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Supersymmetry and the Hopf fibration

Simon Davis

Abstract

The Serre spectral sequence of the Hopf fibration $S^{15} \overset{S^7}{\rightarrow} S^8$ is computed. It is used in a study of supersymmetry and actions based on this fibration.

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1. Introduction

There are compactifications of eleven-dimensional supergravity with an $SU(3)\times SU(2)\times U(1)$ isometry group of the compact space that are known to yield the particle spectrum of the standard model [1], [2]. The fermion multiplets can be included in a spinor space represented by a tensor product of division algebras for each generation. The automorphism group of this product would be $G_2 \times SU(2) \times U(1)$ and it may be demonstrated that there are coset spaces $\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}$ yielding particles and antiparticles with the known quantum numbers [3].

The dimensions of the normed real alternative division algebras correspond to the parallelizability of the spheres. The spheres S^1, S^3 and S^7 in the reduction sequence of the unified field theory represent submanifolds of the higher-dimensional coset space. The representation of unit elements in the components of the spinor space could be related to the fermion bilinears arising in the set of light-like lines in two larger dimensions, yielding S^2, S^4 and S^8. The unit fermions can exist in a fibre of a bundle over the space of light-like lines. Amongst the S^7 bundles over S^8 is the Hopf fibration $S^{15} \overset{S^7}{\rightarrow} S^8$. A classification of physical states described by the Hopf fibrations is given in §2.

It has been demonstrated previously that an S^7 transformation rule cannot be constructed for a pure Yang-Mills theory with the connection taking values only on the four-dimensional base space [4]. Since twistor variables that transform under $Sp(4;\mathbb{O})$ can be combined to transform parameterize S^7 [5], the problem of constructing a model with this invariance may be considered. This can be done only if the space S^8 of lightlike lines of octonionic superparticles is interpreted in terms of fundamental variables in the theory.

If the fermion field is allowed to take values in a one-point compactification of the space identified with the division algebra, an equivalence with the bosonic sector given by the light-like lines can provide a basis for a supersymmetry. This approach can be compared to an algebraic description of the supersymmetric Hopf fibration. When the base space super-sphere S_*^2, a supersymmetric version of the $U(1)$ theory is found [6]. The spectral sequences of the Hopf fibrations of the superspheres and the homology groups are found to unaltered by the introduction of supersymmetry in §3.

The effect of an S^7 transformation on fields in the twistor formalism can be elevated to an invariant action directly because there are anomalous terms in the commutators. Although various spinor bilinears and combinations of supertwistors are found to be invariant, there is a associator term with a spinor field, which must be cancelled for invariance under the composition of these transformations. A method for eliminating the additional terms through the commutator of BRST and gauge transformations is suggested in §4.

2. Spectral Sequences and Hopf Fibrations

It may be recalled that the homology group of the total space of a fibre bundle may be determined from the Serre spectral sequence. For a filtration $X_0 \subset X_1 \subset X_2 \subset ... \subset X$, let $D = \oplus_{m,n} H_m(X_n)$ and $E = \oplus_{m,n} H_m(X_n, X_{n-1})$ define an exact couple such that $im\ j = ker\ k$ where $j : D \to E$ and $k : E \to D$. Let $D' = i(D)$ and $d = jk : E \to E$ with $d^2 = 0$. Suppose that $E' = H(E; d)$, $i' = i|'_D$ and the map $j' : D' \to E'$ is defined by $j'(x)$ is the coset of $j(y)$ in $Z(E)$, where $x = i(y) \in D'$ for $y \in D$. The map k' completes the exact sequence and (D', E', i', j', k') is an exact couple. Further iterations give $D^n, E^n; i_n, j_n, k_n)$ such that $d_n = j_n k_n : E^n \to E^n$, where $E^{n+1} = H(E^n; d^n)$, and $D = \oplus_{p,q} D_{p,q}$ and $E = \oplus_{p,q} E_{p,q}$. Then $i(D_{p,q}) \subset D_{p+1,q-1}$, $j(D_{p,q}) \subset E_{p,q}$ and $k(E_{p,q}) \subset D_{p-1,q}$. If $D^n_{p,q} = D^n \cap D_{p,q}$ and $E^n_{p,q} = H(E^{n-1}_{p,q}; d_n)$, $i(D^n_{p,q}) \subset D^n_{p+1,q-1}$, $j_n(D^n_{p,q}) \subset E^n_{p-n+1,q+n-1}$ and $k_n(E^n_{p,q}) \subset D^n_{p-1,q}$, while $d_n : E^n_{p,q} \to E^n_{p-n,q+n-1}$.

For a fibre bundle, $E \overset{F}{\to} B$, $E^2_{p,q} = H(E^1_{p,q}; d_1) = H_p(B; H_q(F))$ [7]. Consider the Hopf fibration $S^7 \overset{S^3}{\to} S^4$.

$$E^2_{p,q} = H_p(S^4; H_q(S^3)) = \begin{cases} H_p(S^4) & q = 0,3 \\ 0 & otherwise \end{cases} \tag{2.1}$$
$$= \begin{cases} \mathbb{Z} & p = 0,4,\ q = 0,3 \\ 0 & otherwise \end{cases}$$

The boundary mapping d_4 is injective as there would exist an element $y \neq d_4 x$ mapped to zero otherwise, implying that $E^5_{4,0} = H^4(E^4_{4,0}; d_4) \neq 0$. This latter statement would imply

$$E^5_{4,0} \simeq ... \simeq E^\infty_{4,0} \not\simeq 0 \tag{2.2}$$

contrary to $H^4(S^7) \simeq 0$. Also, d_4 is surjective, because $E^5_{0,3} \simeq ... \simeq E^\infty_{0,3} \simeq 0$ since $H_3(S^7) \simeq 0$. It follows that d_4 is an isomorphism and $d_4 : E^4_{4,3} \to E^4_{0,6}$ is surjective. Since $E^5_{4,0} \simeq 0$, $E^5_{3,0} \simeq d_4(E^5_{4,0}) \simeq 0$, removing the $(0,4)$ and $(3,0)$ elements in the sequence for $E^5_{p,q}$. The remaining non-zero entries in the $E^5_{p,q}$ sequence may be deduced from exact sequences derived for filtrations of the total space.

Given that $D_{p,q} = 0$ for $p < 0$ and $E_{p,q} = 0$ for $p < 0$ or $q < 0$, $E^n_{p,q} = 0$ for $p < 0$, $q < 0$ and $p+q < 0$. For large n, there are the following exact sequences:

$$\begin{array}{ccccccccc} \to E^n_{p+n-1,q-n+2} & \overset{k_n}{\to} & D^n_{p+n,q-n+2} & \overset{i_n}{\to} & D^n_{p+n-1,q-n+1} & \overset{j_n}{\to} & E^n_{p,q} \overset{k_n}{\to} & D^n_{p-1,q} & \to \ ... \\ & & i \downarrow & & i\downarrow & & & & \\ 0 & \to & D^{n+1}_{p+n-2,q-n+2} & \overset{i_{n+1}}{\to} & D^{n+1}_{p+n-1,q-n+1} & \overset{j_{n+1}}{\to} & E^{n+1}_{p-1,q+1} & \to & ... \ . \end{array} \tag{2.3}$$

There is a related sequence

$$0 \ \to \ D^{n+1}_{p+n-2,q-n+2} \ \overset{i_{n+1}}{\to} D^{n+1}_{p+n-1,q-n+1} \ \overset{j_n}{\to} \ E^{n+1}_{p,q} \ \to \ ... \tag{2.4}$$

which holds if the domain of j_n can be chosen such that $j_n(D^{n+1}_{p+n-1,q-n+1}) = E^{n+1}_{p,q}$. Since $D^{n+1}_{p,q} = i(D^n_{p,q}) \subseteq D^n_{p,q}$, this is feasible, although j_n has not been defined to have the range $E^{n+1}_{p,q}$. This could be ensured, however, if $E^{n+1}_{p,q} = E^n_{p,q}$. From the sequence (2.4), it follows that the exactness of the sequence

$$E^n_{p+n,q-n+1} \ \overset{d_n}{\to} \ E^n_{p,q} \ \overset{d_n}{\to} \ E^n_{p-n,q+n-1} \tag{2.5}$$

which is equivalent to

$$0 \ \overset{d_n}{\to} \ E^n_{p,q} \ \overset{d_n}{\to} \ 0 \tag{2.6}$$

for $n > p$ and $n > q+1$, implying the constancy of $E^n_{p,q}$ for large n.

From the complex of sequences

$$\begin{array}{cccccccccc} 0 & \to & D^{n+1}_{p+n-2,q-n+2} & \overset{i_{n+1}}{\to} & D^{n+1}_{p+n-1,q-n+1} & \overset{j_n}{\to} & E^{n+1}_{p,q} & \overset{k_n}{\to} & D^{n+1}_{p-1,q} & \to \ ... \\ & & i\downarrow & & i\downarrow & & & & & \\ 0 & \to & D^{n+2}_{p+n-2,q-n+2} & \overset{i_{n+1}}{\to} & D^{n+2}_{p+n-1,q-n+1} & \overset{j_n}{\to} & E^{n+2}_{p,q} & \overset{k_{n+2}}{\to} & D^{n+2}_{p-1,q} & \to \ ... \\ & & ... & & & & & & & \\ 0 & \to & D^\infty_{p+n-2,q-n+2} & \overset{i_\infty}{\to} & D^\infty_{p+n-1,q-n+1} & \overset{j_\infty}{\to} & E^\infty_{p,q} & \overset{k_\infty}{\to} & D^\infty_{p-1,q} & \to \ ... \end{array} \tag{2.7}$$

the last exact sequence does not end. It implies

$$0 \to D^{\infty}_{p-1,q+1} \overset{i_\infty}{\to} D^{\infty}_{p,q} \overset{j_1}{\to} E^{\infty}_{p,q} \overset{k_\infty}{\to} D^{\infty}_{p-1,q} \to ... \tag{2.8}$$

when $n = 1$ is substituted in the final sequence of Eq.(2.7).

For the sequences,

$$\begin{array}{lllllll}
D^{n+1}_{p+n-1,q-n+1} & \overset{j_{n+1}}{\to} & E^{n+1}_{p-1,q+1} & \overset{k_{n+1}}{\to} & D^{n+1}_{p-2,q+1} & \to & ... \\
i\downarrow & & & & & & \\
D^{n+2}_{p+n-1,q-n+1} & \overset{j_{n+2}}{\to} & E^{n+2}_{p-2,q+2} & \overset{k_{n+2}}{\to} & D^{n+2}_{p-3,q+2} & \to & ... \\
i\downarrow & & & & & & \\
\vdots & & & & & & \\
D^{n+p-1}_{p+n-1,q-n+1} & \overset{j_{n+p-1}}{\to} & E^{n+p-1}_{1,p+q-1} & \overset{k_{n+p-1}}{\to} & D^{n+p-1}_{0,p+q-1} & \to & ... \\
i\downarrow & & & & & & \\
D^{n+p}_{p+n-1,q-n+1} & \overset{j_{n+p}}{\to} & E^{n+p}_{0,p+q} & \overset{k_{n+p}}{\to} & D^{n+p}_{-1,p+q} \simeq 0 & &
\end{array} \tag{2.9}$$

and

$$0 \quad \to \quad D^{n+p}_{p+n-2,q-n+2} \quad \overset{i_{n+p}}{\to} \quad D^{n+p}_{p+n-1,q-n+1} \quad \overset{j_{n+p}}{\to} \quad E^{n+p}_{0,p+q} \quad \overset{k_{n+p}}{\to} \quad 0 \tag{2.10}$$

yielding eventually the sequence

$$0 \to \quad D^{\infty}_{p+n-2,q-n+2} \quad \overset{i_\infty}{\to} \quad D^{\infty}_{p+n-1,q-n+1} \quad \overset{j_\infty}{\to} \quad E^{\infty}_{0,p+q} \quad \overset{k_\infty}{\to} \quad 0. \tag{2.11}$$

With $n = 1$ in the indices

$$0 \quad \to \quad D^{\infty}_{p-1,q+1} \quad \overset{i_\infty}{\to} \quad D^{\infty}_{p,q} \quad \overset{j_\infty}{\to} \quad E^{\infty}_{0,p+q} \quad \overset{k_\infty}{\to} \quad 0 \tag{2.12}$$

implying $E^{\infty}_{0,p+q} \simeq D^{\infty}_{p,q}/i_\infty(D^{\infty}_{p-1,q+1})$. Since $H_{p+q}(X_{p-1}) = D^{\infty}_{p-1,q+1} \subset H_{p+q}(X_p) = D^{\infty}_{p,q}$, $D^{\infty}_{p-1,q+1} \subset D^{\infty}_{p,q} \subset D^{\infty}_{p+1,q-1} \subset ... \subset D^{\infty}_{p+n,q-n} \subset ....$ For n sufficiently large, $D^{\infty}_{p+n,q-n} = D^{\infty} \cap D_{p+n,q-n} = D^{\infty} \cap H_{p,q}(X_{p+n}) = D^{\infty} \cap H_{p,q}(X)$ is constant, when the exhaustion of X contains a finite sequence of proper subspaces.

The sequences

$$
\begin{array}{cccccccccc}
0 & \rightarrow & D^{n+p}_{p+n-2,q-n+2} & \overset{i_{n+p}}{\rightarrow} & D^{n+p}_{p+n-1,q-n+1} & \overset{j_{n+p}}{\rightarrow} & E^{n+p}_{0,p+q} & \overset{k_{n+p}}{\rightarrow} & ...0 & \\
 & & i\downarrow & & i\downarrow & & & & & \\
0 & \rightarrow & D^{n+p+1}_{p+n-2,q-n+2} & \overset{i_{n+p+1}}{\rightarrow} & D^{n+p}_{p+n-1,q-n+1} & \overset{j_{n+p+1}}{\rightarrow} & E^{n+p+1}_{0,p+q} & \overset{k_{n+p+1}}{\rightarrow} & 0 & \\
 & & i\downarrow & & i\downarrow & & & & & \\
 & & & \vdots & & & & & & \\
0 & \rightarrow & D^{n+n'}_{p+n-2,q-n+2} & \overset{i_{n+n'}}{\rightarrow} & D^{n+n'}_{p+n-1,q-n+1} & \overset{j_{n+n'}}{\rightarrow} & E^{n+n'}_{0,p+q} & \overset{k_{n+n'}}{\rightarrow} & 0 & \\
 & & & \vdots & & & & & & \\
0 & \rightarrow & D^{\infty}_{p+n-2,q-n+2} & \overset{i_{\infty}}{\rightarrow} & D^{\infty}_{p+n-1,q-n+1} & \overset{j_{\infty}}{\rightarrow} & E^{\infty}_{0,p+q} & \overset{k_{\infty}}{\rightarrow} & 0 &
\end{array}
\tag{2.13}
$$

yield the isomorphisms

$$
\begin{aligned}
E^{n+p}_{0,p+q} &\simeq D^{n+p}_{p+n-1,q-n+1}/i_{n+p}(D^{n+p}_{p++n-2,q-n+2}) \\
&\vdots \\
E^{n+n'}_{0,p+q} &\simeq D^{n+n'}_{p+n-1,q-n+1}/i_{n+n'}(D^{n+n'}_{p+n-2,q-n+2}) \\
&\vdots \\
E^{\infty}_{0,p+q} &\simeq D^{\infty}_{p+n-1,q-n+1}/i_{\infty}(D^{\infty}_{p+n-2,q-n+2}).
\end{aligned}
\tag{2.14}
$$

It is apparent that

$$
\begin{aligned}
D^{n+n'}_{p+n-1,q-n+1} &\simeq D'\cap D_{p+n-1,q-n+1} = D'\cap H_{p+q}(X_{p+n-1}) \\
D^{n+n'}_{p+n-2,q-n+2} &\simeq D'\cap D_{p+n-2,q-n+2} = D'\cap H_{p+q}(X_{p+n-2}).
\end{aligned}
\tag{2.15}
$$

For sufficiently large n, $H_{p+q}(X_{p+n-1}) \simeq H_{p+q}(X_{p+n-2}) \simeq H_{p+q}(X)$ and

$$
\begin{aligned}
E^{n+p}_{0,p+q} &\simeq [D'\cap H_{p+q}(X)]/i_{n+p}(D'\cap H_{p+q}(X)) \\
&\vdots \\
E^{n+n'}_{0,p+q} &\simeq [D'\cap H_{p+q}(X)]/i_{n+n'}(D'\cap H_{p+q}(X)) \\
&\vdots \\
E^{\infty}_{0,p+q} &\simeq [D'\cap H_{p+q}(X)]/i_{\infty}(D'\cap H_{p+q}(X))
\end{aligned}
\tag{2.16}
$$

are the quotient groups related to $H_{p+q}(X)$, and $D' = i(D) = i(\oplus_{p,q}D_{p,q})$
$= i(\oplus_{p,q}H_{p+q}(X_p))$.

Fixing $p+q$,

$$\begin{aligned}\oplus_{p+q=constant} H_{p+q}(X_p) &= H_{p+q}(X_0) \oplus H_{p+q}(X_1) \oplus ... \oplus H_{p+q}(X_m) \oplus ... \\ &\oplus H_{p+q}(X) \\ &\simeq H_{p+q}(X) \\ \oplus_{p,q} H_{p+q}(X_p) &\simeq \oplus^{\infty}_{p+q=-\infty} H_{p+q}(X) \\ \oplus^{\infty}_{p'+q'=0} H_{p'+q'}(X) \cap H_{p+q}(X) &\simeq H_{p+q}(X)\end{aligned} \tag{2.17}$$

and the following isomorphisms hold:

$$\begin{aligned}E^{n+p}_{0,p+q} &\simeq H_{p+q}(X)/i_{n+p}(H_{p+q}(X)) \\ &\vdots \\ E^{\infty}_{0,p+q} &\simeq H_{p+q}(X)/i_{\infty}(H_{p+q}(X)).\end{aligned} \tag{2.18}$$

From the sequences (2.8) and

$$0 \to D^{\infty}_{p-1,q+1} \overset{\iota_\infty}{\to} D^{\infty}_{p,q} \overset{j_2}{\to} E^{\infty}_{p-1,q+1} \overset{k_\infty}{\to} D^{\infty}_{p-2,q+1} \to .. \tag{2.19}$$

isomorphisms of the form $E^{\infty}_{p-1,q+1} \simeq E^{\infty}_{p,q} \simeq ...$ may be deduced, and $E^{\infty}_{p,q} \simeq D^{\infty}_{p,q}/i_{\infty}(D^{\infty}_{p-1,q+1})$. By the exact sequence (2.12), $i_{\infty}(D^{\infty}_{p-1,q+1}) = ker\ j_{\infty}(D^{\infty}_{p,q})$ consists of the identity element, because j_∞ must be an injective homomorphism as $j_{\infty}(D^{\infty}_{p,q}) \subset E^{\infty}_{p,q} = E^{\infty}_{0,p+q}$, and $E^{\infty}_{p,q} = H_{p+q}(X)$.

It follows that, for the Hopf fibration $S^7 \overset{S^3}{\to} S^4$, $E^{\infty}_{p,q} \simeq H_{p+q}(S^7)$ and

$$H_r(S^7) \simeq \begin{cases} \mathbb{Z} & r = 0,7 \\ 0 & \text{otherwise} \end{cases}. \tag{2.20}$$

For the Hopf fibration $S^3 \overset{S^1}{\to} S^2$, there exist multi-soliton solutions parameterized by the homotopy group $\pi_3(S^2)$ [8]. Similarly the homotopy group $\pi_7(S^4)$ could be used to parameterize the Hopf number of soliton solutions to theories based on the next Hopf fibration, as the one-soliton solutions can be combined to give N-soliton solutions. From the Hurewicz theorem [9], the k^{th} homology and homotopy groups of the sphere S^k are isomorphic to $\mathbb{Z}$. By the exact sequence $S^3 \to S^7 \to S^4$, it follows that $\pi_7(S^4) = \pi_7(S^7) \oplus \pi_6(S^3) = \mathbb{Z} \oplus \mathbb{Z}_{12}$ [10] [11], implying that there would be twelve varieties of each N-soliton.

For the Hopf fibration $S^{15} \overset{S^7}{\to} S^8$,

$$E^2_{p,q} = H_p(S^8; H_q(S^7)) = \begin{cases} \mathbb{Z} & p = 0,8,\ q = 0,7,\ p+q = 15 \\ 0 & \text{otherwise} \end{cases}. \tag{2.21}$$

Since every element in $E^8_{7,0}$ is $d_8 x$, $x \in E^8_{0,8}$, And $E^9_{7,0} \simeq H_7(E^8_{7,0}; d_8) \simeq 0$. Similarly, $d_8 : E^{7,0}_8 \to E^{0,8}_8$. Therefore,

$$E^{0,8}_9 \simeq H^8(E^{0,8}_8; d_8) \simeq 0 \tag{2.22}$$

Therefore, $E^9_{0,8} \simeq 0$. Again $E^9_{p,q} \simeq ... \simeq E^\infty_{p,q}$ and

$$E^\infty_{p,q} \simeq H_{p+q}(S^{15}) \simeq \begin{cases} \mathbb{Z} & p=q=0,\ p=8,\ q=7,\ p+q=15 \\ 0 & \text{otherwise} \end{cases}, \tag{2.23}$$

which is consistent with $H_r(S^{15}) \simeq \mathbb{Z}$ $r = 0,\ 15$ and $H_r(S^{15}) \simeq 0$ otherwise.

An instanton solution to the Yang-Mills equations related to the last Hopf fibration has been found with the Euler number equal to $N\int_{S^8}(F \wedge F \wedge F \wedge F \wedge F)dV$ [12]. However, there is no reference to the gauge instanton in the Euler number, which is a topological invariant that is entirely characteristic of the spheres in the fibration. This result has been explained through the equivalence of this integral with that of the Pfaffian of $\frac{1}{2\pi}\hat{F}$, where $\hat{F}_{\mu\nu}$ is the field of the spinor connection [13].

The expression for the Euler number is derived from the curvature form. However, the formula for the curvature form of the spinor connection is given by $\Omega^{\mu\nu} = e^\mu \wedge e^\nu$ [13], and it would appear that equivalence with a volume form would follow. The Hopf invariant, given by the integral $\int_{S^{15}} \alpha \wedge d\alpha$, where α is a volume form on S^7, can be projected to $\int_{S^8} d\alpha_s$, where α_s is a singular form as a result of the intersections of the seven-spheres, which has integer values. While the Hopf invariant is equal to the number of links of seven-spheres in S^{15}, its integrality is similar, therefore, to that of the Euler class, which is a generator of a homology group isomorphic to $\mathbb{Z}$.

Upon deriving an N-soliton configuration from an N-instanton solution, the classification would be given by the homotopy group $\pi_{15}(S^8) = \pi_{15}(S^{15}) \oplus \pi_{14}(S^7) = \mathbb{Z} \oplus \mathbb{Z}_{120}$ [11].

3. Supersymmetric Hopf Fibrations

It has been shown that there is a generalization of the Hopf fibration $S^3 \overset{S^1}{\rightarrow} S^2$ to $SU(2)_* \overset{U(1)_*}{\rightarrow} S^2_*$, where each of the spaces is a supersphere [14]. The analogue of an element of $SU(2)$ represented by $s(t) = exp\,(iT^a\epsilon_a(t))$, $T^a = \frac{\sigma_a}{2}$, is

$$\begin{aligned} s_*(t,\theta) &= exp\,(iT^a\eta_a(t,\theta)) \\ \eta_a(t,\theta) &= \epsilon_a(t) - 2\theta i\xi_a(t) \end{aligned} \tag{3.1}$$

From Eq.(3.1),

$$\begin{aligned} s_*(t,\theta) &= 1 + iT^a(\epsilon_a(t) - 2\theta i\xi_a(t)) - \frac{1}{2!}T^a(\epsilon_a(t) - 2\theta i\xi_a(t))T^b(\epsilon_b(t) - 2\theta i\xi_b(t)) + ... \\ &= (1+\theta\sigma^a\xi_a(t))(1 + iT^a\epsilon_a(t) - \frac{1}{2!}T^aT^b\epsilon_a(t)\epsilon_b(t) + ...) \\ &= (1+\theta\xi)s(t) \qquad \xi = \sigma^a\xi_a(t) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} s^\dagger_*(t,\theta)s_*(t,\theta) &= s(t)^\dagger(1-\theta\xi)(1+\theta\xi)s(t) \\ &= s(t)^\dagger(1 - \theta\xi + \theta\xi + \theta\xi\theta\xi)s(t) \\ &= s(t)^\dagger s(t) = 1 \end{aligned} \tag{3.3}$$

The action of $U(1)_*$ on $SU(2)_*$ is $s_* \to s_* e^{i\sigma_3 \alpha}$ and the projection from $SU(2)$ to S^2, $s(t) \to s(t)\sigma_3 s(t)^\dagger$ is generalized such that

$$\hat{x}_* = \hat{x}_{*a}\sigma = s_*\sigma_3\sigma_*^\dagger \tag{3.4}$$

parameterizes S^2_*.

This space may be compared with the supersphere $S^{2,2}$ defined as $OSp(1|2)/U(1)$, which has even coordinates x_i and odd coordinates θ_α satisfying

$$\sum_i x_i x_i + \sum_{\alpha,\beta} C_{\alpha\beta}\theta_\alpha\theta_\beta \tag{3.5}$$

where $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ [15]. The coordinates of S^2 are given by

$$y_i = \left(1 + \frac{\theta C \theta}{2r^2}\right) \tag{3.6}$$

and $\sum_i y_i y_i = r^2$. The action of the $U(1)_*$ is right multiplication by an unitary group element and therefore identified with the action of $U(1)$.

Since the supersymmetry algebra has the form $\{D^*, D\} = 0$, where

$$D = \frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t} \tag{3.7}$$

is similar to the exterior derivative operator, it might be considered useful to determine de Rham cohomology for the supersymmetric Hopf fibration. The graded differential calculus on a supersphere can be constructed such that

$$\begin{aligned}(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^p \omega \wedge d\omega' \\ \omega \in \Omega^p(S^{m,n}),\ \omega' \in \Omega^{p'}(S^{m,n})\end{aligned} \tag{3.8}$$

where $\Omega^p(S^{m,n})$ and $\Omega^{p'}(S^{m,n})$ are exterior form algebra. and $d^2 = 0$. As the dimension of a supermanifold belongs to $\mathbb{Z}[\epsilon]/(\epsilon^2 - 1) = \mathbb{Z} \oplus \mathbb{Z}\epsilon$, there is an isomorphism of the de Rham cohomology of a supermanifold with that of the underlying manifold [16][17]. The de Rham cohomology groups of the spheres have given

$$H^k_{dR}(S^n) \sim \begin{cases} \mathbb{R} & \text{if } k = 0, n \\ 0 & \text{if } k \neq 0, n \end{cases} \tag{3.9}$$

whereas,

$$H^k_{dR}(S^n; \mathbb{Z}) \sim \begin{cases} \mathbb{Z} & \text{if } k = 0, n \\ 0 & \text{if } k \neq 0, n \end{cases} \tag{3.10}$$

By de Rham's theorem, there is an isomorphism between the de Rham cohomology group $H^k_{dR}(M)$ and the cohomology groups $H^k(M; \mathbb{R})$ for any smooth manifold. From the commutative diagram of isomorphisms, it follows that the spectral sequences based on the homology groups of spheres could be adapted to the superspheres after specializing to a specific coefficient field. Consequently, the results of §2 may be used for the superspheres and each of the supersymmetric Hopf fibrations, based on the exact sequences.

$$\begin{array}{l} 0 \rightarrow S_*^3 \rightarrow S_*^7 \rightarrow S_*^4 \rightarrow 0 \\ 0 \rightarrow S_*^7 \rightarrow S_*^{15} \rightarrow S_*^8 \rightarrow 0 \end{array} \tag{3.11}$$

4. The Action of S^7 and its Supersymmetric Generalization

Although it has been demonstrated that the principal bundle structure of gauge theories is dependent on a Lie group structure, the action of S^7 has been developed for twistor variables. In ten dimensions, the momentum vector of a massless particle can be expressed as $p = \psi\psi^\dagger$, where ψ is a spinor that traces out S^8 . By the action of S^7 on S^8, consistent with that of the Hopf fibration, there exists a transformation $\delta\psi^\alpha = T\psi^\alpha = \psi^\alpha o^{(\alpha)}$ such that

$$\begin{aligned} [T, T']\psi^\alpha &= o^{(\alpha)}(o'^{(\alpha)}\psi^\alpha) - o'^{(\alpha)}(o^\alpha\psi^\alpha) \\ &= ([o^{(\alpha)}, o'^{(\alpha)}] - 2[o^{(\alpha)}, o'^{(\alpha)}, e^{(\alpha)}]\bar{e}^{(\alpha)})\psi^\alpha \\ e^{(\alpha)} &= |\psi^\alpha|^{-1}\psi^\alpha \end{aligned} \tag{4.1}$$

the action is not covariant, and this prevents the construction of an entirely invariant action [18].

For a supersymmetric particle, the variables

$$\begin{aligned} \xi &= \psi\theta^\dagger + \theta\psi^\dagger \\ \omega &= X\psi + i\xi\theta \\ Z^A &= (\psi^\alpha, \omega_{\dot{\alpha}}) \end{aligned} \tag{4.2}$$

may be used to construct invariants under the action of S^7, $\psi \to \psi o$, $\omega \to \omega o$, $|o| = 1$,

$$J^{MN} = \left[\frac{1}{2} Z^\dagger \Gamma^{MN} Z\right] \tag{4.3}$$

where the square brackets refer to the selection of the e^0 component. While the components of J^{MN}, $\psi^1\bar{\psi}^2$, $\omega^1\bar{\omega}^2$, $\psi^1\bar{\omega}^2$ and $\psi^2\bar{\omega}^1$ are separately invariant, the repeated action of the S^7 transformations generates additional terms through the nonvanishing associator. . One method for eliminating the extra term may be based on the construction of a charge which could cancel the associator containing either $e^{(1)}$ or $e^{(2)}$. If this is included in the S^7 transformation, it would cause the transformations to be covariant. The BRST charge, for example, is typically constructed such that an exact term does not affect the invariance of the Lagrangian under local gauge transformations. However, this would depend on the associativity of the operations of the gauge transformation and BRST transformation. Through the variables transforming under $SL(2; \mathbb{O})$, an additional term derived from an associator containing the two transformations would be introduced.

Since $d = jk$ in the exact couple, a correspondence between spectral sequences and the theories with an operator satisfying $d^2 = 0$ can be established. The BRST cohomology of quantum field theories has been calculated previously with spectral sequences [19]. Similarly, in supersymmetric models, the anticommutator of the supercharge satisfies $\{Q_\alpha, Q^\dagger_{\dot{\alpha}}\} = \sigma^\mu_{\alpha\dot{\alpha}} P_\mu$, and the trace will be non-zero except for a vacuum with zero energy. The existence of a spectral sequence only for the ground state is indicative of a connection with the index $Tr\ (-1)^F$.

The necessity of the BRST charge in the study of the action of S^7 transformations on components of the supertwistor and the BRST cohomology of ten-dimensional supergravity and superstring theories implies that the role of the anomaly, which can be determined through spectral sequences, is similar to that of the quantum terms representing lack of closure of the classical S^7 algebra on spinor fields.

5. Conclusion

The homology groups of the total spaces in the Hopf fibrations are calculated with the spectral sequence. The conditions for the equality of $E^{\infty}_{p,q} = H_{p+q}(X)$ will be satisfied if $E^n_{p,q}$ is constant for sufficiently large n. The derived homology groups $E^n_{p,q}$ are found to be constant if $n \geq 5$ for the fibration $S^7 \overset{S^3}{\to} S^4$. and $n \geq 9$ for the fibration $S^{15} \overset{S^7}{\to} S^8$. The classification of solitons resulting from these fibrations are given by the homotopy groups $\pi_7(S^4)$ and $\pi_{15}(S^8)$ respectively.

With the introduction of supersymmetry, the spectral sequences for the superspheres would have the same form. The homotopy groups determining the classification of the soliton states would follow. The description of the spaces in the third and fourth supersymmetric fibrations may be given together with a derivation of the homology and homotopy groups of the superspheres.

It is suggested that an invariant action under S^7 transformations can be developed if a BRST charge is added to the theory. This BRST charge generates another cohomology, which may be evaluated for a general quantum field theory. The consistency of the theory through a vanishing anomaly provides conditions on the matter content.

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Simon Davis (sbdavis@resfdnsca.org)
Research Foundation of Southern California, 8837 Villa La Jolla Drive #13595, La Jolla, CA 92039, USA.