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Topological conditions for the representation of preorders by continuous utilities

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Abstract

We remove the Hausdorff condition from Levin's theorem on the representation of preorders by families of continuous utilities. We compare some alternative topological assumptions in a Levin's type theorem, and show that they are equivalent to a Polish space assumption.

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1. Introduction

A topological preordered space is a triple $(E, \mathscr{T}, \le)$, where $(E, \mathscr{T})$ is a topological space endowed with a *preorder* $\le$, that is, $\le$ is a reflexive and transitive relation [18]. A function $f : E \to \mathbb{R}$ is *isotone* if $x \le y \Rightarrow f(x) \le f(y)$, and a *utility* if it is isotone and additionally "$x \le y$ and $y \not\le x \Rightarrow f(x) < f(y)$".

In this work we wish to establish sufficient topological conditions on $(E, \mathscr{T})$ for the representability of the preorder through the family $\mathcal{U}$ of continuous utility functions with value in $[0, 1]$. That is, we look for topological conditions that imply the validity of the following property

$$x \le y \Leftrightarrow \forall f \in \mathcal{U}, f(x) \le f(y).$$

Economists have long been interested in the representation of preorders by utility functions [4]. More recently, this mathematical problem has found application in other fields such as spacetime physics [16] and dynamical systems [1].

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To start with, it will be convenient to recall some notions from the theory of topological preordered spaces [18]. A *semiclosed preordered space* E is a topological preordered space such that, for every point $x \in E$, the increasing hull $i(x) = \{y \in E : x \le y\}$ and the decreasing hull $d(x) = \{y : y \le x\}$, are closed. A *closed preordered space* E is a topological preordered space endowed with a *closed preorder*, that is, the graph $G(\le) = \{(x, y) : x \le y\}$ is closed in the product topology on $E \times E$.

Let E be a topological preordered space. A subset $S \subset E$ is called *increasing* if $i(S) = S$ and *decreasing* if $d(S) = S$, where $i(S) = \bigcup_{s\in S} i(s)$ and analogously for $d(S)$. A subset $S \subset E$ is *convex* if it is the intersection of an increasing and a decreasing set, in which case we have $S = i(S) \cap d(S)$.

A topological preordered space E is *convex* if for every $x \in E$, and open set $O \ni x$, there are an open decreasing set U and an open increasing set V such that $x \in U \cap V \subset O$. Notice that according to this terminology the statement "the topological preordered space E is convex" differs from the statement "the subset E is convex" (which is always true). The terminology is not uniform in the literature, for instance Lawson [14] calls *strongly order convexity* what we call convexity. The topological preordered space E is *locally convex* if for every point $x \in E$, the set of convex neighborhoods of x is a base for the neighborhoods system of x [18]. Clearly, convexity implies local convexity.

A topological preordered space is a *normally preordered space* if it is semiclosed preordered and for every closed decreasing set A and closed increasing set B which are disjoint, it is possible to find an open decreasing set U and an open increasing set V which separate them, namely $A \subset U$, $B \subset V$, and $U \cap V = \varnothing$.

A *regularly preordered space* is a semiclosed preordered space such that if $x \notin B$, where B is a closed increasing set, then there is an open decreasing set $U \ni x$ and an open increasing set $V \supset B$, such that $U \cap V = \varnothing$, and analogously, a dual property must hold for $y \notin A$ where A is a closed decreasing set.

We have the implications: normally preordered space $\Rightarrow$ regularly preordered space $\Rightarrow$ closed preordered space $\Rightarrow$ semiclosed preordered space.

For normally preordered spaces a natural generalization of Urysohn's lemma holds [18, Theor. 1]: If A and B are respectively a closed decreasing set and a closed increasing set such that $A \cap B = \varnothing$, then there is a continuous isotone function $f : E \to [0, 1]$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.

A trivial and well known consequence of this fact is (take $A = d(y)$ and $B = i(x)$ with $x \nleq y$)

Proposition 1.1. *Let E be a normally preordered space and let $\mathcal{I}$ be the family of continuous isotone functions with value in* $[0, 1]$*, then*

$$x \le y \Leftrightarrow \forall f \in \mathcal{I}, f(x) \le f(y). \tag{1.1}$$

This result almost solves our original problem but for the fact that the family of continuous utility functions is replaced by the larger family of continuous isotone functions. Moreover, we have still to identify some topological conditions on $(E, \mathscr{T})$ in order to guarantee that E is a normally preordered space. It is worth noting that Eq. (1.1) is one of the two conditions which characterize the *completely regularly preordered spaces* [18].

Let us recall that a k_ω-space is a topological space characterized through the following property [9]: there is a countable (*admissible*) sequence K_i of compact sets such that $\bigcup_{i=1}^{\infty} K_i = E$ and for every subset $O \subset E$, O is open if and only if $O \cap K_i$ is open in K_i for every i (here E is not required to be Hausdorff).

Recently, the author proved the following results [17]

Theorem 1.2. *Every k_ω-space equipped with a closed preorder is a normally preordered space.*

Theorem 1.3. *Every second countable regularly preordered space admits a countable continuous utility representation, that is, there is a countable set $\{f_k\}$ of continuous utility functions $f_k : E \to [0,1]$ such that*

$$x \le y \Leftrightarrow \forall k, f_k(x) \le f_k(y).$$

Using the previous results we obtain the following improvement of Levin's theorem.[1,2] [15] [4, Lemma 8.3.4]

Corollary 1.4. *Let $(E, \mathscr{T}, \le)$ be a second-countable k_ω-space equipped with a closed preorder, then there is a countable family $\{u_k\}$ of continuous utility functions $u_k : E \to [0,1]$ such that*

$$x \le y \;\Leftrightarrow\; \forall k,\; u_k(x) \le u_k(y).$$

Proof. Every closed preordered k_ω-space is a normally preordered space (Theor. 1.2). Since E is a second countable regularly preordered space it admits a countable continuous utility representation (Theor. 1.3). □

With respect to the references we have removed the Hausdorff condition.[3] Another interesting improvement can be found in [5]. In the remainder of the work we wish to compare this result with other reformulations which use different topological assumptions.

1.1. **Topological preliminaries.** Since in this work we do not assume Hausdorffness of E it is necessary to clarify that in our terminology a topological space is *locally compact* if every point admits a compact neighborhood.

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[1] These references do not consider the representation problem but rather the existence of just one continuous utility. Nevertheless, the argument for the existence of the whole representation is contained at the end of the proof of [4, Lemma 8.3.4].

[2] The result [8, Theor. 1] should not be confused with this one, since their definition of utility differs from our. That theorem can instead be deduced from the stronger theorem 1.2. Also note that in their proof they tacitly use a k_ω-space assumption which can nevertheless be justified.

[3] In [10] it was first suggested that the Hausdorff condition could be removed. This generalization is non trivial and requires some care in the reformulation and generalization of some extendibility results [17].

Definition 1.5. A topological space $(E, \mathscr{T})$ is *hemicompact* if there is a countable sequence K_i, called *admissible*, of compact sets such that every compact set is contained in some K_i (since points are compact we have $\bigcup_{i=1}^{\infty} K_i = E$, and without loss of generality we can assume $K_i \subset K_{i+1}$).

The following facts are well known (Hausdorffness is not required). Every compact set is hemicompact and every hemicompact set is σ-compact. Every locally compact Lindelöf space is hemicompact, and every first countable hemicompact space is locally compact.[4,5]

Definition 1.6. A topological space E is a *k-space* if for every subset $O \subset E$, O is open if and only if, for every compact set $K \subset E$, $O \cap K$ is open in K.

We remark that we use the definition given in [21] and so do not include Hausdorffness in the definition as done in [7, Cor. 3.3.19].

Every first countable or locally compact space is a k-space.[6] Thus under second countability "hemicompact k-space" is equivalent to local compactness.

It is easy to prove that an hemicompact k-space is a k_ω-space and the converse can be proved under T_1 separability (see [20, Lemma 9.3]). Further, in an hemicompact k-space every admissible sequence K_i, $K_i \subset K_{i+1}$, in the sense of the hemicompact definition is also an admissible sequence in the sense of the k_ω-space definition. The mentioned results imply the chain of implications

$$\text{compact} \Rightarrow \text{hemicompact } k\text{-space} \Rightarrow k_\omega\text{-space} \Rightarrow \sigma\text{-compact} \Rightarrow \text{Lindelöf}$$

and the fact that local compactness makes the last four properties coincide.

A continuous function $f : X \to Y$ between topological spaces is said to be a *quasi-homeomorphism* if the following conditions are satisfied [11, 6]:

(i) For any closed set C in X, $f^{-1}(\overline{f(C)}) = C$.

(ii) For any closed set F in Y, $\overline{f(f^{-1}(F))} = F$.

Every quasi-homeomorphism establishes a bijective correspondence $\psi_f : CL(Y) \to CL(X)$ between the closed sets of Y and X through the definition $\psi_f(C) = f^{-1}(C)$.

Remark 1.7. If f is surjective (ii) is satisfied. Furthermore, a quotient (hence surjective) map which satisfies $f^{-1}(f(C)) = C$ for every closed set C (or equivalently, for every open set) is a quasi-homeomorphism. Indeed, if C is closed then $f(C)$ is closed, because of the identity $f^{-1}(f(C)) = C$ and the definition of quotient topology. Thus both properties (i)-(ii) hold, and f is a quasi-homeomorphism. The given argument also shows that f is closed (and open). Furthermore, it can be shown that a quasi-homeomorphism is surjective if and only if it is closed, if and only if it is open [6, Prop. 2.4].

[4] In order to prove the last claim, modify slightly the proof given in [2, p. 486] replacing "Suppose no neighborhood V_i has a compact closure" with "Suppose x has no compact neighborhood".

[5] A first countable Hausdorff hemicompact k-space space need not be second countable. Indeed, as stressed in [9] not even compactness is sufficient as the unit square with a suitable topology provides a counterexample [19, p. 73].

[6] Modify slightly the proof in [21, Theor. 43.9]

2. Ordered quotient and local convexity

On a topological preordered space E the relation $\sim$, defined by $x \sim y$ if $x \le y$ and $y \le x$, is an equivalence relation. Let $E/\!\sim$ be the quotient space, $\mathscr{T}/\sim$ the quotient topology, and let $\lesssim$ be defined by, $[x] \lesssim [y]$ if $x \le y$ for some representatives (with some abuse of notation we shall denote with $[x]$ both a subset of E and a point on $E/\sim$). The quotient preorder is by construction an order. The triple $(E/\sim, \mathscr{T}/\sim, \lesssim)$ is a topological ordered space and $\pi : E \to E/\!\sim$ is the continuous quotient projection.

Remark 2.1. Taking into account the definition of quotient topology we have that every open (closed) increasing set on E projects to an open (resp. closed) increasing set on $E/\!\sim$ and all the latter sets can be regarded as such projections. The same holds replacing *increasing* by *decreasing*. As a consequence, $(E, \mathscr{T}, \le)$ is a normally preordered space (semiclosed preordered space, regularly preordered space) if and only if $(E/\!\sim, \mathscr{T}/\!\sim, \lesssim)$ is a normally ordered space (resp. semiclosed ordered space, regularly ordered space). The effect of the quotient $\pi : E \to E/\!\sim$ on the topological preordered properties has been studied in [13].

Remark 2.2. A set $S \subset E$ is convex if and only if $\pi(S)$ is convex. Indeed, let U and V be respectively decreasing and increasing sets, we have $\pi(U \cap V) = \pi(U) \cap \pi(V)$ because: $U \cap V \subset \pi^{-1}(\pi(U \cap V)) \subset \pi^{-1}(\pi(U) \cap \pi(V)) = \pi^{-1}(\pi(U)) \cap \pi^{-1}(\pi(V)) = U \cap V$.

Proposition 2.3. *Let $(E, \mathscr{T}, \le)$ be a topological preordered space. If local convexity holds at $x \in E$ then $[x]$ is compact and every open neighborhood of x is also an open neighborhood of $[x]$. If E is locally convex then every open set is saturated with respect to π (that is $\pi^{-1}(\pi(O)) = O$ for every open set O). Hence π is a (surjective) quasi-homeomorphism, in particular π is open, closed and proper.*

Proof. Let O be an open neighborhood of x and let C be a convex set such that $x \in C \subset O$, then $[x] = d(x) \cap i(x) \subset d(C) \cap i(C) = C \subset O$, thus O is also an open neighborhood for $[x]$. The compactness of $[x]$ follows easily.

Let $O \subset E$ be an open set and let $x \in O$. We have already proved that $[x] \subset O$. Since this is true for every $x \in O$, we have $\pi^{-1}(\pi(O)) = O$. Therefore, by remark 1.7, since π is a quotient map it is a quasi-homeomorphism which is open and closed. Every such map is easily seen to be proper. □

Remark 2.4. By the previous result under local convexity the quotient π establishes a bijection between the respective families in E and $E/\sim$ of open sets, closed sets, compact sets, increasing sets, decreasing sets and convex sets. Continuous isotone functions on E pass to the quotient on $E/\!\sim$ and conversely, continuous isotone functions on $E/\!\sim$ can be lifted to continuous isotone functions on E. As a consequence, many properties are shared between E and $E/\!\sim$ regarded as topological preordered spaces (one should not apply this observation carelessly, otherwise one would conclude that $\le$ is an order and that $\mathscr{T}$ is Hausdorff). For instance, we have

Proposition 2.5. *If E is a locally convex closed preordered space then $E/\sim$ is a locally convex closed ordered space.*

Proof. We just prove closure to show how the argument works. If $[x] \not\lesssim [y]$ then $x \nleq y$. The representatives x and y are separated by open sets [18, Prop. 1, Chap. 1] U_x and U_y such that $i(U_x) \cap d(U_y) = \varnothing$. By local convexity the increasing neighborhood of x, $i(U_x)$, projects into an increasing neighborhood $\pi(i(U_x))$ of $[x]$. Analogously, $\pi(d(U_y))$ is a decreasing neighborhood of $[y]$ which is disjoint from $\pi(i(U_x))$. We conclude that $\lesssim$ is closed [18, Prop. 1, Chap. 1]. □

The property of closure for the graph of the preorder does not pass to the quotient without additional assumptions [13]. For instance, the previous result holds with "locally convex" replaced by k_ω-space [17].

Remark 2.6. In a topological space $(E, \mathscr{T})$ the *specialization preorder* is defined by $x \preceq y$ if $\overline{x} \subset \overline{y}$. Two points x, y are indistinguishable according to the topology if $x \preceq y$ and $y \preceq x$, denoted $x \simeq y$, since in this case they have the same neighborhoods. The quotient under $\simeq$ of the topological space is called *Kolmogorov quotient* or T_0*-identification* and gives a T_0-space, sometimes called the T_0*-reflection* of E. The Kolmogorov quotient is by construction open, closed and a quasi-homeomorphism.

The first statement of proposition 2.3 implies that under local convexity if $x \sim y$ then x and y have the same neighborhoods, that is, $x \simeq y$. If the preorder $\leq$ on E is semiclosed the converse holds because $\overline{y} = \overline{x} \subset i(x) \cap d(x)$, which implies, $y \sim x$. Thus in a locally convex semiclosed preordered space, π is the Kolmogorov quotient and $E/\sim$ is the T_0-identified space. Actually, $E/\sim$ is a T_1-space because it is a semiclosed ordered space (remark 2.1) thus $\overline{[x]} \subset i_{E/\sim}([x]) \cap d_{E/\sim}([x]) = \{[x]\}$. If additionally E is a closed preordered space we already know that $E/\sim$ is a closed ordered space.

Another way to prove that $\lesssim$ is closed is to observe that the T_0-reflection of a product is the product of the T_0-reflections, that is, $\pi \times \pi$ is the Kolmogorov quotient of $E \times E$, and since the Kolmogorov quotient is closed it sends the closed graph $G(\leq)$ into the graph $G(\lesssim)$ which is therefore closed. In summary we have proved

Proposition 2.7. *Let E be a locally convex semiclosed preordered space then $\pi : E \to E/\sim$ is the T_0-identification of E and $E/\sim$ is T_1. Furthermore, if E is also a closed preordered space then $E/\sim$ is a closed ordered space and hence T_2.*

The next proposition will be useful (see Prop. 3.1) and is an immediate corollary of remark 2.4.

Proposition 2.8. *If $(E, \mathscr{T}, \leq)$ is (locally) convex then $(E/\sim, \mathscr{T}/\sim, \lesssim)$ is (resp. locally) convex. If $(E, \mathscr{T}, \leq)$ is locally convex and locally compact then $(E/\sim, \mathscr{T}/\sim, \lesssim)$ is locally compact, and if additionally E is a closed preordered space then every point of E admits a base of closed compact neighborhoods (but E need not be T_1).*

3. Equivalence of some topological assumptions

We wish to clarify the relative strength of some topological conditions that can be used in a Levin's type theorem.

Let us recall that a *Polish space* is a topological space which is homeomorphic to a separable complete metric space [3, Part II, Chap. IX, Sect. 6]. A *pseudo-metric* is a metric for which the condition $d(x, y) = 0 \Rightarrow x = y$, has been dropped [12]. The relation $x \approx y$ if $d(x, y) = 0$, is an equivalence relation and the quotient $E/\approx$ is a metric space.

A pseudo-metrizable space is a topological space with a topology which comes from some pseudo-metric. In particular, it is Hausdorff if and only if it is metrizable because the Hausdorff property holds if and only if the equivalence classes are trivial. We say that a space is a pseudo-Polish space if it is homeomorphic to a pseudo-metric space and the quotient under $\approx$ is a Polish space. Note that every pseudo-Polish space is separable.

The next result is purely topological (see Prop. 2.7) but at some places it makes reference to a preorder. This is done because it is meant to clarify the topological conditions underlying a Levin's type theorem in which the presence of a closed preorder is included in the assumptions.

Proposition 3.1. *Let us consider the following properties for a topological space* $(E, \mathscr{T})$ *and let* $\le$ *be any preorder on* E *(e.g. the discrete-order)*

(i) *second-countable* k_ω*-space,*
(ii) *second-countable locally compact,*
(iii) *pseudo-metrizable hemicompact* k*-space,*
(iv) *locally compact pseudo-Polish space.*

Then $(iv) \Leftrightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$. *Furthermore, if* $(E, \mathscr{T}, \le)$ *is a locally convex semiclosed preordered space we have* $(i) \Rightarrow (ii)$*, and if* $(E, \mathscr{T}, \le)$ *is a locally convex closed preordered space we have* $(ii) \Rightarrow (iii)$ *(note that the discrete-order is locally convex thus the former implication holds also under* T_1 *separability of* $\mathscr{T}$ *and the latter implication holds also under* T_2 *separability of* $\mathscr{T}$*). In particular they are all equivalent if* $(E, \mathscr{T}, \le)$ *is a locally convex closed preordered space (e.g. under Hausdorffness).*

Proof. We shall make extensive use of results recalled in the introduction.

(ii) $\Rightarrow$ (i). Every second countable locally compact space is an hemicompact k-space and hence a k_ω-space.

(i) $\Rightarrow$ (ii). Assume that $(E, \mathscr{T}, \le)$ is a locally convex semiclosed preordered space. If we prove that E is hemicompact we have finished because first countability and hemicompactness imply local compactness. We have already proved that $(E/\!\sim, \mathscr{T}/\!\sim)$ is T_1 (Prop. 2.7). But $(E/\!\sim, \mathscr{T}/\!\sim)$ is a k_ω-space by a non-Hausdorff generalization of Morita's theorem [17] thus $E/\!\sim$ is hemicompact [20, Lemma 9.3]. Let $\tilde{K}_i$ be an admissible sequence on $E/\!\sim$, since π is proper (Prop. 2.3) the sets $K_i = \pi^{-1}(\tilde{K}_i)$ are compact. They give an admissible sequence for the hemicompact property, indeed if K is any compact on E then

$\pi(K)$ is compact on $E/\sim$ thus there is some $\tilde{K}_i$ such that $\pi(K) \subset \tilde{K}_i$. Finally, $K \subset \pi^{-1}(\pi(K)) \subset \pi^{-1}(\tilde{K}_i) = K_i$.

(ii) $\Rightarrow$ (iii). A second countable locally compact space is an hemicompact k-space. Since $(E, \mathscr{T}, \le)$ is a locally convex closed preordered space, $E/\sim$ is Hausdorff (Prop. 2.7). Local convexity, local compactness, and second countability pass to the quotient $E/\sim$ (see Prop. 2.3,2.8) which is therefore metrizable by Urysohn's theorem. Thus E is pseudo-metrizable with the pullback by π of the metric on $E/\sim$.

(iii) $\Rightarrow$ (ii). A pseudo-metrizable space is second countable if and only if it is separable [12, Theor. 11 Chap. 4] thus it suffices to prove separability. In particular, since E is σ-compact it suffices to prove separability on each compact set K_n (of the hemicompact decomposition) with the induced topology (which comes from the induced pseudo-metric). It is known that every compact pseudo-metrizable space is second countable [12, Theor. 5, Chap. 5] and hence separable, thus we proved that E is second countable. As first countability and the hemicompact property imply local compactness we get the thesis.

(iv) $\Rightarrow$ (iii). E is a separable pseudo-metrizable space thus second countable [12, Theor. 11 Chap. 4]. Second countability and local compactness imply the hemicompact k-space property.

(iii) $\Rightarrow$ (iv). Since (iii) $\Rightarrow$ (ii), E is second countable and locally compact. Let d be a compatible pseudo-metric on E and let $E/\approx$ be the metric quotient. Since $\pi_{\approx} : E \to E/\approx$ is an open continuous map (actually a quasi-homeomorphism) and E is second countable and locally compact then $E/\approx$ is second countable and locally compact too. We conclude by [21, 23C] that the one point compactification of $E/\approx$ is metrizable, and by compactness the one point compactification of $E/\approx$ is completely metrizable. Further, since $E/\approx$ is separable its one point compactification is also separable. We conclude that the one point compactification of $E/\approx$ is Polish, and since $E/\approx$ is Hausdorff and locally compact, $E/\approx$ is an open subset of a Polish space hence Polish [3, Part II, Chap. IX, Sect. 6]. □

Remark 3.2. If E is a locally convex closed preordered space and (iv) holds, as is implied by the assumptions of corollary 1.4, then the local compactness mentioned in (iv) implies, despite the lack of Hausdorffness, the stronger versions of local compactness (Prop. 2.8).

4. Conclusions

We have deduced an improved version of Levin's theorem in which the Hausdorff condition has been removed. Furthermore, some alterative topological assumptions underlying a Levin's type theorem have been compared and we we have shown that the original Levin's theorem included a Polish space assumption.

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References

[1] E. Akin and J. Auslander, *Generalized recurrence, compactifications and the Lyapunov topology*, Studia Mathematica **201** (2010), 49–63.

[2] R. F. Arens, *A topology for spaces of transformations*, Ann. of Math. **47** (1946), 480–495.

[3] N. Bourbaki, *Elements of Mathematics: General topology I*, Reading: Addison-Wesley Publishing (1966).

[4] D. S. Bridges and G. B. Mehta, *Representations of preference orderings*, vol. 442 of *Lectures Notes in Economics and Mathematical Systems.* Berlin: Springer-Verlag (1995).

[5] A. Caterino, R. Ceppitelli and F. Maccarino, *Continuous utility functions on submetrizable hemicompact k-spaces*, Applied General Topology **10** (2009), 187–195.

[6] O. Echi and S. Lazaar, *Quasihomeomorphisms and lattice equivalent topological spaces*, Applied General Topology **10** (2009), 227–237.

[7] R. Engelking, *General Topology*, Berlin: Helderman Verlag (1989).

[8] O. Evren and E. A. Ok, *On the multi-utility representation of preference relations*, J. Math. Econ. **47** (2011), 554–563.

[9] S. T. Franklin and B. V. Smith Thomas, *A survey of k_ω-spaces*, Topology Proceedings **2** (1977), 111–124.

[10] G. Herden and A. Pallack, *On the continuous analogue of the Szpilrajn theorem I*, Mathematical Social Sciences **43** (2002), 115–134.

[11] Y. Kai-Wing, *Quasi-homeomorphisms and lattice-equivalences of topological spaces*, J. Austral. Math. Soc. **14** (1972), 41–44.

[12] J. L. Kelley, *General Topology*, New York: Springer-Verlag (1955).

[13] H.-P. A. Künzi and T. A. Richmond, *T_i-ordered reflections*, Applied General Topology **6** (2005), 207–216.

[14] J. D. Lawson, *Order and strongly sober compactifications*, Oxford: Clarendon Press, vol. Topology and Category Theory in Computer Science, pages 171–206 (1991).

[15] V. L. Levin, *A continuous utility theorem for closed preorders on a σ-compact metrizable space*, Soviet Math. Dokl. **28** (1983), 715–718.

[16] E. Minguzzi, *Time functions as utilities*, Commun. Math. Phys. **298** (2010), 855–868.

[17] E. Minguzzi, *Normally preordered spaces and utilities*, Order (2011), `DOI: 10.1007/s11083-011-9230-4, arXiv:1106.4457v2`.

[18] L. Nachbin, *Topology and order*, Princeton: D. Van Nostrand Company, Inc. (1965).

[19] L. A. Steen and J. A. Seebach, Jr., *Counterexamples in topology*, New York: Holt, Rinehart and Winston, Inc. (1970).

[20] N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152.

[21] S. Willard, *General topology*, Reading: Addison-Wesley Publishing Company (1970).

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