of dispersion order 2 and we proceed to show that it is neither regular nor weakly Whyburn.

It is easy to see that X is not regular since the open neighbourhood $W_{V,z}$ of $(z, 1)$ contains no closed neighbourhood of that point. To prove that X is semiregular, it suffices to show that each of the sets $W_{V,z}$ is regular open. To see this, suppose that $(t, 1) \in \mathrm{cl}_X(W_{V,z})$ where $t \neq z$; then since $\phi(z)$ is dense in Z, each neighbourhood of $(t, 1)$ meets the set $\phi(z) \times \{0\}$ showing that $(t, 1) \notin \mathrm{int}_X(\mathrm{cl}(W_{V,z}))$.

Finally, to show that (X, τ) is not Whyburn, it suffices to prove that there is some $A \subseteq Z$ such that $A \times \{0\}$ is Whyburn closed but not closed in X. However, Z is not weakly Whyburn and hence there is some $A \subseteq Z$ which is Whyburn closed but not closed in Z and so if $B \subseteq A$ is such that $\mathrm{cl}_Z(B) \setminus A$ is nonempty, we must have $\mathrm{cl}_Z(B) \setminus A$ has no isolated points (and hence is infinite). We claim that if $B \subseteq A$ is such that $\mathrm{cl}_Z(B) \setminus A$ is nonempty then $\mathrm{cl}_X(B \times \{0\}) \setminus (A \times \{0\})$ is infinite. To prove our claim, suppose that $s \in \mathrm{cl}_Z(B) \setminus A$; then either $(s, 1) \in \mathrm{cl}_X(B \times \{0\}) \setminus (A \times \{0\})$ or not. If $(s, 1) \notin \mathrm{cl}_X(B \times \{0\}) \setminus (A \times \{0\})$ then there is some open neighbourhood U of s in Z such that $\mathrm{cl}_Z(U) \cap B \subseteq \phi(s)$ and U contains infinitely many points of $\mathrm{cl}_Z(B) \setminus A$. If $s \neq t \in U \cap (\mathrm{cl}_Z(B) \setminus A)$, then since $B \cap U \not\subseteq \phi(t)$, it follows that $t \in \mathrm{cl}_X(B \times \{0\}) \setminus (A \times \{0\})$, showing that $\mathrm{cl}_X(B \times \{0\}) \setminus (A \times \{0\})$ is infinite. □

4. Some open questions

The space constructed in Example 2.3 is not regular, thus we are led to ask:

Question 4.1. Does every (weakly) Whyburn pseudocompact Tychonoff space have a convergent sequence?

A number of dense pseudocompact subspaces of $\{0,1\}^{\mathfrak{c}}$ and $\mathbb{I}^{\mathfrak{c}}$ have been constructed which do not possess a non-trivial convergent sequence (for example see [6]); however, the question of whether such constructions can produce a weakly Whyburn space has apparently not been studied.

Question 4.2. Is the bound $\psi(x, X) < \aleph_\omega$ necessary in Theorem 2.6?

Question 4.3. Suppose that $|X| > 2^{d(X)}$; can X be weakly Whyburn?

Question 4.4. Does there exist in ZFC a dense Whyburn subspace of $\beta\omega \setminus \omega$?

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