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Random selection of Borel sets II

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Abstract

The theory of random Borel sets as presented in part I of this paper is developed further. Special attention is payed to the reconstruction of the topology of the underlying space from our presentation of the measure algebra, to an analysis of capacities in context of random Borel sets, to inspection processes on the unit segment and to the Markov process of random allocation.

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1. Introduction

We present four topics from the theory of random Borel sets as laid out in the first part of this paper:

1. It is well known [5, thm.13.1] that the topology of a Polish space can be modified such that an arbitrarily prescribed Borel subset is turned into a closed and open subset without modifying the Borel algebra as a whole, and in consequence it is not possible to characterize open subsets or closed subsets in the standard Borel algebra by *invariant* properties. In spite of this deficiency, we are going to demonstrate in section 2 that our "dyadic" algebra Y [2, §5] provides us with a special representation of the standard Borel algebra that easily distinguishes open sets from general Borel sets. This discrepancy should not come as a surprise, because Y has built in G_∞-invariance [2, §3] but fails the invariance under the larger group consisting of *all* measure preserving homeomorphisms [2, §9], characteristic for the standard measure algebra. Evidently, Y has a more rigid structure, and the isomorphism [2, §6] *forgets* certain properties.

2. In section 3 we investigate the possibility to apply Choquet capacities in context of random Borel sets in a manner similar to the closed case. This turns out to be infeasible. However, a simple description of random Borel sets in terms of stochastic processes can be given, leading to an elegant and abstract characterization. For practical and numerical purposes the "dyadic algebra" is superior, as will be seen for instance in section 5.

3. In [8] Straka and Štěpán used inspection processes to deal with random Borel sets on the unit segment, but they seem to have overlooked the fact that not every inspection process corresponds to a random set. The problem will be dealt with in section 4.

4. The combinatorial concept of random allocation describes the probabilistic allotment of objects to certain places and the observation of the time evolution of their distribution [6]. In each round one may allocate a single object, or a fixed finite number of objects, or an infinite set. An example for the second case consists of the numbered lots drawn in each round in a lottery drawing, where one may observe which numbers appeared at least once after r rounds. Thus in the i-th run we select a finite random set B_i consisting of the numbers drawn, and observe the growth of the accumulated set $C_r := \bigcup_{i=1}^{r} B_i$ over the Markov time r. If the lottery is fair, the random variable B_i assumes each set containing the correct number of elements with equal probability. In more general examples, one may relax the assumption that the sets B_i have a predetermined size but still require all sets of the same size to have the same probability.

We may play the same game using infinite sets for B_i, in our case Borel sets. Having equal probability for all sets of the same measure would require location independence of the distribution of B_i, but we know from [2, §9] that the best we can achieve is G_∞-invariance. In section 5 we compute expected value and second moment of the size of the accumulated sets C_r in a numerically feasible way.

This paper uses the same terms and notations as part I; in particular X denotes a compact metric space carrying a non atomic measure μ with $\mathrm{Supp}(\mu) = X$; $Y(\mu)$ denotes the Borel algebra of μ-measurable sets, identifying two sets if they differ only in a 0-set. Y denotes the "coordinate representation" of $Y(\mu)$ introduced in [2, §5].

2. Reconstruction of the topology

The isomorphism between the algebras Y and $Y(\mu)$ was proved using resolutions of either of two species, labelled type I and type II [2, Def.4.4], and it was already observed that the type I resolutions establish a close link between measure and topology. This is illustrated by the following proposition:

Proposition 2.1. *If a type I resolution is used for the isomorphism theorem* [2, Thm.6.1]*, then:*
i) The Borel set corresponding to a sequence (x_{nm}) can be chosen as open *if and only if $x_{nm} = \sup_{N\geq n} 2^{n-N} \# \left\{k : 2^{N-n}m \leq k < 2^{N-n+1}, x_{Nk} = 1\right\}$ for*

all n, m.
ii) The Borel set corresponding to a sequence (x_{nm}) *can be chosen as* closed *if and only if* $x_{nm} = \inf_{N \geq n} 2^{n-N} \# \left\{k : 2^{N-n}m \leq k < 2^{N-n+1}, x_{Nk} > 0\right\}$ *for all* n, m.
iii) A Borel set is Jordan measurable (i.e. its boundary is a 0*-set* [3, Ch.3]*) if and only if it satisfies both (i) and (ii).*

Proof. To prove (i) let us assume $x_{nm} = \sup_{N \geq n} 2^{n-N} \# \left\{k : 2^{N-n}m \leq k < 2^{N-n+1}, x_{Nk} = 1\right\}$ for all n, m and set $B_n := \left(\bigcup_{m: x_{nm}=1} A_{nm}\right)^{\circ}$. We observe $B_{n+1} \supseteq B_n$ because $x_{nm} = 1 \Rightarrow x_{n+1,2m} = x_{n+1,2m+1} = 1$ and consider the open set $B := \bigcup_n B_n$. Therefore $\mu\left(A_{nm} \cap B\right) = \lim_{N\to\infty} \mu\left(A_{nm} \cap B_N\right) = \lim_{N\to\infty} \sum_{k: x_{NK}=1, 2^{N-n}m \leq k < 2^{N-n+1}m} \mu\left(A_{NK}\right) = \lim_{N\to\infty} 2^{-N} \# \{k : x_{NK} = 1, 2^{N-n}m \leq k < 2^{N-n+1}m\} = 2^n x_{nm}$ hence the sequence (x_{nm}) corresponds to the open set B under our isomorphism theorem (if necessary we replace μ by $\frac{1}{\varphi}\mu$).

Conversely, if B is open we can use inner regularity to find a compact subset $K \subseteq A_{nm} \cap B$ whose measure is as close to that of $A_{nm} \cap B$ as we please and then choose $N \geq n$ such that $\max_k \operatorname{diam} A_{Nk}$ is a Lebesgue number for the open covering consisting of B and $\complement K$. The limit condition follows.

(ii) is dual to (i) and does not require separate proof. Sufficiency of (iii) follows from (i) and (ii). It remains to establish necessity, so let us consider a sequence (x_{nm}) satisfying both (i) and (ii). We construct the sets $C_n := \left(\bigcup_{m: x_{nm}=0} A_{nm}\right)^{\circ}$ and $C := \bigcup_n C_n$, and just like above we see that C is an open set representing the complement of B, i.e. $\mu\left(B \triangle \complement C\right) = 0$. But by construction $B \cap C = \varnothing$ and therefore $\partial \overline{B} \subseteq \complement\left(B \cup C\right) = B \triangle \complement C$ must be a 0-set. □

Remark 2.2. This means that (x_{nm}) represents an open set if and only if $\lim_{N\to\infty} 2^{n-N} \sum_{\substack{2^{N-n}m \leq k < 2^{N-n+1}m \\ 0 < x_{Nk} < 1}} x_{Nk} = 0$ for each n, m and a closed set if and only if $\lim_{N\to\infty} 2^{n-N} \sum_{\substack{2^{N-n}m \leq k < 2^{N-n+1}m \\ 0 < x_{Nk} < 1}} \left(1 - x_{Nk}\right) = 0$ for each n, m.

Corollary 2.3. *If the measures* ν_N *in* [2, Thm.7.1] *satisfy* $\nu_N\left(\partial I^{2^N}\right) = 0$ *(e.g.* [2, Ex.7.4] *and* [2, Ex.7.5]*), then open sets and closed set have probability* 0*. In contrast, the "Sierpiński" example* [2, §10] *corresponds to Jordan measurable sets.*

Corollary 2.4. *For any* $0 < \vartheta < 1$ *the exists a dense subset of* X *with empty interior and measure arbitrarily close to* ϑ*.*

Proof. We claim that the Borel A set corresponding to the element $(x_{nm})_{0 \leq m < 2^n, n \in \mathbb{N}_0} \in Y$ with $x_{nm} = \vartheta$ for all m, n has the required properties. Its measure can be assumed as close to $\vartheta = x_{00} = \int\limits_A \frac{1}{\varphi g} d\mu$ as we please (cf. [2, Thm.6.1]). If the element $(x'_{nm})_{0 \leq m < 2^n, n \in \mathbb{N}_0} \in Y$ represents a *closed* set A' containing A, then $x'_{nm} \geq x_{nm} > 0$ for all m, n, and remark 2.2 implies $0 = \lim_{N\to\infty} 2^{n-N} \sum_{2^{N-n}m \leq k < 2^{N-n+1}m} \left(1 - x'_{Nk}\right) = 1 - x'_{nm}$ for all

m, n, in particular $\mu(A') = 1$ since $x'_{00} = 1$. Hence $\complement A'$ is an open subset of X with measure 0, and since $\mathrm{Supp}(\mu) = X$ we must have $A' = X$, i.e. A is dense. Similarly, if the element $(x''_{nm})_{0 \le m < 2^n, n \in \mathbb{N}_0} \in Y$ represents an *open* subset A'' contained in A, then $x''_{nm} \le x_{nm} < 1$ and remark 2.2 implies $0 = \lim_{N\to\infty} 2^{n-N} \sum_{2^{N-n}m \le k < 2^{N-n+1}m} x''_{Nk} = x''_{nm}$, in particular $\mu(A'') = 0$ since $x''_{00} = 0$. Since A'' is open it must be empty and A has empty interior. □

3. The capacity problem

Our construction of probability measures for random Borel sets [2, Thm.7.1] requires resolutions and thus utilizes a particular "coordinate representation" of the standard Borel algebra. In contrast, random *closed* sets can be defined by capacities in a coordinate independent way. A coordinate free presentation can be given in the Borel setting too, but rather different from the closed case. We start by preparing some tools.

3.1. Finite partitions.

Definition 3.1. A finite partition $\mathcal{B} = \{B_1, \dots B_n\}$ of X is a finite set of Borel subsets $B_i \subseteq X$ such that

(1) $\forall i : \mu(B_i) > 0$
(2) $\forall i \ne j : \mu(B_i \cap B_i) = 0$
(3) $\sum_i \mu(B_i) = 1$.

We set $\mathrm{mesh}\,\mathcal{B} := \max_i \mathrm{diam}\, B_i$.

We observe that the mesh of a partition is defined by metric properties, not by set theoretic ones.

Proposition 3.2. *For any two Borel subsets $A, B \subseteq X$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that for each finite partition $\mathcal{C} = \{C_1, \dots C_n\}$ of X with* $\mathrm{mesh}\,\mathcal{C} < \delta$

$$\left| \mu(A \cap B) - \sum_i \frac{\mu(A \cap C_i)\,\mu(B \cap C_i)}{\mu(C_i)} \right| < \varepsilon \tag{3.1}$$

Proof. For every finite partition $\mathcal{C}$ of X we define an orthogonal projection operator $T_{\mathcal{C}}$ in the Hilbert space $\mathcal{L}^2(\mu)$ by

$$T_{\mathcal{C}} f := \sum_i \frac{\int_{C_i} f d\mu}{\mu(C_i)} \chi_{C_i} \tag{3.2}$$

A short calculation shows that $T_{\mathcal{C}}$ is the orthogonal projection onto the finite dimensional subspace spanned by the indicator functions χ_{C_i}.

Now suppose $\varepsilon > 0$ and a *continuous* function f are given; we choose $\delta > 0$ such that $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$, observing that f must be uniformly continuous by compactness of X. Then for $\mathrm{mesh}\,\mathcal{C} < \delta$ we must have $|T_{\mathcal{C}} fx - fx| < \varepsilon$ uniformly and therefore $\|T_{\mathcal{C}} f - f\|_2 \le \varepsilon$.

Now, finally, suppose A and B are given and choose continuous function f, g such that $\|\chi_A - f\|_2 < \varepsilon$ and $\|\chi_B - g\|_2 < \varepsilon$; then using the case above pick

$\delta > 0$ such that $\|T_{\mathcal{C}}f - f\|_2 \le \varepsilon$ and $\|T_{\mathcal{C}}g - g\|_2 \le \varepsilon$ for $\operatorname{mesh}\mathcal{C} < \delta$. Observing $\|T_{\mathcal{C}}\| \le 1$ we obtain $\|T_{\mathcal{C}}\chi_A - \chi_A\|_2 \le \|T_{\mathcal{C}}(\chi_A - f)\|_2 + \|T_{\mathcal{C}}f - f\|_2 + \|f - \chi_A\|_2 \le \|\chi_A - f\|_2 + \|T_{\mathcal{C}}f - f\|_2 + \|f - \chi_A\|_2 \le 3\varepsilon$ and similarly $\|T_{\mathcal{C}}\chi_B - \chi_B\|_2 \le 3\varepsilon$. Now $|\langle\chi_A, \chi_B\rangle - \langle T_{\mathcal{C}}\chi_A, T_{\mathcal{C}}\chi_B\rangle| \le |\langle\chi_A - T_{\mathcal{C}}\chi_A, \chi_B\rangle| + |\langle T_{\mathcal{C}}\chi_A, \chi_B - T_{\mathcal{C}}\chi_B\rangle| \le \|\chi_A - T_{\mathcal{C}}\chi_A\|_2 \|\chi_B\|_2 + \|T_{\mathcal{C}}\chi_A\|_2 \|\chi_B - T_{\mathcal{C}}\chi_B\|_2 \le 6\varepsilon$. Since $\langle\chi_A, \chi_B\rangle = \mu(A \cap B)$ and $\langle T_{\mathcal{C}}\chi_A, T_{\mathcal{C}}\chi_B\rangle = \sum_i \frac{\mu(A\cap C_i)\mu(B\cap C_i)}{\mu(C_i)}$ condition (3.1) follows. □

Corollary 3.3. *For every Borel subset $A \subseteq X$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that for each finite partition $\mathcal{B} = \{B_1, \dots B_n\}$ with* $\operatorname{mesh}\mathcal{B} < \delta$

$$0 \le \mu(A) - \sum_i \frac{\mu(A \cap B_i)^2}{\mu(B_i)} < \varepsilon \tag{3.3}$$

Proof. $\sum_i \frac{\mu(A\cap B_i)^2}{\mu(B_i)} = \sum_i \mu(B_i) \left(\frac{\mu(A\cap B_i)}{\mu(B_i)}\right)^2 \le \sum_i \mu(B_i) \frac{\mu(A\cap B_i)}{\mu(B_i)} = \sum_i \mu(A \cap B_i) = \mu(A)$.

The rest follows from an application of proposition 3.2 to the case $A = B$. □

Observe that

$$M_{\varepsilon\delta} := \bigcap_{\mathcal{B}:\operatorname{mesh}\mathcal{B}<\delta} \left\{ A \,\middle|\, \sum_i \frac{\mu(A \cap B_i)^2}{\mu(B_i)} \ge \mu(A) - \varepsilon \right\} \subseteq Y(\mu) \tag{3.4}$$

is an event because the uncountable intersection may be replaced by an intersection over a dense countable family of finite partitions without affecting the result. Our proposition states $\forall\varepsilon > 0 : \bigcup_n M_{\varepsilon\frac{1}{n}} = Y(\mu)$.

3.2. **Generating systems of the Borel algebra.** The following proposition clearly distinguishes random Borel sets from their closed counterpart, because in the closed setting the subsets of the form $Y_B = \left\{A \subseteq X \,\middle|\, A \cap B \neq \varnothing\right\}$ generate the Borel algebra and thus pave the way for capacities:

Proposition 3.4. *The σ-algebra generated by the subsets*

$$Y_B := \left\{A \subseteq X \,\middle|\, \mu(A \cap B) > 0\right\} \subseteq Y(\mu) \tag{3.5}$$

with B ranging over all Borel subsets of X is strictly smaller than the Borel algebra on $Y(\mu)$; for instance it does not contain the event $M := \left\{A \subseteq X \,\middle|\, \mu(A) > \frac{1}{2}\right\}$.

Proof. Suppose the contrary. Then by [4, Ch.I,§5,Thm.D] there exists a sequence of Borel sets B_n such that M is contained in the σ-algebra $\mathfrak{F}$ generated by the countably family of events Y_{B_n}. Consider $A' \in M$, i.e. $A' \subseteq X$ with $\mu(A') > \frac{1}{2}$. Clearly $\mathfrak{F}$ cannot distinguish A' from any other subset A'' such that for all n: $\mu(A' \cap B_n) > 0 \Leftrightarrow \mu(A'' \cap B_n) > 0$; hence any such subset must be contained in M and therefore should satisfy $\mu(A'') > \frac{1}{2}$.

Set $\Gamma := \left\{n \,\middle|\, \mu(A' \cap B_n) > 0\right\} \subseteq \mathbb{N}$. Using the intermediate value theorem for non atomic measures we can find a subset $C_n \subseteq A' \cap B_n$ for each $n \in \Gamma$, such that $0 < \mu(C_n) < 2^{-n-2}$, and then in particular $\mu\left(\bigcup_{n\in\Gamma} C_n\right) < \frac{1}{4}$ and thus $A'' := \bigcup_{n\in\Gamma} C_n \subseteq A'$ with $\mu(A'') < \frac{1}{4}$. For $n \in \Gamma$ we have $A'' \cap B_n \supseteq C_n$ and therefore $\mu(A'' \cap B_n) > 0$, while for $n \in \mathbb{N}\setminus\Gamma$ we obtain $A'' \cap B_n \subseteq A' \cap B_n$ and $\mu(A' \cap B_n) = 0$. From the above we conclude $A'' \in M$, a contradiction. □

Proposition 3.5. *The Borel algebra on* $Y(\mu)$ *is generated by the events*

$$Y_{t,B} := \left\{A \subseteq X \middle| \mu(A \cap B) < t\right\} \subseteq Y(\mu) \tag{3.6}$$

with B *ranging over all Borel subsets of* X *and* $0 < t < 1$. *Actually it is sufficient to restrict* B *to a dense subset of* $Y(\mu)$ *(such as all open or all compact subsets of* X*) and* t *to a dense subset of* $]0,1[$.

Proof. Define metric d on $Y(\mu)$ by $d(A,B) := \max(\mu(A\setminus B), \mu(B\setminus A))$. Since $\frac{1}{2}\mu(A\triangle B) \leq d(A,B) \leq \mu(A\triangle B)$ this generates the customary topology. Since $\mu(B\setminus A) = \mu(B) - \mu(B\cap A)$ we obtain $\mu(B\setminus A) < \varepsilon \Leftrightarrow \forall\delta > \mu(B) - \varepsilon : \mu(A\cap B) < \delta$. Hence the ε-ball around B with respect to the metric d is given by

$$\begin{aligned}\left\{A \subseteq X \middle| \mu\left(A \cap \complement B\right) < \varepsilon\right\} \cap \bigcap_n \left\{A \subseteq X \middle| \mu(A\cap B) < \mu(B) - \varepsilon + \frac{1}{n}\right\}\\ = Y_{\varepsilon,\complement B} \cap \bigcap_n Y_{\mu(B)-\varepsilon+\frac{1}{n},B}\end{aligned} \tag{3.7}$$

Therefore these sets generate the topology and hence all Borel sets in $Y(\mu)$. □

Unfortunately, the events $Y_{t,B}$ do not constitute a *ring*, i.e. they are are incompatible with unions or intersections. Therefore the construction of measures will require us to consider finite intersections:

$$Y_{t_1,\dots t_r,B_1,\dots B_r} := \left\{A \subseteq X \middle| \mu(A\cap B_1) < t_1 \text{ and } \dots \mu(A\cap B_r) < t_r\right\} = \bigcap_{i=1}^{r} Y_{t_i,B_i}.$$

3.3. **The probability space.** For every finite partition $\mathcal{B} = \{B_1, \dots B_n\}$ we consider the map

$$q_{\mathcal{B}} : Y(\mu) \to I^n \tag{3.8}$$

$$A \mapsto \left(\frac{\mu(A\cap B_1)}{\mu(B_1)}, \dots \frac{\mu(A\cap B_n)}{\mu(B_n)}\right) \tag{3.9}$$

and denote by $\nu_{\mathcal{B}}$ the image of the probability measure on $Y(\mu)$ on I^n under the map $q_{\mathcal{B}}$. I^n will be equipped with the metric $d_{\mathcal{B}}\left(t_j, t'_j\right) := \sqrt{\sum_j \mu(B_j)\left(t_j - t'_j\right)^2}$.

Consider a finite partition $\mathcal{C} = \{C_1, \dots C_m\}$. Then for any other finite partition $\mathcal{B} = \{B_1, \dots B_n\}$, not necessarily a refinement of $\mathcal{C}$, we define a map

$$q_{\mathcal{C}}^{\mathcal{B}} : I^n \to I^m \tag{3.10}$$

$$(t_1, \dots t_n) \mapsto (u_1, \dots u_m) \tag{3.11}$$

$$u_i := \sum_j \frac{\mu(B_j \cap C_i)}{\mu(C_i)} t_j \tag{3.12}$$

Notice that the map $q_{\mathcal{C}}^{\mathcal{B}}$ is a contraction: by convexity for each i:

$$\left(\sum_j \frac{\mu(B_j \cap C_i)}{\mu(C_i)} (t_j - t_j') \right)^2 \leq \sum_j \frac{\mu(B_j \cap C_i)}{\mu(C_i)} (t_j - t_j')^2,$$

hence $d_{\mathcal{C}}\left(q_{\mathcal{C}}^{\mathcal{B}}(t_j), q_{\mathcal{C}}^{\mathcal{B}}(t_j')\right) = \sqrt{\sum_i \mu(C_i) \left(\sum_j \frac{\mu(B_j \cap C_i)}{\mu(C_i)} (t_j - t_j') \right)^2} \leq \sqrt{\sum_{i,j} \mu(C_i) \frac{\mu(B_j \cap C_i)}{\mu(C_i)} (t_j - t_j')^2} = \sqrt{\sum_j \mu(B_j)(t_j - t_j')^2} = d_{\mathcal{B}}(t_j, t_j')$.

$\mathcal{B} = \{B_1, \dots B_n\}$ is called refinement of $\mathcal{C} = \{C_1, \dots C_m\}$ if for any two indices i, j either $\mu(C_i \cap B_j) = 0$ or $\mu(C_i \setminus B_j) = 0$. Under this condition equation (3.12) assumes the form $u_i = \sum_{j: \mu(C_i \cap B_j) \neq 0} \frac{\mu(B_j)}{\mu(C_i)} t_j$.

If $\mathcal{B}$ is a refinement of $\mathcal{C}$ and $\mathcal{A}$ an arbitrary finite partition, then $q_{\mathcal{C}}^{\mathcal{B}} q_{\mathcal{B}}^{\mathcal{A}} = q_{\mathcal{C}}^{\mathcal{A}}$ and $q_{\mathcal{C}}^{\mathcal{B}} q_{\mathcal{B}} = q_{\mathcal{C}}$

Theorem 3.6. *The system of measures $\nu_{\mathcal{B}}$ satisfies the following two conditions:*

(1) $q_{\mathcal{C}}^{\mathcal{B}} \nu_{\mathcal{B}} = \nu_{\mathcal{C}}$ *for every refinement $\mathcal{B}$ of $\mathcal{C}$.*
(2) *For each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\nu_{\mathcal{B}}\left((t_k) \in I^n : \sum_k \mu(B_k) \left(t_k - \frac{1}{2}\right)^2 > \frac{1}{4} - \varepsilon \right) > 1 - \varepsilon \tag{3.13}$$

for each finite partition $\mathcal{B} = \{B_1, \dots B_n\}$ with $\operatorname{mesh} \mathcal{B} < \delta$.

Conversely: every system of probability measures satisfying these properties derives from a probability measure P on $Y(\mu)$ as system of images $\nu_{\mathcal{B}} = q_{\mathcal{B}} P$.

Proof. We extend definition (3.8) to a map $q_{\mathcal{B}} : Z(\mu) \to I^n$, $q_{\mathcal{B}}(f) = (t_1, \dots t_n)$, $t_k := \frac{1}{\mu(B_k)} \int_{B_k} f d\mu$; by reasons of compactness this is easily checked to be an inverse limit representation. It now suffices to show that the subset $Y(\mu) \subseteq Z(\mu)$ corresponds precisely to the set M of all those $f \in Z(\mu)$, whose representations $(t_1, \dots t_n) = q_{\mathcal{B}}(f)$ satisfy the condition $\forall \varepsilon > 0 \exists \delta > 0 \forall \operatorname{mesh} \mathcal{B} < \delta : \sum_k \mu(B_k) \left(t_k - \frac{1}{2}\right)^2 > \frac{1}{4} - \varepsilon$. corollary 3.3 readily implies $Y(\mu) \subseteq M$. But for $f \in Z(\mu) \setminus Y(\mu)$ we can find $\varepsilon > 0$ with $\mu(\varepsilon < f < 1 - \varepsilon) > \varepsilon$; let $\mathcal{C} = \{C_1, C_2, C_3\}$ be the partition $C_1 := \{f \leq \varepsilon\}$, $C_2 := \{\varepsilon < f < 1 - \varepsilon\}$, $C_3 := \{f \geq 1 - \varepsilon\}$. If $\mathcal{B}$ is any refinement of $\mathcal{C}$, then $(t_1, \dots t_n) = q_{\mathcal{B}}(f)$ satisfies $\sum_k \mu(B_k) \left(t_k - \frac{1}{2}\right)^2 \leq \frac{1}{4} - \varepsilon^2$ and therefore $f \notin M$. □

Example 3.7. Suppose we are given a random *closed* set A with capacity functional $\tau(K) = P(A \cap K \neq 0)$. Considering this as random Borel set we get for any finite partition $\mathcal{B} = \{B_1, \dots B_n\}$ the corresponding capacity as $\nu_{\mathcal{B}}\left(\prod_i [0, t_i[\right) = P\left(\forall i : \mu\left(A \cap B_i\right) < t_i \mu\left(B_i\right)\right) = 1 - \sup\left\{\tau\left(\bigcup_i K_i\right) \middle| \forall i : K_i \text{ compact}, K_i \subseteq B_i, \mu\left(K_i\right) > \left(1 - t_i\right)\mu\left(B_i\right)\right\}$.

3.4. **The process aspect.** We may now describe random Borel sets as stochastic processes as follows:

Theorem 3.8. *For any random Borel set A consider the process assigning to each compact set (each open set, each Borel set) K the random variable $Y_K := \frac{\mu(A\cap K)}{\mu(K)}$ with values in I. (For $\mu(K) = 0$ we may define Y_K arbitrarily or disregard it entirely.) Then:*

1. $Y_{K_1 \cup K_2} = \frac{\mu(K_1)}{\mu(K_1 \cup K_2)} Y_{K_1} + \frac{\mu(K_2)}{\mu(K_1 \cup K_2)} Y_{K_2}$ *provided* $K_1 \cap K_2 = \varnothing$.
2. *For each finite partition $\mathcal{K} = \{K_1, \dots K_n\}$ consider the sum $\sum_i \mu\left(K_i\right) Y_{K_i}^2$. Then $\sum_i \mu\left(K_i\right) Y_{K_i}^2 \to Y_X$ a.s. if* $\operatorname{mesh} \mathcal{K} \to 0$. *(Notice $Y_X = \mu(A)$.)*

Conversely, any process satisfying these two properties derives from a random Borel set, whose distribution is uniquely characterized by these properties.

The proof is a direct application of theorem 3.6 making a few observations: Consider two sets K, L. Then $\left|Y_K - Y_L\right| = \left|\frac{\mu(K \setminus L)}{\mu(K)} Y_{K\setminus L} + \frac{\mu(K\cap L)}{\mu(K)} Y_{K\cap L} - \frac{\mu(L\setminus K)}{\mu(L)} Y_{L\setminus K} - \frac{\mu(K\cap L)}{\mu(L)} Y_{K\cap L}\right| \leq \frac{\mu(K\setminus L)}{\mu(K)} + \frac{\mu(L\setminus K)}{\mu(L)} + \mu\left(K \cap L\right)\left|\frac{1}{\mu(K)} - \frac{1}{\mu(L)}\right|$. This immediately implies $Y_K = Y_L$ if $\mu\left(K \triangle L\right) = 0$; moreover $Y_{K_n} \to Y_K$ uniformly if $K_n \to K$ in the topology of $Y(\mu)$. Since the compact sets and the open sets are dense in $Y(\mu)$ they suffice to define Y_B for any Borel set B as uniform limit. Furthermore, just like in the proof of [2, Lem.4.1] it can be shown that the compact sets, whose boundary is of measure 0, (or the open sets, whose boundary is of measure 0) are dense in $Y(\mu)$. This readily implies that any finite partition of the compact space X into *Borel* sets can be arbitrarily closely approximated by a finite partition into compact sets or into open sets. Condition (2) means $\sum_i \mu\left(K_i\right)\left(Y_{K_i} - \frac{1}{2}\right)^2 = \sum_i \mu\left(K_i\right) Y_{K_i}^2 - \sum_i \mu\left(K_i\right) Y_{K_i} + \frac{1}{4} = \sum_i \mu\left(K_i\right) Y_{K_i}^2 - Y_X + \frac{1}{4}$ and is preserved by the extension to Borel sets. Now observe corollary 3.3. Then define the measure $\nu_{\mathcal{B}}$ as the joint distribution of the random variables $Y_{B_1}, \dots Y_{B_n}$.

Lemma 3.9. *For any Borel set B, the random variable Y_B is the a.s.-limit $\sum_i \frac{\mu(B_i)}{\mu(B)} Y_{B_i}^2 \to Y_B$ with $\mathcal{B} = \{B_1, \dots B_n\}$ ranging over the partitions of B such that* $\operatorname{mesh} \mathcal{B} \to 0$.

Proof. Since $0 \leq Y_{B_i}$ we have $\sum_i \frac{\mu(B_i)}{\mu(B)} Y_{B_i}^2 \leq \sum_i \frac{\mu(B_i)}{\mu(B)} Y_{B_i} = Y_B$ for each partition of B. Now set $C := \complement B$ and supplement every partition of B with one of C to obtain a partition of X without increasing the mesh. Then also $\sum_i \frac{\mu(C_i)}{\mu(C)} Y_{C_i}^2 \leq Y_C$. Furthermore $\mu(B) \sum_i \frac{\mu(B_i)}{\mu(B)} Y_{B_i}^2 + \mu(C) \sum_i \frac{\mu(C_i)}{\mu(C)} Y_{C_i}^2 \to Y_X$ by theorem 3.8, which can happen only if $\sum_i \frac{\mu(B_i)}{\mu(B)} Y_{B_i}^2 \to Y_B$ and $\sum_i \frac{\mu(C_i)}{\mu(C)} Y_{C_i}^2 \to Y_C$. ☐

Proposition 3.10. *For any two random Borel sets represented by processes Y_K resp. Y'_K we can define a new process*

$$Y_K \wedge Y'_K := \lim \sum_i \frac{\mu(K_i)}{\mu(K)} Y_{K_i} Y'_{K_i} \tag{3.14}$$

as a.s.-limit taken over all finite partitions $\mathcal{K} = \{K_1, \dots K_n\}$ of K such that $\operatorname{mesh}\mathcal{K} \to 0$. *This new process represents the* intersection *of the two given random Borel sets.*

Proof. Apply proposition 3.2. □

Lemma 3.11. *Two random Borel sets represented by processes Y_K resp. Y'_K are independent as random variables if and only if for any two Borel sets (compact sets, open sets) K and L the random variables Y_K and Y'_L are independent.*

Proof. Apply proposition 3.5. □

Definition 3.12. A random Borel set represented by a process Y_K is called *isotropic*, if for every $\varepsilon > 0$ there exists a finite partition $\mathcal{B} = \{B_1, \dots B_n\}$ of X with $\operatorname{mesh}\mathcal{B} < \varepsilon$, such that for any two indices $1 \le i \ne j \le n$ there exists a permutation $\pi \in \mathfrak{S}_n$ with $\pi(i) = j$, such that the joint distribution of $Y_{K_{\pi(1)}}, \dots Y_{K_{\pi(n)}}$ coincides with the joint distribution of $Y_{K_1}, \dots Y_{K_n}$.

Notice that under the condition above in particular $E(Y_{K_i}) = E(Y_{K_j})$ for any two indices i, j, and since $Y_X = \sum_i \mu(K_i) Y_{K_i}$ we obtain $E(Y_X) = \sum_i \mu(K_i) E(Y_{K_i}) = E(Y_{K_j}) \sum_i \mu(K_i) = E(Y_{K_j})$ for each j.

Proposition 3.13. *Consider two independent random Borel sets A and B, at least one of which is assumed isotropic. Then $E(\mu(A \cap B)) = E(\mu(A)) E(\mu(B))$.*

Proof. Assume Y'_K is isotropic and apply proposition 3.10: $E(\mu(A \cap B)) = E(Y_X \wedge Y'_X) = \lim \sum_i \mu(K_i) E(Y_{K_i}) E(Y'_{K_i}) = (\lim \sum_i \mu(K_i) E(Y_{K_i})) E(Y'_X) = (\lim E(\sum_i \mu(K_i) Y_{K_i})) E(Y'_X) = E(Y_X) E(Y'_X) = E(\mu(A)) E(\mu(B))$. □

4. Random Borel sets on the unit segment and inspection processes

In [8] random Borel sets on the unit segment were uniquely characterized by their inspection processes, neglecting the fact that not every inspection process corresponds to a random Borel set. Every random variable φ satisfying conditions (1–2) below can be interpreted as an inspection process of a random variable with values in a certain function space and hence is more general than a random set; the special case or random sets is determined by condition (3) below.

For any random Borel subset X of the unit segment I we consider the inspection process $X_t := \lambda(X \cap [0, t]) = \int_0^t \chi_X d\lambda$, where λ is Lebesgue measure on I. Observe that X_t as a function of t is continuous, increasing and almost everywhere differentiable with derivative almost surely equal to χ_X [7, Thm8.8,p.168]. That means X_t is the primitive (syn. antiderivative) of the indicator function χ_X.

Observe that this implies that the paths X_t are much better behaved than the paths of Brownian motion, which are almost surely nowhere differentiable [1, Thm.12.25,p.261].

We consider the set of functions $\varphi : I \to \mathbb{R}$ subject to the conditions

(1) $\varphi(0) = 0$
(2) $\forall 0 \leq x \leq y \leq 1 : 0 \leq \varphi(y) - \varphi(x) \leq y - x$
(3) $\sup \sum_k \frac{[\varphi(t_{k+1}) - \varphi(t_k)]^2}{t_{k+1} - t_k} = \varphi(1)$, where the supremum is taken over all finite decompositions of the unit segment $0 = t_0 < t_1 < \cdots < t_n = 1$.

Notice that the functions satisfying conditions (1) and (2) constitute a compact convex set Z of the space of all continuous functions on I equipped with the topology of uniform convergence by the theorem of Arzela-Ascoli. By $Y \subset Z$ we denote the subspace of functions satisfying all three conditions (1–3).

Lemma 4.1. *For all $x, y \in \mathbb{R}$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$ the inequality $(x+y)^2 \leq \frac{x^2}{\alpha} + \frac{y^2}{\beta}$ holds.*

Proof. $0 \leq \left(\sqrt{\frac{\beta}{\alpha}}x - \sqrt{\frac{\alpha}{\beta}}y\right)^2 = \frac{\beta}{\alpha}x^2 - 2xy + \frac{\alpha}{\beta}y^2 = \left(\frac{1}{\alpha} - 1\right)x^2 - 2xy + \left(\frac{1}{\beta} - 1\right)y^2 = \frac{1}{\alpha}x^2 + \frac{1}{\beta}y^2 - (x+y)^2$. □

Lemma 4.2. *All functions $\varphi \in Z$ satisfy $\sup \sum_k \frac{[\varphi(t_{k+1}) - \varphi(t_k)]^2}{t_{k+1} - t_k} \leq \varphi(1)$. Moreover, if the decomposition $0 = u_0 < u_1 < \cdots < u_N = 1$ is a refinement of $0 = t_0 < t_1 < \cdots < t_n = 1$, then $\sum_{k=0}^{N-1} \frac{[\varphi(u_{k+1}) - \varphi(u_k)]^2}{u_{k+1} - u_k} \geq \sum_{k=0}^{n-1} \frac{[\varphi(t_{k+1}) - \varphi(t_k)]^2}{t_{k+1} - t_k}$.*

Proof. $\sum_{k=0}^{n-1} \frac{[\varphi(t_{k+1}) - \varphi(t_k)]^2}{t_{k+1} - t_k} = \sum_{k=0}^{n-1} (t_{k+1} - t_k) \left[\frac{\varphi(t_{k+1}) - \varphi(t_k)}{t_{k+1} - t_k}\right]^2 \leq \sum_{k=0}^{n-1} (t_{k+1} - t_k) \left[\frac{\varphi(t_{k+1}) - \varphi(t_k)}{t_{k+1} - t_k}\right] = \sum_{k=0}^{n-1} [\varphi(t_{k+1}) - \varphi(t_k)] = \varphi(1)$.

Now consider three points $0 \leq v_1 < v_2 < v_3 \leq 1$; then an application of lemma 4.1 to $x := \varphi(v_2) - \varphi(v_1)$, $y := \varphi(v_3) - \varphi(v_3)$, $\alpha := \frac{v_2 - v_1}{v_3 - v_1}$ and $\beta := \frac{v_3 - v_2}{v_3 - v_1}$ yields $\frac{[\varphi(v_3) - \varphi(v_1)]^2}{v_3 - v_1} \leq \frac{[\varphi(v_2) - \varphi(v_1)]^2}{v_2 - v_1} + \frac{[\varphi(v_3) - \varphi(v_2)]^2}{v_3 - v_2}$, and this concludes the proof. □

Remark 4.3. For every function $\varphi \in Z$ there is a Lebesgue measurable function $\psi : I \to I$ with $\varphi(t) = \int_0^t \psi(v)dv$ for all $t \in I$ (consider positive measure defined on intervals $[x, y]$ by $\varphi(y) - \varphi(x)$ and apply the Radon-Nikodym theorem).

Lemma 4.4. *Suppose φ is the antiderivative $\varphi(t) = \int_0^t \psi(v)dv$ of a measurable function $\psi : I \to I$, and suppose a sequence of decompositions $\zeta_n = \left\{0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{N_n}^{(n)} = 1\right\}$ of the unit segment is given with $\lim_{n\to\infty} \operatorname{mesh} \zeta_n = 0$. Then $\lim_{n\to\infty} \sum_{k=0}^{N_n - 1} \frac{\left[\varphi\left(t_{k+1}^{(n)}\right) - \varphi\left(t_k^{(n)}\right)\right]^2}{t_{k+1}^{(n)} - t_k^{(n)}} = \int_0^1 \psi(v)^2 dv$.*

Proof. We define a sequence of functions $f_n : I \to I$ by $f_n(t) := \frac{\varphi\left(t_{k+1}^{(n)}\right)-\varphi\left(t_k^{(n)}\right)}{t_{k+1}^{(n)}-t_k^{(n)}}$ if $t_k^{(n)} \leq t < t_{k+1}^{(n)}$. By [7, Thm8.8,p.168] $f_n \to \psi$ almost everywhere, and consequently $f_n^2 \to \psi^2$ almost everywhere. Hence by Lebesgue's dominated convergence theorem $\lim_{n\to\infty} \int_0^1 f_n(v)^2 dv = \int_0^1 \psi(v)^2 dv$. □

Proposition 4.5. *A function $\varphi \in Z$ is the antiderivative of an indicator function if and only if it satisfies condition (3).*

Proof. In view of lemma 4.4 condition 3 translates to $\int_0^1 \psi(1-\psi)d\lambda = 0$. This means that ψ can assume values different from 0 or 1 only on a 0-set, i.e. ψ almost surely equals an indicator function. □

By proposition 4.5 we may identify inspection processes satisfying condition (3) with random Borel sets. Evidently, this condition is a relation among the increments of the inspection process, and one suspects that these increments cannot be independent.

Proposition 4.6. *Suppose the inspection process $X_t := \lambda\left(X \cap [0,t]\right) = \int_0^t \chi_X d\lambda$ corresponding to a random Borel set X has independent increments. Then there exists a fixed (deterministic) Borel set B such that $X \equiv B$ a.s.*

Proof. We consider two numbers $0 \leq x < y \leq 1$ and an integer $n \in \mathbb{N}$. Setting $h := \frac{y-x}{n}$, $t_k := x + kh$, $0 \leq k \leq n$, we telescope $X_y - X_x = \sum_{k=0}^n \left(X_{t_{k+1}} - X_{t_k}\right)$. Furthermore set $b_t := E\left(X_t\right)$. Independence of increments then implies

(4.1)

$$\mathrm{Var}\left(X_y - X_x\right) = \mathrm{Var}\sum_{k=0}^n \left(X_{t_{k+1}} - X_{t_k}\right)$$

$$= \sum_{k=0}^n \mathrm{Var}\left(X_{t_{k+1}} - X_{t_k}\right) \qquad \text{(by independence)} \tag{4.2}$$

$$= \sum_{k=0}^n \left(E\left(\left(X_{t_{k+1}} - X_{t_k}\right)^2\right) - \left(E\left(X_{t_{k+1}} - X_{t_k}\right)\right)^2\right) \tag{4.3}$$

$$= hE\sum_{k=0}^n \frac{\left(X_{t_{k+1}} - X_{t_k}\right)^2}{t_{k+1}-t_k} - h\sum_{k=0}^n \frac{\left(b_{t_{k+1}} - b_{t_k}\right)^2}{t_{k+1}-t_k} \tag{4.4}$$

$$\to hE\left(X_{t_{k+1}} - X_{t_k}\right) - h\sum_{k=0}^n \frac{\left(b_{t_{k+1}} - b_{t_k}\right)^2}{t_{k+1}-t_k} \qquad \text{(by condition 3)} \tag{4.5}$$

$$= h\left(b_{t_{k+1}} - b_{t_k}\right) - h\sum_{k=0}^n \frac{\left(b_{t_{k+1}} - b_{t_k}\right)^2}{t_{k+1}-t_k} \tag{4.6}$$

Now $\mathrm{Var}\left(X_y - X_x\right) = 0$ is obtained by letting $n \to \infty$ and hence $h \to 0$; this means $X_y - X_x \equiv b_y - b_x$ a.s. Hence the function $t \mapsto b_t$ must also satisfy condition 3 and defines the Borel set B required by our theorem. (On a sideline we may observe that the function $t \mapsto b_t$ satisfies conditions 1 and 2, therefore $\sum_{k=0}^{n} \frac{\left(b_{t_{k+1}} - b_{t_k}\right)^2}{t_{k+1} - t_k} \leq y - x$.) □

5. An application to random allocation

Our random allocation experiment will be developed from a random Borel set that is a stronger version of example [2, Ex.7.5]; that means we construct a probability measure on the space of Borel sets B by prescribing the distribution of the composite random variable $B \mapsto \mu(B)$ (condition (1) below) and then inductively lifting the measure over the fibers of the projection maps $p_n^{n+1} : I^{2^{n+1}} \to I^{2^n}$ familiar from [2, §7] (condition (2) below). The intended computation of first and second moments requires some detailed preparations.

So we suppose we are given

(1) a random variable x_{00} distributed on I and
(2) another random variable u symmetrically distributed on $[-1, 1]$ such that

$$\forall \varepsilon > 0 : \qquad P\left(|u| \geq 1 - \varepsilon\right) > 0 \tag{5.1}$$

Then we define random variables x_{nm} for $n > 0$, $0 \leq m < 2^n$ inductively by picking identically distributed copies u_{nm} of u, independent from one another and from all x_{rk}, $r \leq n$, and setting

$$x_{n+1,2m} = x_{nm} + u_{nm} \min\left(x_{nm}, 1 - x_{nm}\right) \tag{5.2}$$

$$x_{n+1,2m+1} = x_{nm} - u_{nm} \min\left(x_{nm}, 1 - x_{nm}\right) \tag{5.3}$$

Observe that the scaling factor $\min\left(x_{nm}, 1 - x_{nm}\right)$ is chosen such that it ensures $0 \leq x_{n+1,2m} \leq 1$ and $0 \leq x_{n+1,2m+1} \leq 1$. Also notice that this is almost the same as [2, Ex.7.5], with the difference that our random variables need not possess a density, and with [2, Eq.7.8] replaced by the *much weaker* assumption (5.1). The system of random variables is invariant under G_∞.

For fixed n, the joint distribution of the random variables $x_{n0}, \ldots x_{n,2^n-1}$ defines a probability measure ν_n on I^{2^n}, and we claim that [2, Eq.7.1] is satisfied and hence a probability measure on the space of Borel sets $Y(\mu)$ is obtained. By G_n-invariance $E\left(\left(x_{nm} - \frac{1}{2}\right)^2\right)$ is independent of m and we can define

$$a_n := E\left(\left(x_{nm} - \frac{1}{2}\right)^2\right) = E\left(2^{-n} \sum_{m=0}^{2^n-1} \left(x_{nm} - \frac{1}{2}\right)^2\right) \tag{5.4}$$

By Jensen's inequality the sum $S_n := 2^{-n} \sum_{m=0}^{2^n-1} \left(x_{nm} - \frac{1}{2}\right)^2$ increases with n; hence, if we can show $a_n \to \frac{1}{4}$, we can conclude that S_n converges to $\frac{1}{4}$ almost surely and therefore in probability. Thus [2, Eq.7.1] will be proved.

We visualize our random variables as tree with root x_{00}, such that each x_{nm} has two descendents $x_{n+1,2m}$ and $x_{n+1,2m+1}$. Fix $0 < \varepsilon < \frac{1}{2}$ and set $q := \frac{1}{2}P\left(|u| \geq 1 - 2\varepsilon\right) > 0$. We claim

$$P\left(\left|x_{nm} - \frac{1}{2}\right| \geq \frac{1}{2} - \varepsilon\right) \geq q \tag{5.5}$$

for all $n > 0$. For, assume $m = 2k$ is even and $x_{n-1,k} \geq \frac{1}{2}$, then $x_{nm} = x_{n-1,k} + u_{n-1,k}\left(1 - x_{n-1,k}\right) = \left(1 - u_{n-1,k}\right)x_{n-1,k} + u_{n-1,k} \geq \frac{1}{2}\left(1 - u_{n-1,k}\right) + u_{n-1,k} = \frac{1}{2}\left(1 + u_{n-1,k}\right)$. But since $u_{n-1,k} \geq 1 - 2\varepsilon$ with probability at least q we obtain $x_{nm} \geq 1 - \varepsilon$ with probability at least q provided $x_{n-1,k} \geq \frac{1}{2}$. The cases $x_{n-1,k} \leq \frac{1}{2}$ or m odd are handled similarly. Now observing the independence of $u_{n-1,k}$ from $x_{n-1,k}$ we conclude that the event A_{nm} that x_{nm} itself or at least one of its ancestors x_{rk} in our tree satisfies $\left|x_{rk} - \frac{1}{2}\right| \geq \frac{1}{2} - \varepsilon$ has probability $P\left(A_{nm}\right) \geq 1 - (1-q)^n$.

We claim $a_n \geq \left(\frac{1}{2} - \varepsilon\right)^2\left[1 - (1-q)^n\right]$. $\liminf_n a_n \geq \left(\frac{1}{2} - \varepsilon\right)^2$ will follow, and therefore, since ε was arbitrary, $\lim_n a_n = \frac{1}{4}$. To this end consider the event $A_{nm}^{(r)} \subseteq A_{nm}$ such that the first ancestor of x_{nm} satisfying $\left|x_{rk} - \frac{1}{2}\right| \geq \frac{1}{2} - \varepsilon$ occurs at level r and observe that $A_{nm}^{(r)}$ is invariant under the subgroup of G_n leaving the range $2^{n-r}k \leq \ell < 2^{n-r}(k+1)$ invariant; therefore

$$\int\limits_{A_{nm}^{(r)}} \left(x_{nm} - \frac{1}{2}\right)^2 dP = \int\limits_{A_{nm}^{(r)}} 2^{r-n} \sum_{\ell=2^{n-r}k}^{2^{n-r}(k+1)-1} \left(x_{n\ell} - \frac{1}{2}\right)^2 dP \tag{5.6}$$

However, by Jensen's inequality we have $2^{r-n}\sum_{\ell=2^{n-r}k}^{2^{n-r}(k+1)-1}\left(x_{n\ell} - \frac{1}{2}\right)^2 \geq \left(2^{r-n}\sum_{\ell=2^{n-r}k}^{2^{n-r}(k+1)-1} x_{n\ell} - \frac{1}{2}\right)^2 = \left(x_{rk} - \frac{1}{2}\right)^2 \geq \left(\frac{1}{2} - \varepsilon\right)^2$ on $A_{nm}^{(r)}$ and therefore $\int_{A_{nm}^{(r)}} \left(x_{nm} - \frac{1}{2}\right)^2 dP \geq \left(\frac{1}{2} - \varepsilon\right)^2 P\left(A_{nm}^{(r)}\right)$. Hence $\int_{A_{nm}} \left(x_{nm} - \frac{1}{2}\right)^2 dP = \sum_r \int_{A_{nm}^{(r)}} \left(x_{nm} - \frac{1}{2}\right)^2 dP \geq \left(\frac{1}{2} - \varepsilon\right)^2 \sum_r P\left(A_{nm}^{(r)}\right) = \left(\frac{1}{2} - \varepsilon\right)^2 P\left(A_{nm}\right) \geq \left(\frac{1}{2} - \varepsilon\right)^2\left[1 - (1-q)^n\right]$. This concludes the proof.

Assumption 1. For the remainder of this subsection we assume that the random variable u is uniformly distributed on $[-1, 1]$.

Observe that with this assumption $E\left(h\left(x_{nm}\right)\Big|x_{10}=t\right)=E\left(h\left(x_{n-1,m}\right)\Big|x_{00}=t\right)$ for any function h! In consequence

$$E\left(h\left(x_{nm}\right)\Big|x_{00}=t\right) \tag{5.7}$$

$$=\frac{1}{2}\int_{-1}^{+1}E\left(h\left(x_{nm}\right)\Big|x_{10}=t+u\min\left(t,1-t\right)\right)du \tag{5.8}$$

$$=\frac{1}{2}\int_{-1}^{+1}E\left(h\left(x_{n-1,m}\right)\Big|x_{00}=t+u\min\left(t,1-t\right)\right)du \tag{5.9}$$

$$=\frac{1}{2\min(t,1-t)}\int_{\max(0,2t-1)}^{\min(1,2t)}E\left(h\left(x_{n-1,m}\right)\Big|x_{00}=v\right)dv \tag{5.10}$$

Applying the foregoing to $h(t)=t^2$ leads to the functions $f_n(t)=E\left(x_{nm}^2\Big|x_{00}=t\right)$, whose special role has already been investigated in [2, §8]:

$$f_0(t):=t^2 \tag{5.11}$$

$$f_{n+1}(t):=\frac{1}{2\min(t,1-t)}\int_{\max(0,2t-1)}^{\min(1,2t)}f_n(u)du \tag{5.12}$$

$$f_1(t)=t^2+\frac{1}{3}\left[\min(t,1-t)\right]^2 \tag{5.13}$$

By induction one can easily prove $f_n(t)\le t$ for all n. Since $f_1\ge f_0$ we also infer by induction $f_{n+1}\ge f_n$ for all n. Consequently there must be a limit function $f_\infty(t)=\lim_{n\to\infty}f_n(t)$, at least lower semicontinuous, and satisfying $t^2\le f_\infty(t)\le t$ with

$$f_\infty(t)=\frac{1}{2\min(t,1-t)}\int_{\max(0,2t-1)}^{\min(1,2t)}f_\infty(u)du \tag{5.14}$$

Equation (5.14) immediately implies that f_∞ is continuous on the whole interval I, even at 0 and 1.

By induction one can prove

$$f_n(t)-f_n(1-t)=2t-1 \tag{5.15}$$

for all n, thus $g_n(t):=t-f_n(t)$ is symmetric around $\frac{1}{2}$, $g_n(1-t)=g_n(t)$, $g_0(t)=t(1-t)$, $g_{n+1}(t)=\frac{1}{2t}\int_0^{2t}g_n(u)du$ for $0\le t\le\frac{1}{2}$ and $g_n\downarrow 0$ for $n\to\infty$.

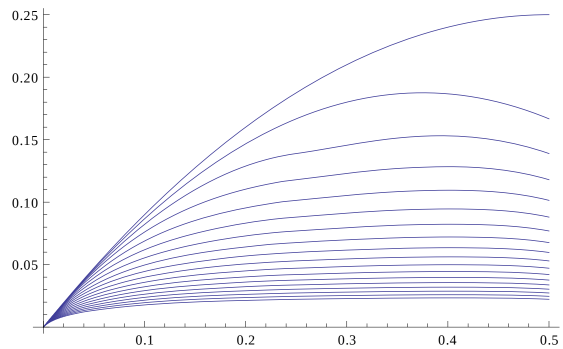

FIGURE 1. First few iterates of g_n.

Lemma 5.1. *Under assumption 1, $f_\infty(t) = t$ for all $t \in I$.*

Proof. Suppose the contrary and consider the function $h(t) := t - f_\infty(t)$; then h is continuous, $h \geq 0$, $h(0) = h(1) = 0$, $h(t) = \frac{1}{2\min(t,1-t)} \int\limits_{\max(0,2t-1)}^{\min(1,2t)} h(u)du$ and there exists $t_0 \in]0,1[$ such that $h(t_0) = \max\limits_{u \in I} h(u) > 0$. If $t_0 \leq \frac{1}{2}$, then $h(t_0) = \frac{1}{2t_0} \int\limits_0^{2t_0} h(u)du < \frac{1}{2t_0} \int\limits_0^{2t_0} h(t_0)\, du = h(t_0)$, a contradiction. Similarly, if $t_0 \geq \frac{1}{2}$, then $h(t_0) = \frac{1}{2(1-t_0)} \int\limits_{1-2t_0}^{1} h(u)du < \frac{1}{2(1-t_0)} \int\limits_{1-2t_0}^{1} h(t_0)\, du = h(t_0)$ and we arrive again at a contradiction. □

Observe that lemma 5.1 provides an independent proof that our construction leads to a well defined probability on Y. The first few iterates f_n of the Fredholm equation (5.12) (or equivalently, of g_n, cf. figure 1) can be evaluated in closed form, later ones can be obtained numerically.

Now we repeat our random Borel set drawing experiment independently and with identical distribution, at the i-th run obtaining a Borel set B_i. Then $C_r := \bigcup_{i=1}^r B_i$ models the set of elements allocated up to run r; the set valued random variables C_r constitute a Markov chain over the time variable r.

Theorem 5.2. *Let P be the probability measure on I obtained as the distribution of x_{00}, i.e. of the measure of the Borel set obtained in a single run of our random experiment. Then at Markov time r the accumulated Borel set has the expected measure $1 - \left[\int (1-x)dP(x)\right]^r$ and variance*

(5.16)
$$\sum_{m=0}^{\infty} 2^{-m-1} \left[\int \left(2f_m(1-x) - f_{m+1}(1-x)\right) dP(x)\right]^r - \left[\int (1-x)dP(x)\right]^{2r}$$

(The case of a Dirac measure P located at the "fixed weight" x is displayed in figure 2.)

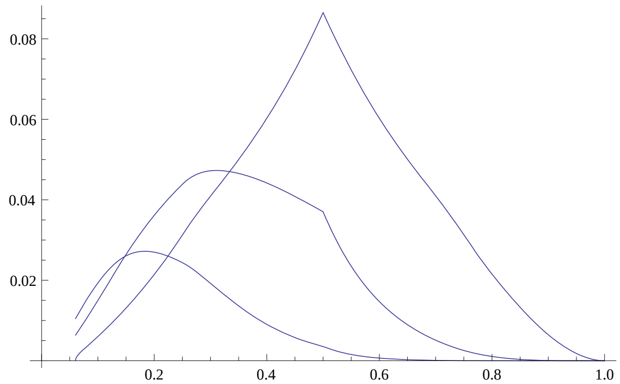

FIGURE 2. Square root of variance for Markov times $r = 2, 5, 10$.

The proof requires a few preparations, starting with a stronger continuity property of the $\wedge$-product than given in [2, Lem.5.2]:

Proposition 5.3. *The $\wedge$-product is a jointly continuous map $\wedge : Z_h \times Z_h \to Z_h$ in the Hilbert space topology and $\|\mathbf{a} \wedge \mathbf{x} - \mathbf{a} \wedge \mathbf{y}\|_2 \leq \sqrt{\|\mathbf{a}\|_2 \cdot \|\mathbf{x} - \mathbf{y}\|_2}$.*

Proof. We recall that for all $\mathbf{z} = (z_{nm})_{0 \leq m < 2^n, n \geq 0} \in Y$ the sequence $2^{-n} \sum_{m=0}^{2^n-1} z_{nm}^2$ is increasing with $\sup_{n \in \mathbb{N}_0} 2^{-n} \sum_{m=0}^{2^n-1} z_{nm}^2 = \|\mathbf{z}\|_2^2$ (cf. [2, eq.5.4]). For any two vectors $\mathbf{z}', \mathbf{z}'' \in Z$ we have $\|\mathbf{z}' - \mathbf{z}''\|_2^2 \leq 1$ because all $|z'_{nm} - z''_{nm}| \leq 1$.

This applies in particular to $\mathbf{z} = \mathbf{a} \wedge \mathbf{x} - \mathbf{a} \wedge \mathbf{y}$. Hence for each $0 \leq \alpha < \|\mathbf{a} \wedge \mathbf{x} - \mathbf{a} \wedge \mathbf{y}\|_2^2 \leq 1$ we can find $n \in \mathbb{N}_0$ such that $2^{-n} \sum_{m=0}^{2^n-1} z_{nm}^2 > \alpha$. Now pick numbers $0 \leq \alpha_m < 1$ such that $2^{-n} \sum_{m=0}^{2^n-1} \alpha_m \geq \alpha$ and $z_{nm}^2 > \alpha_m$ for each $0 \leq m < 2^n$; by definition of the $\wedge$-product there exists $N_0 \geq n$ such that for all $N \geq N_0$ and each $0 \leq m < 2^n$ the inequality $2^{n-N} \sum_{k=m2^{N-n}}^{(m+1)2^{N-n}-1} a_{Nk} |x_{Nk} - y_{Nk}| > \sqrt{\alpha_m} \geq \alpha_m$ holds (remember $\alpha_m \leq 1$, hence $\sqrt{\alpha_m} \geq \alpha_m$). By summing over all m we obtain $2^{-N} \sum_{k=0}^{2^N-1} a_{Nk} |x_{Nk} - y_{Nk}| > 2^{-n} \sum_{m=0}^{2^n-1} \alpha_m \geq \alpha$ and hence, using the Cauchy-Schwarz inequality: $\|\mathbf{a}\|_2 \cdot \|\mathbf{x} - \mathbf{y}\|_2 \geq \sqrt{2^{-N} \sum_{k=0}^{2^N-1} a_{Nk}^2} \cdot \sqrt{2^{-N} \sum_{k=0}^{2^N-1} (x_{Nk} - y_{Nk})^2} > \alpha$. Since $\alpha < \|\mathbf{a} \wedge \mathbf{x} - \mathbf{a} \wedge \mathbf{y}\|_2^2$ was arbitrary this implies $\|\mathbf{a}\|_2 \cdot \|\mathbf{x} - \mathbf{y}\|_2 \geq \|\mathbf{a} \wedge \mathbf{x} - \mathbf{a} \wedge \mathbf{y}\|_2^2$.

This implies continuity of the $\wedge$-product with respect to the Hilbert space topology because $\|\mathbf{x} \wedge \mathbf{y} - \mathbf{x}_0 \wedge \mathbf{y}_0\|_2 \leq \sqrt{\|\mathbf{x}_0\|_2 \cdot \|\mathbf{y} - \mathbf{y}_0\|_2} + \sqrt{\|\mathbf{y}\|_2 \cdot \|\mathbf{x} - \mathbf{x}_0\|_2}$ for $\mathbf{x}, \mathbf{x}_0, \mathbf{y}, \mathbf{y}_0 \in Z$. □

Lemma 5.4. *For each $\mathbf{x} = (x_{nm}) \in Z$ consider the sequence $\mathbf{x}^{(r)} = \left(x_{nm}^{(r)}\right) \in Z$ defined by*

$$x_{nm}^{(r)} := \begin{cases} x_{r, \lfloor m2^{r-n} \rfloor} & r \leq n \\ x_{nm} & r \geq n \end{cases} \tag{5.17}$$

Then $\lim_{r\to\infty} \mathbf{x}^{(r)} = \mathbf{x}$ *in the Hilbert space topology of* Z *(cf. [2, §5]).*

Observe that $\mathbf{x} \in Z \Rightarrow \mathbf{x}^{(r)} \in Z$, but in general $\mathbf{x}^{(r)} \notin Y$ even if $\mathbf{x} \in Y$.

Proof. $x^{(r)}_{n,2m} = x^{(r)}_{n,2m+1}$ for $n > r$, therefore $\left\|\mathbf{x} - \mathbf{x}^{(r)}\right\|^2 = \sum_{n=r+1}^{\infty} \sum_{m=0}^{2^{n-1}-1} 2^{-n-1} \left(x_{n,2m} - x_{n,2m+1}\right)^2$. For $r \to \infty$ this is the remainder of a convergent series $x_{00}^2 + \sum_{n=1}^{\infty} \sum_{m=0}^{2^{n-1}-1} 2^{-n-1} \left(x_{n,2m} - x_{n,2m+1}\right)^2 = \|\mathbf{x}\|^2$ and hence converges to 0 for $r \to \infty$. □

Corollary 5.5. *For each finite sequence of vectors* $\mathbf{x}_i = \left(x_{i;nm}\right)_{0\le m<2^n, n\in\mathbb{N}_0} \in Z$, $1 \le i \le s$*:*

$$\left(\bigwedge_{i=1}^{s} \mathbf{x}_i\right)_{nm} = \lim_{N\to\infty} 2^{n-N} \sum_{k=m2^{N-n}}^{(m+1)2^{N-n}-1} \prod_{i=1}^{s} x_{i;Nk} \tag{5.18}$$

Proof. By induction on s one can easily show

$$\left(\bigwedge_{i=1}^{s} \mathbf{x}_i^{(r)}\right)_{nm} = \begin{cases} 2^{n-r} \sum\limits_{k=m2^{r-n}}^{(m+1)2^{r-n}-1} \prod\limits_{i=1}^{s} x_{i;rk} & r \ge n \\ \prod\limits_{i=1}^{s} x_{i;r,\lfloor m2^{r-n}\rfloor} & r \le n \end{cases} \tag{5.19}$$

The corollary now follows from continuity of the $\wedge$-product taking the limit $r \to \infty$. □

Proof of theorem 5.2. We switch to complements, setting $B_i' := \complement B_i$, $C_r' := \complement C_r = \bigcap_{i=1}^{r} B_i'$, thus translating unions into intersections, which is equivalent but in better agreement with our preparations. Now the proof resembles that of [2, Thm.8.2].

In coordinate representation, let the random Borel set B_i' correspond to the process $\xi^{(i)}_{nm}$, then by corollary 5.5 C_r' corresponds to $\xi_{nm} = \lim_{N\to\infty} 2^{n-N} \sum_{k=m2^{N-n}}^{(m+1)2^{N-n}-1} \prod_{i=1}^{r} \xi^{(i)}_{Nk}$. Therefore

$$E\left(\xi_{00}|\xi^{(1)}_{00}, \ldots \xi^{(r)}_{00}\right) = \lim_{N\to\infty} 2^{-N} \sum_{k=0}^{2^N-1} E\left(\prod_{i=1}^{r} \xi^{(i)}_{Nk} \middle| \xi^{(1)}_{00}, \ldots \xi^{(r)}_{00}\right) \tag{5.20}$$

$$= \lim_{N\to\infty} 2^{-N} \sum_{k=0}^{2^N-1} \prod_{i=1}^{r} E\left(\xi^{(i)}_{Nk}|\xi^{(i)}_{00}\right) \tag{5.21}$$

$$= \prod_{i=1}^{r} \xi^{(i)}_{00} \tag{5.22}$$

and with $\xi_{00} = \mu(C_r') = 1 - \mu(C_r)$, $\xi_{00}^{(i)} = \mu(B_i') = 1 - \mu(B_i)$ this leads to

(5.23)

$$1 - E(\mu(C_r)) = \int E\left(\xi_{00}\middle|\xi_{00}^{(1)} = 1 - t_1, \dots \xi_{00}^{(r)} = 1 - t_r\right)(P \otimes \cdots \otimes P)(dt_1 \cdots dt_r)$$

$$= \int \prod_{i=1}^{r} (1 - t_i)(P \otimes \cdots \otimes P)(dt_1 \cdots dt_r) \tag{5.24}$$

$$= \left[1 - \int tP(dt)\right]^r \tag{5.25}$$

Now the second moments are obtained as follows:

(5.26)

$$E\left(\xi_{00}^2|\xi_{00}^{(1)}, \dots \xi_{00}^{(r)}\right) = \lim_{N\to\infty} 2^{-2N}\left[\sum_{k=0}^{2^N-1}\prod_{i=1}^{r} E\left(\xi_{Nk}^{(i)2}|\xi_{00}^{(i)}\right) + \sum_{a\neq b=0}^{2^N-1}\prod_{i=1}^{r} E\left(\xi_{Na}^{(i)}\xi_{Nb}^{(i)}|\xi_{00}^{(i)}\right)\right]$$

$$\begin{aligned} &= \lim_{N\to\infty} 2^{-2N}\left[\sum_{k=0}^{2^N-1}\prod_{i=1}^{r} f_N\left(\xi_{00}^{(i)}\right)\right. \\ &\qquad \left. + \sum_{a\neq b=0}^{2^N-1}\prod_{i=1}^{r}\left(2f_{v(a,b)-1}\left(\xi_{00}^{(i)}\right) - f_{v(a,b)}\left(\xi_{00}^{(i)}\right)\right)\right] \end{aligned} \tag{5.27}$$

Integrating the summand $2^{-2N}\sum_{k=0}^{2^N-1}\prod_{i=1}^{r} f_N\left(\xi_{00}^{(i)}\right)$ over the product measure $P \otimes \cdots \otimes P$ leads to $2^{-N}\left[\int f_N(1-t)P(dt)\right]^r$ and drops out in the limit $N \to \infty$. This leaves us with

(5.28)

$$E\left(\mu(C_r')^2\right) = \lim_{N\to\infty} 2^{-2N}\sum_{a\neq b=0}^{2^N-1}\left[\int\left(2f_{v(a,b)-1}(1-t) - f_{v(a,b)}(1-t)\right)P(dt)\right]^r$$

Observing $\#\{(a,b)|v(a,b) = m\} = 2^{2N-m}$ we arrive at

$$E\left(\mu(C_r')^2\right) = \sum_{m=1}^{\infty} 2^{-m}\left[\int (2f_{m-1}(1-t) - f_m(1-t))P(dt)\right]^r \tag{5.29}$$

(5.30)

$$\begin{aligned} \mathrm{Var}\left(\mu(C_r)^2\right) = \mathrm{Var}\left(\mu(C_r')^2\right) &= \sum_{m=1}^{\infty} 2^{-m}\left[\int (2f_{m-1}(1-t) - f_m(1-t))P(dt)\right]^r \\ &\quad - \left[\int (1-t)P(dt)\right]^{2r} \end{aligned}$$

□

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