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New matrix partial order based on spectrally orthogonal matrix decomposition

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Abstract

We investigate partial orders on the set of complex square matrices and introduce a new order relation based on spectrally orthogonal matrix decompositions. We also establish the relation of this concept with the known orders.

Keywords. Spectrum, sharp order, minus order, spectrally orthogonal matrix decomposition

1 Introduction

The spectral theorem is a well-known technique mainly used in the literature for working with diagonalizable matrices. However, for general matrices (that is, diagonalizable or not) this tool (see [13, pp. 603]) has not been so much exploited in relation to partial orders. In this paper we apply matrix canonical forms defined in [4], namely spectrally orthogonal matrix decompositions, which are useful for general matrices. This gives us the possibility to introduce and investigate some new partial orders.

Let \mathbb{C} be the field of complex numbers and $M_n(\mathbb{C})$ be the set of $n \times n$ matrices over \mathbb{C} . We denote by I_n the identity matrix of size n and by O_n the $n \times n$ zero matrix. We omit the subscripts for I_n and O_n if their size is clear from the context. Let E_{ij} be the matrix with 1 in (i, j) -th position and zeros elsewhere. Two matrices $A, B \in M_n(\mathbb{C})$ are said to be orthogonal, and denoted by $A \perp B$, if $AB = BA = O$. The symbols rk and Spec will stand for the rank and the spectrum of a square matrix, respectively.

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We recall that a matrix $M \in M_n(\mathbb{C})$ has *index* ℓ ($\text{Ind } M = \ell$) if $\text{rk } M^\ell = \text{rk } M^{\ell+1}$ and ℓ is the smallest nonnegative integer with this property. The *core-nilpotent decomposition* of an arbitrary matrix $M \in M_n(\mathbb{C})$ is given by the sum $M = C_M + N_M$, where $C_M \perp N_M$, $\text{Ind } C_M \leq 1$, and $N_M = O$ or N_M is a nilpotent matrix. By convention, if $M = O$, we set $C_M = N_M = O$. This decomposition always exists and is unique (see [3, Chapter 4.8]).

Let $M \in M_n(\mathbb{C})$. A matrix $X \in M_n(\mathbb{C})$ is called a *g-inverse* of M if it satisfies $MXM = M$ and will be denoted by $X = M^-$. If X is a *g-inverse* of M that satisfies $XXM = X$ and $MX = XM$, it is called a *group inverse* of M and is denoted by $X = M^\#$. If X satisfies $XXM = X$, $MX = XM$, and $M^{\ell+1}X = M^\ell$, with $\text{Ind } M = \ell$, it is called a *Drazin inverse* of M and is denoted by $X = M^D$. The Drazin inverse is unique and its existence is guaranteed for every square matrix [3]. In the particular case $\text{Ind } M \leq 1$, the Drazin inverse becomes the group inverse.

Some generalized inverse matrices provide a method to define partial orders (see [15] and references therein, [1, 2, 8, 9, 10, 11, 12, 17]).

Definition 1.1. [6, 16] The *minus partial order* on $M_n(\mathbb{C})$ is defined by $A \bar{\leq} B$ if and only if $A^-A = A^-B$ and $AA^- = BA^-$ for some *g-inverse* A^- of A .

Definition 1.2. [14] Let $A, B \in M_n(\mathbb{C})$ of index less than or equal to 1. It is said that A is below B under the *sharp partial order*, and denoted by $A \stackrel{\#}{\leq} B$, if and only if $A^\#A = A^\#B$ and $AA^\# = BA^\#$.

When the Drazin inverse is used, the corresponding binary relation ($A^D A = A^D B$ and $AA^D = BA^D$) does not give a partial order but a pre-order. However, using the core-nilpotent decomposition of both A and B , the following notion is recalled.

Definition 1.3. [7] Let $A, B \in M_n(\mathbb{C})$. It is said that A is below B under the *cn-order*, and denoted by $A \stackrel{\text{cn}}{\leq} B$, if and only if $C_A \stackrel{\#}{\leq} C_B$ and $N_A \bar{\leq} N_B$.

Alternative ways to define the minus and the sharp orders are the following:

Lemma 1.4. [14, 16] Let $A, B \in M_n(\mathbb{C})$. The following statements hold:

- (a) $A \bar{\leq} B$ if and only if $\text{rk}(B - A) = \text{rk } B - \text{rk } A$.
- (b) $A \stackrel{\#}{\leq} B$ if and only if $A^2 = BA = AB$, for A and B being of index at most 1.

This paper is organized as follows. In Section 2 we present the spectrally orthogonal matrix decompositions and introduce two new binary relations based on these decompositions, namely $\stackrel{1}{\leq}$ and $\stackrel{2,3}{\leq}$. We prove that both relations are partial orders on $M_n(\mathbb{C})$. In

Section 3, we establish that \leq^1 is equivalent to the cn-order, which gives a characterization for the latter via a spectrally orthogonal decomposition. Also this section contains various examples, among them, there is an example showing that $\leq^{2,3}$ indeed provides a new relation on $M_n(\mathbb{C})$, in particular, it is different from the cn-order and other known order relations. In Section 4, sets of matrices which are majorized by an idempotent are considered. Finally, we establish that the $\leq^{2,3}$ -order on $M_n(\mathbb{C})$ is not \mathcal{G} -based for any $n > 1$.

2 Partial orders and spectrally orthogonal decompositions

The notion of spectrally orthogonal matrix decompositions was introduced in [4]. These decompositions are used in the present paper, so we recall below some basic definitions and properties.

We consider the *counting function* $k_M: \mathbb{C} \times \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$ defined by the following rule: for $\lambda \in \mathbb{C}$ and $r \in \mathbb{N}$ the value $k_M(\lambda, r)$ equals to the number of Jordan blocks of $M \in M_n(\mathbb{C})$ of size r , corresponding to the eigenvalue λ . If there are no Jordan blocks of M with λ of size r then $k_M(\lambda, r) = 0$.

Moreover, the function $K_M: \mathbb{C} \rightarrow \mathbb{N} \cup \{0\}$, given by $K_M(\lambda) = \sum_{r=1}^{\infty} k_M(\lambda, r)$, determines the total number of Jordan blocks of M , corresponding to the eigenvalue λ .

Observe that $\text{Spec } M = \{\lambda \in \mathbb{C} \mid K_M(\lambda) > 0\}$. Now we are ready to define the following matrix functions:

Definition 2.1. Let $M = C_M + N_M$ be the core-nilpotent decomposition of $M \in M_n(\mathbb{C})$. We define the maps $S_M^i: \mathbb{C} \rightarrow M_n(\mathbb{C})$, $i = 1, 2, 3$, by

- (i) $S_M^1(0) = N_M$ and for any $\lambda \neq 0$ the matrix $S_M^1(\lambda) = X_\lambda$ where X_λ is the unique (by [4, Lemma 2.14]) matrix such that the following three conditions hold
 - a) $X_\lambda^\sharp X_\lambda = X_\lambda^\sharp M = M X_\lambda^\sharp$,
 - b) $K_{X_\lambda}(\lambda) = K_M(\lambda)$,
 - c) $K_{X_\lambda}(\mu) = 0$ for all $\mu \in \mathbb{C} \setminus \{0, \lambda\}$.
- (ii) $S_M^2(\lambda) = S_{M+I}^1(\lambda + 1) - S_M^1(\lambda)$ for all $\lambda \in \mathbb{C}$.
- (iii) $S_M^3(\lambda) = S_M^1(\lambda) - \lambda S_M^2(\lambda)$ for all $\lambda \in \mathbb{C}$.

In the sequel we need several properties of these decompositions proved in [4], see also [5]:

Theorem 2.2. [4, Remark 2.16, Theorem 2.17] *Let $M \in M_n(\mathbb{C})$.*

- (i) If $\lambda \notin \text{Spec } M$ then $S_M^i(\lambda) = O$ for $i = 1, 2, 3$.
- (ii) $\text{rk}(S_M^2(\lambda)) = \text{deg}_{\chi_M}(z - \lambda)$ is the multiplicity of λ in the characteristic polynomial χ_M .
- (iii) $S_M^i(\lambda) \perp S_M^j(\mu)$ for all $\lambda \neq \mu$, $i, j = 1, 2, 3$.
- (iv) $S_{P^{-1}MP}^i(\lambda) = P^{-1}S_M^i(\lambda)P$ for all $\lambda \in \mathbb{C}$, $M \in M_n(\mathbb{C})$, $P \in GL_n(\mathbb{C})$, and $i = 1, 2, 3$.
- (v) The matrix $S_M^2(\lambda)$ is idempotent for all $\lambda \in \mathbb{C}$.
- (vi) The matrix $S_M^3(\lambda)$ is nilpotent or null for all $\lambda \in \mathbb{C}$.
- (vii) $M = \sum_{\lambda \in \mathbb{C}} S_M^1(\lambda) = \sum_{\lambda \in \mathbb{C}} (\lambda S_M^2(\lambda) + S_M^3(\lambda))$, $I = \sum_{\lambda \in \mathbb{C}} S_M^2(\lambda)$.

These matrix functions and their properties allow us to give the following definition.

Definition 2.3. The decompositions $M = \sum_{\lambda \in \mathbb{C}} S_M^1(\lambda)$ and $M = \sum_{\lambda \in \mathbb{C}} (\lambda S_M^2(\lambda) + S_M^3(\lambda))$ stated in Theorem 2.2 are called the *spectrally orthogonal matrix decompositions* of $M \in M_n(\mathbb{C})$.

The following property of the matrix $S_M^2(\lambda)$ is very important.

Lemma 2.4. Let $M \in M_n(\mathbb{C})$. Then for $\lambda \in \mathbb{C}$ it holds that $\lambda \in \text{Spec}(M)$ if and only if $S_M^2(\lambda) \neq O$.

Proof. Let $S_M^2(\lambda) \neq O$. Hence, by item (ii) of Theorem 2.2, $\text{deg}_{\chi_M}(z - \lambda) > 0$. Then $\lambda \in \text{Spec}(M)$. The converse follows similarly. \square

From now on, we introduce and investigate the following binary relations on matrices based on spectrally orthogonal matrix decompositions.

Definition 2.5. For $A, B \in M_n(\mathbb{C})$ it is said that $A \stackrel{1}{\leq} B$ if $S_A^1(0) \bar{\leq} S_B^1(0)$ and $S_A^1(\lambda) \stackrel{\#}{\leq} S_B^1(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

Definition 2.6. For $A, B \in M_n(\mathbb{C})$ it is said that $A \stackrel{2,3}{\leq} B$ if $S_A^2(\lambda) \stackrel{\#}{\leq} S_B^2(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ and $S_A^3(\lambda) \bar{\leq} S_B^3(\lambda)$ for all $\lambda \in \mathbb{C}$.

It is clear that the cn-order coincides with the sharp partial order for matrices of index at most 1 and also coincides with the minus partial order for nilpotent matrices. This leads to the following lemmas.

Lemma 2.7. Let $A, B \in M_n(\mathbb{C})$. Then $A \stackrel{1}{\leq} B$ if and only if $S_A^1(\lambda) \stackrel{cn}{\leq} S_B^1(\lambda)$ for all $\lambda \in \mathbb{C}$.

Proof. Follows directly from the definition of the cn-order and from the fact that $S_M^1(0)$ is nilpotent and $S_M^1(\lambda)$ is of index at most 1 for all $\lambda \in \mathbb{C} \setminus \{0\}$ and for any $M \in M_n(\mathbb{C})$. In particular, apply this reasoning to $M \in \{A, B\}$. \square

Lemma 2.8. *Let $A, B \in M_n(\mathbb{C})$. Then $A \stackrel{2,3}{\leq} B$ if and only if $S_A^2(\lambda) \stackrel{\bar{}}{\leq} S_B^2(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ and $S_A^3(\lambda) \stackrel{\bar{}}{\leq} S_B^3(\lambda)$ for all $\lambda \in \mathbb{C}$.*

Proof. Since $S_A^2(\lambda)$ and $S_B^2(\lambda)$ are idempotent matrices, $S_A^2(\lambda) \stackrel{\#}{\leq} S_B^2(\lambda)$ is equivalent to $S_A^2(\lambda) \stackrel{\bar{}}{\leq} S_B^2(\lambda)$. \square

Now, we can prove that these new binary relations are indeed partial orders.

Theorem 2.9. $\stackrel{1}{\leq}$ and $\stackrel{2,3}{\leq}$ are partial order relations on $M_n(\mathbb{C})$.

Proof. It is straightforward that $A \stackrel{1}{\leq} A$ and $A \stackrel{2,3}{\leq} A$ for all $A \in M_n(\mathbb{C})$.

Let $A \stackrel{1}{\leq} B$ and $B \stackrel{1}{\leq} A$. Then $S_A^1(\lambda) = S_B^1(\lambda)$ for all $\lambda \in \mathbb{C}$. Hence, $A = B$ by the first decomposition from Definition 2.3.

Let $A \stackrel{2,3}{\leq} B$ and $B \stackrel{2,3}{\leq} A$. Then $S_A^3(\lambda) = S_B^3(\lambda)$ for all $\lambda \in \mathbb{C}$ and $S_A^2(\lambda) = S_B^2(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. However, $S_M^2(0)$ is included to the decomposition of M given by Definition 2.3 with the 0 coefficient. Thus, again $A = B$.

Let $A \stackrel{1}{\leq} B \stackrel{1}{\leq} C$. Then for any $\lambda \in \mathbb{C}$ it holds that $S_A^1(\lambda) \stackrel{\text{cn}}{\leq} S_B^1(\lambda) \stackrel{\text{cn}}{\leq} S_C^1(\lambda)$. Hence, $S_A^1(\lambda) \stackrel{\text{cn}}{\leq} S_C^1(\lambda)$ since $\stackrel{\text{cn}}{\leq}$ is a partial order relation. Thus, by Lemma 2.7, $A \stackrel{1}{\leq} C$.

Let $A \stackrel{2,3}{\leq} B \stackrel{2,3}{\leq} C$. Then for any $\lambda \in \mathbb{C}$ it holds that $S_A^3(\lambda) \stackrel{\bar{}}{\leq} S_B^3(\lambda) \stackrel{\bar{}}{\leq} S_C^3(\lambda)$ and for any $\lambda \in \mathbb{C} \setminus \{0\}$ it holds that $S_A^2(\lambda) \stackrel{\#}{\leq} S_B^2(\lambda) \stackrel{\#}{\leq} S_C^2(\lambda)$. Hence, $A \stackrel{2,3}{\leq} C$ by definition since $\stackrel{\bar{}}{\leq}$ and $\stackrel{\#}{\leq}$ are partial order relations. \square

It is well known that if $A \stackrel{\text{cn}}{\leq} B$ then $\text{Spec } A \subseteq \text{Spec } B \cup \{0\}$ (see [15, Theorem 4.4.18]). Similar relations are valid for $\stackrel{1}{\leq}$ - and $\stackrel{2,3}{\leq}$ -orders.

Lemma 2.10. *Let $A, B \in M_n(\mathbb{C})$ such that $A \stackrel{1}{\leq} B$ or $A \stackrel{2,3}{\leq} B$. Then $\text{Spec } A \subseteq \text{Spec } B \cup \{0\}$. Moreover, if $0 \notin \text{Spec } A$ or A has a Jordan cell of size $k \geq 2$ with 0 eigenvalue, then $\text{Spec } A \subseteq \text{Spec } B$.*

Proof. Let $\lambda \in \text{Spec } A \setminus \{0\}$. If $A \stackrel{1}{\leq} B$ or $A \stackrel{2,3}{\leq} B$ then $S_A^i(\lambda) \stackrel{\#}{\leq} S_B^i(\lambda)$ for $i = 1, 2$, respectively. If we suppose $\lambda \notin \text{Spec } B$ then $O \neq S_A^i(\lambda) \stackrel{\#}{\leq} S_B^i(\lambda) = O$, which is not possible. Hence, $\text{Spec } A \subseteq \text{Spec } B \cup \{0\}$. In addition, if A has a Jordan cell of size $k \geq 2$ with 0 eigenvalue then $S_A^1(0) \neq O$ and $S_A^3(0) \neq O$. Hence, considering the minus order, we have $S_B^1(0) \neq O$ and $S_B^3(0) \neq O$. Thus, $0 \in \text{Spec } B$. \square

The introduced orders are invariant under similarities.

Lemma 2.11. *Let $A, B \in M_n(\mathbb{C})$. For all nonsingular $P \in M_n(\mathbb{C})$, it follows that*

$$(a) \ A \stackrel{1}{\leq} B \text{ implies } PAP^{-1} \stackrel{1}{\leq} PBP^{-1}.$$

$$(b) \ A \stackrel{2,3}{\leq} B \text{ implies } PAP^{-1} \stackrel{2,3}{\leq} PBP^{-1}.$$

Proof. It is straightforward using item (iv) of Theorem 2.2 and the fact that minus, sharp and cn-orders are invariant under similarities. \square

3 Relationships with other orders

In this section we show that a matrix A is below another matrix B under the cn-order if and only if A is below B under the $\stackrel{1}{\leq}$ order. We will also prove that although this equivalence does not occur between the cn-order and the $\stackrel{2,3}{\leq}$ -order, the implication

$$A \stackrel{\text{cn}}{\leq} B \implies A \stackrel{2,3}{\leq} B$$

remains valid.

We quote here the following result which is useful for our considerations.

Theorem 3.1. [15, Theorem 4.4.18] *Let $A, B \in M_n(\mathbb{C})$. Then $A \stackrel{\text{cn}}{\leq} B$ if and only if there exist invertible matrices $P \in M_n(\mathbb{C})$, $C_1 \in M_{k_1}(\mathbb{C})$, $C_2 \in M_{k_2}(\mathbb{C})$ and nilpotent matrices $N_1, N_2 \in M_{n-(k_1+k_2)}$, such that*

$$A = P \begin{pmatrix} C_1 & & \\ & O & \\ & & N_1 \end{pmatrix} P^{-1} \quad \text{and} \quad B = P \begin{pmatrix} C_1 & & \\ & C_2 & \\ & & N_2 \end{pmatrix} P^{-1}$$

with $N_1 \stackrel{-}{\leq} N_2$.

Now, we show that $\stackrel{1}{\leq}$ -order follows from $\stackrel{\text{cn}}{\leq}$ -order.

Lemma 3.2. *Let $A, B \in M_n(\mathbb{C})$. If $A \stackrel{\text{cn}}{\leq} B$ then $A \stackrel{1}{\leq} B$.*

Proof. Let $A \stackrel{\text{cn}}{\leq} B$. Clearly, the result is trivial for $A = O$. Hence, assume further that $A \neq O$, so $B \neq O$. Then by Theorem 3.1

$$A = P \begin{pmatrix} C_1 & & \\ & O & \\ & & N_1 \end{pmatrix} P^{-1} \quad \text{and} \quad B = P \begin{pmatrix} C_1 & & \\ & C_2 & \\ & & N_2 \end{pmatrix} P^{-1}$$

with $N_1 \bar{\leq} N_2$ and C_1, C_2 invertible. Then from the uniqueness of the core-nilpotent decomposition of a matrix it follows that

$$C_A = P \begin{pmatrix} C_1 & & \\ & O & \\ & & O \end{pmatrix} P^{-1}, \quad N_A = P \begin{pmatrix} O & & \\ & O & \\ & & N_1 \end{pmatrix} P^{-1},$$

$$C_B = P \begin{pmatrix} C_1 & & \\ & C_2 & \\ & & O \end{pmatrix} P^{-1}, \quad \text{and} \quad N_B = P \begin{pmatrix} O & & \\ & O & \\ & & N_2 \end{pmatrix} P^{-1}$$

are the core-nilpotent decompositions of A and B , correspondingly, and we have $C_A \stackrel{\#}{\leq} C_B$ and $N_A \bar{\leq} N_B$.

In order to see that $A \stackrel{1}{\leq} B$ holds we have to demonstrate the inequalities $S_A^1(\lambda) \stackrel{\#}{\leq} S_B^1(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ and $S_A^1(0) \bar{\leq} S_B^1(0)$. By definition, $S_A^1(0) = N_A$ and $S_B^1(0) = N_B$, thus

$$S_A^1(0) \bar{\leq} S_B^1(0) \tag{3.1}$$

is obtained directly.

Since similarities preserves the $\stackrel{\#}{\leq}$ order, it follows from $C_A \stackrel{\#}{\leq} C_B$ that

$$\begin{pmatrix} C_1 & & \\ & O & \\ & & O \end{pmatrix} \stackrel{\#}{\leq} \begin{pmatrix} C_1 & & \\ & C_2 & \\ & & O \end{pmatrix}.$$

Let us compute the Jordan canonical forms of C_1 and C_2 , that is, $C_1 = S_1 J_1 S_1^{-1}$ and $C_2 = S_2 J_2 S_2^{-1}$. Hence,

$$\begin{pmatrix} S_1 J_1 S_1^{-1} & & \\ & O & \\ & & O \end{pmatrix} \stackrel{\#}{\leq} \begin{pmatrix} S_1 J_1 S_1^{-1} & & \\ & S_2 J_2 S_2^{-1} & \\ & & O \end{pmatrix},$$

which can be re-written as

$$\begin{pmatrix} S_1 & & \\ & S_2 & \\ & & I \end{pmatrix} \begin{pmatrix} J_1 & & \\ & O & \\ & & O \end{pmatrix} \begin{pmatrix} S_1^{-1} & & \\ & S_2^{-1} & \\ & & I \end{pmatrix} \stackrel{\#}{\leq} \begin{pmatrix} S_1 & & \\ & S_2 & \\ & & I \end{pmatrix} \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & O \end{pmatrix} \begin{pmatrix} S_1^{-1} & & \\ & S_2^{-1} & \\ & & I \end{pmatrix}$$

and is equivalent to

$$A' = \begin{pmatrix} J_1 & & \\ & O & \\ & & O \end{pmatrix} \stackrel{\#}{\leq} \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & O \end{pmatrix} = B'. \tag{3.2}$$

We consider the diagonal concatenation of blocks with the same eigenvalue in the Jordan form:

$$J_1 = \begin{pmatrix} A_1 & \dots & O \\ \vdots & \ddots & \vdots \\ O & \dots & A_p \end{pmatrix}, \quad J_2 = \begin{pmatrix} B_1 & \dots & O & O & \dots & O \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ O & \dots & B_p & O & \dots & O \\ O & \dots & O & B_{p+1} & \dots & O \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ O & \dots & O & \dots & \dots & B_q \end{pmatrix}. \quad (3.3)$$

Here, for $i, j = 1, \dots, p$, we have $\text{Spec } A_i = \{\lambda_i\}$, $0 \notin \text{Spec } A_i$, and $\text{Spec } A_i \cap \text{Spec } A_j = \emptyset$ if $i \neq j$. Analogously, for $i = 1, \dots, p$, we have $\text{Spec } B_i = \{\lambda_i\}$ and moreover, for $i, j = 1, \dots, q$, $\text{Spec } B_i \cap \text{Spec } B_j = \emptyset$ if $i \neq j$ and for $k = p+1, \dots, q$, $0 \notin \text{Spec } B_k$. We observe that some (or even all) of the blocks B_k , $k = p+1, \dots, q$, may be absent. Also, the size of B_i , $i = 1, \dots, p$, and the one of A_i may not be the same.

We get by (3.2) and (3.3) that

$$\left(\begin{array}{ccc|c|c|c|c} A_1 & \dots & O & & & & \\ \vdots & \ddots & \vdots & & & & \\ O & \dots & A_p & & & & \\ \hline & & & O & & & \\ \hline & & & & O & & \\ \hline & & & & & O & \end{array} \right) \stackrel{\#}{=} \left(\begin{array}{ccc|ccc|ccc|c} A_1 & \dots & O & & & & & & & & & \\ \vdots & \ddots & \vdots & & & & & & & & & \\ O & \dots & A_p & & & & & & & & & \\ \hline & & & B_1 & \dots & O & & & & & & \\ & & & \vdots & \ddots & \vdots & & & & & & \\ & & & O & \dots & B_p & & & & & & \\ \hline & & & & & & B_{p+1} & \dots & O & & & \\ & & & & & & \vdots & \ddots & \vdots & & & \\ & & & & & & O & \dots & B_q & & & \\ \hline & & & & & & & & & & & O \end{array} \right) \quad (3.4)$$

Note that, by definition, for each $i = 1, \dots, p$,

$$S_{A'}^1(\lambda_i) = \begin{pmatrix} O & & & & & & \\ & \ddots & & & & & \\ & & O & & & & \\ & & & A_i & & & \\ & & & & O & & \\ & & & & & \ddots & \\ & & & & & & O & \\ & & & & & & & O & \\ & & & & & & & & O & \end{pmatrix} \quad \text{and} \quad S_{B'}^1(\lambda_i) = \begin{pmatrix} O & & & & & & \\ & \ddots & & & & & \\ & & O & & & & \\ & & & A_i & & & \\ & & & & O & & \\ & & & & & \ddots & \\ & & & & & & O & \\ & & & & & & & O & \\ & & & & & & & & O & \\ & & & & & & & & & O & \end{pmatrix},$$

that is, in $S_{B'}^1(\lambda_i)$ the block A_i is located exactly at the same place as in $S_{A'}^1(\lambda_i)$.

Then it is straightforward to check that for each $i = 1, \dots, p$ it holds that $S_{A'}^1(\lambda_i) \stackrel{\#}{\leq} S_{B'}^1(\lambda_i)$. Therefore,

$$S_A^1(\lambda_i) \stackrel{\#}{\leq} S_B^1(\lambda_i), \quad (3.5)$$

since the $\stackrel{\#}{\leq}$ -order is preserved under similarities (see item (iv) of Theorem 2.2).

For any $\lambda \in \mathbb{C} \setminus \{0, \lambda_1, \dots, \lambda_p\}$,

$$S_A^1(\lambda) = O \stackrel{\#}{\leq} S_B^1(\lambda). \quad (3.6)$$

Combining (3.1), (3.5), and (3.6), we get $A \stackrel{1}{\leq} B$. \square

The following lemma establishes that the cn-order also follows from the $\stackrel{1}{\leq}$ -order.

Lemma 3.3. *Let $A, B \in M_n(\mathbb{C})$. If $A \stackrel{1}{\leq} B$ then $A \stackrel{cn}{\leq} B$.*

Proof. By Lemma 2.7, $A \stackrel{1}{\leq} B$ is equivalent to $S_A^1(\lambda) \stackrel{cn}{\leq} S_B^1(\lambda)$ for all $\lambda \in \mathbb{C}$. Hence, $S_A^1(0) \stackrel{\bar{\leq}}{\leq} S_B^1(0)$, i.e. $N_A \stackrel{\bar{\leq}}{\leq} N_B$.

We have to prove now that $S_A^1(\lambda) \stackrel{\#}{\leq} S_B^1(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ implies $C_A \stackrel{\#}{\leq} C_B$, i.e.

$$\sum_{\lambda \in \mathbb{C} \setminus \{0\}} S_A^1(\lambda) \stackrel{\#}{\leq} \sum_{\lambda \in \mathbb{C} \setminus \{0\}} S_B^1(\lambda).$$

It follows from the proof of Lemma 3.2 that for any $\lambda \neq 0$ and for any $X \in M_n(\mathbb{C})$, $S_X^1(\lambda) = S_{C_X}^1(\lambda)$. By definition, $S_X^1(\lambda) \stackrel{\#}{\leq} X$ for any λ . It then follows that $S_A^1(\lambda) \stackrel{\#}{\leq} S_B^1(\lambda) = S_{C_B}^1(\lambda) \stackrel{\#}{\leq} C_B$ for any $\lambda \neq 0$, i.e., $S_A^1(\lambda) \stackrel{\#}{\leq} C_B$ for any $\lambda \neq 0$. Denote by $\lambda_1, \dots, \lambda_p$ the set of nonzero eigenvalues of A . Since $S_A^1(\lambda_i) \perp S_A^1(\lambda_j)$ (see Theorem 2.2), it follows by [4, Theorem 2.22] that $S_A^1(\lambda_1) + S_A^1(\lambda_2) \stackrel{\#}{\leq} C_B$. In addition, $S_A^1(\lambda_1) + S_A^1(\lambda_2) \perp S_A^1(\lambda_3)$. Hence, $(S_A^1(\lambda_1) + S_A^1(\lambda_2)) + S_A^1(\lambda_3) \stackrel{\#}{\leq} C_B$. Arguing in the same way we obtain that $(S_A^1(\lambda_1) + \dots + S_A^1(\lambda_{p-1})) + S_A^1(\lambda_p) \stackrel{\#}{\leq} C_B$. Using that $C_A = S_A^1(\lambda_1) + \dots + S_A^1(\lambda_p)$, we arrive at $C_A \stackrel{\#}{\leq} C_B$ and the result follows. \square

From Lemmas 3.2 and 3.3 we derive the following result which provides a characterization of the cn-order in terms of the spectrally orthogonal decomposition.

Theorem 3.4. *Let $A, B \in M_n(\mathbb{C})$. Then $A \stackrel{1}{\leq} B$ if and only if $A \stackrel{cn}{\leq} B$.*

In what follows, the relationship between the $\stackrel{cn}{\leq}$ - and $\stackrel{2,3}{\leq}$ -partial orders is analyzed.

Theorem 3.5. *Let $A, B \in M_n(\mathbb{C})$. If $A \stackrel{cn}{\leq} B$ then $A \stackrel{2,3}{\leq} B$.*

Proof. Suppose that $A \stackrel{cn}{\leq} B$. Clearly, $S_A^3(0) = S_A^1(0) = N_A \stackrel{\bar{\leq}}{\leq} N_B = S_B^1(0) = S_B^3(0)$. Moreover, as we have obtained in the proof of Lemma 3.2, we can also get the expression (3.4). Then, for $i = 1, \dots, p$,

$$S_{A'}^2(\lambda_i) = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & I_{l_i} & & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix} \quad \text{and} \quad S_{B'}^2(\lambda_i) = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & I_{l_i} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 & \\ & & & & & & & I \\ & & & & & & & & 0 \end{pmatrix}.$$

In both matrices, the block I_{l_i} is located exactly at the same place, i.e., is a successor of the $l_1 + \dots + l_{i-1}$ square zero block. Hence, it is straightforward to see that $S_{A'}^2(\lambda_i) \stackrel{\sharp}{\leq} S_{B'}^2(\lambda_i)$.

Similarly,

$$S_{A'}^3(\lambda_i) = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & N_{l_i} \\ & & & & 0 \\ & & & & & \ddots \end{pmatrix} \quad \text{and} \quad S_{B'}^3(\lambda_i) = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & N_{l_i} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \\ & & & & & & & \tilde{N}_{\tilde{l}_i} \\ & & & & & & & & 0 \end{pmatrix}$$

have both nonzero nilpotent blocks in the position of A_i and, additionally, $S_{B'}^3(\lambda_i)$ may have also some other nonzero diagonal block $\tilde{N}_{\tilde{l}_i}$, which is not located on the position occupied by A_i . Hence, $S_{A'}^3(\lambda_i) \stackrel{\bar{\leq}}{\leq} S_{B'}^3(\lambda_i)$.

Since minus and sharp partial orders are preserved under similarity transformations, $S_A^2(\lambda_i) \stackrel{\sharp}{\leq} S_B^2(\lambda_i)$ and $S_A^3(\lambda_i) \stackrel{\bar{\leq}}{\leq} S_B^3(\lambda_i)$. \square

3.1 Examples and counterexamples

Below we provide a number of examples showing that the $\stackrel{2,3}{\leq}$ -order introduced in this paper is essentially different from the extensions of the $\stackrel{\sharp}{\leq}$ -order. In particular, it is different from the cn -order.

We start with an example showing that in general the $\stackrel{\text{cn}}{\leq}$ -order does not follow from the $\stackrel{2,3}{\leq}$ -order.

Example 3.6. Let $A \in M_n(\mathbb{C})$ be a Jordan cell with 1 on the main diagonal, i.e.

$$A = J_n(1) = I + E_{12} + E_{23} + \dots + E_{n-1,n}.$$

Then it follows from Theorem 3.1, that A is a minimal element of $M_n(\mathbb{C}) \setminus \{O\}$ with respect to the $\stackrel{\sharp}{\leq}$ - and $\stackrel{\text{cn}}{\leq}$ -orders. Indeed, a Jordan cell can not be similar to a block matrix of type $\begin{pmatrix} C_1 & & \\ & C_2 & \\ & & O_k \end{pmatrix}$, $k > 0$, with invertible blocks C_1 and C_2 . So, there is no nonzero $B \neq A$ such that $B \stackrel{\text{cn}}{\leq} A$. However $E_{ii} \stackrel{2,3}{\leq} A$ for all $i = 1, \dots, n$. Indeed, $S_A^2(\lambda) = S_A^3(\lambda) = O$ for all $\lambda \in \mathbb{C} \setminus \{1\}$,

$$S_A^2(1) = I_n, \quad \text{and} \quad S_A^3(1) = J_n(0) = E_{12} + E_{23} + \dots + E_{n-1,n}.$$

For any i , $1 \leq i \leq n$, we have $S_{E_{ii}}^2(1) = E_{ii}$, $S_{E_{ii}}^2(\lambda) = O$ for all $\lambda \in \mathbb{C} \setminus \{0, 1\}$ and $S_{E_{ii}}^3(\lambda) = O$ for all $\lambda \in \mathbb{C}$. Since $E_{ii} \stackrel{\sharp}{\leq} I_n$ and $O \stackrel{\bar{\leq}}{\leq} S_A^3(\lambda)$ for all $\lambda \in \mathbb{C}$, it follows that $E_{ii} \stackrel{2,3}{\leq} A$ for all $i = 1, \dots, n$.

Example 3.7. In the previous example it was shown that $E_{ii} \stackrel{2,3}{\leq} J_n(1)$. Observe that E_{ii} is not below $J_n(1)$ with respect to the minus order. This shows that the $\stackrel{2,3}{\leq}$ -order does not imply the minus order, in contrast with sharp and cn-orders.

Let us show that the $\stackrel{2,3}{\leq}$ -order also does not follow from the minus order. We provide two different examples of matrices of different sizes in order to show that it is a general situation.

Example 3.8. 1) Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

It is not hard to see that $A \bar{\leq} B$ but A is not below B under the $\stackrel{2,3}{\leq}$ -order because $S_A^3(0) = A$ is not below $S_B^3(0) = O$ under the minus order.

2) Similarly, the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 & -2 & 0 \\ 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 2 & 4 & 0 \end{pmatrix},$$

satisfy that $A \bar{\leq} B$ but A is not below B under the $\stackrel{2,3}{\leq}$ -order because

$$S_A^3(0) = A, \quad S_B^3(0) = -2E_{43},$$

and $\text{rk}(S_B^3(0) - S_A^3(0)) \neq \text{rk} S_B^3(0) - \text{rk} S_A^3(0)$.

A well-known property for the minus, sharp and cn-orders is that if A is strictly below B with respect to one of these orders then $\text{rk} A < \text{rk} B$. The following example remarks that the $\stackrel{2,3}{\leq}$ -order does not satisfy such a property.

Example 3.9. It is not hard to see that $I_n \stackrel{2,3}{\leq} J_n(1)$ but $\text{rk} I_n = \text{rk} J_n(1) = n$.

Remark 3.10. The above examples show that in contrast to all known extensions of the sharp order, see [15], the $\stackrel{2,3}{\leq}$ -order is unrelated with the minus order and has nonstandard behavior with respect to the rank function.

Observe that some properties of the $\stackrel{2,3}{\leq}$ -order introduced in this paper are just opposite to the properties of the sharp and cn-orders.

Example 3.11. In Example 3.6, the matrix A has been pointed out to be a minimal element of $M_n(\mathbb{C}) \setminus \{O\}$ with respect to the $\overset{\#}{\leq}$ - and $\overset{\text{cn}}{\leq}$ -orders. Now, we establish that A is a maximal element of $M_n(\mathbb{C})$ with respect to the $\overset{2,3}{\leq}$ -order. Indeed, if there exists a matrix $B \in M_n(\mathbb{C})$ such that $A \overset{2,3}{\leq} B$ then $S_B^2(1) = I_n$ (that is, $\lambda = 1$ is the only eigenvalue of B) and $S_B^2(\lambda) = O$ for all $\lambda \neq 1$. Moreover, from $S_A^3(1) = A - I_n \overset{\bar{\leq}}{S_B^3(1)}$ and $S_B^3(1)$ is nilpotent we get $\text{rk } S_B^3(1) = n - 1$. Hence, $S_B^3(1) = A - I_n$ since $\text{rk}(A - I_n) = n - 1$. Consequently, $B = S_B^2(1) + S_B^3(1) = A$.

4 Some algebraic properties of the introduced order

We start with the investigation of the behavior of the class of idempotent matrices under the considered orders.

Theorem 4.1. *Let $A, B \in M_n(\mathbb{C})$ such that $A \overset{2,3}{\leq} B$. If B is idempotent then A is idempotent.*

Proof. It is clear that $B^2 = B$ assures the existence of a nonsingular $P \in M_n(\mathbb{C})$ such that $B = P(I \oplus O)P^{-1}$. By Lemma 2.10, we obtain that $\text{Spec}(A) \subseteq \text{Spec}(B) \cup \{0\} \subseteq \{0, 1\}$. On the other hand, $A \overset{2,3}{\leq} B$ also implies $S_A^2(1) \overset{\#}{\leq} S_B^2(1) = B$, $S_A^3(1) \overset{\bar{\leq}}{S_B^3(1)} = O$ and $S_A^3(0) \overset{\bar{\leq}}{S_B^3(0)} = O$. Thus, $S_A^3(0) = O$ and $S_A^3(1) = O$. From items (v) and (vii) of Theorem 2.2, we arrive at $A = S_A^2(1)$, which is idempotent. \square

However, the converse of the previous theorem is not valid. Indeed, Example 3.6 shows that $E_{ii} \overset{2,3}{\leq} J_n(1)$ where E_{ii} is idempotent and $J_n(1)$ is not.

We can say even more.

Theorem 4.2. *Let $A, B \in M_n(\mathbb{C})$ be idempotent. The following conditions are equivalent:*

- (a) $A \overset{2,3}{\leq} B$.
- (b) $A \overset{\text{cn}}{\leq} B$.
- (c) $A \overset{\#}{\leq} B$.

Proof. It is straightforward from [15, Theorem 4.2.8(iii)]. \square

The most studied orders in the literature, namely minus, star, sharp, are \mathcal{G} -based. This means that $A \overset{\text{G}}{\leq} B$ if and only if there exists a g -inverse $G \in M_n(\mathbb{C})$ of A such that $AG = BG$ and $GA = GB$.

Note that, since the $\overset{1}{\leq}$ -order is equivalent to the cn-order, it is not \mathcal{G} -based. We close this paper showing that, in general, the $\overset{2,3}{\leq}$ -partial order is not \mathcal{G} -based on $M_n(\mathbb{C})$ as well.

Remark 4.3. The order $\leq^{2,3}$ is not \mathcal{G} -based for any $n > 1$. We consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus O_{n-2} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus O_{n-2},$$

where $X \oplus Y$ denotes the block diagonal matrix with the blocks X and Y on the diagonal. It is straightforward to check that $A \leq^{2,3} B$. Suppose that there is a g -inverse $G \in M_n(\mathbb{C})$ of A such that $AG = BG$ and $GA = GB$. Let $T = \frac{1}{2}(A + B)$. Then $T - A = \frac{1}{2}(B - A)$ and so, $AG = TG$ and $GA = GT$. Hence, $A \leq^{2,3} T$, which is a contradiction due to $S_A^2(1)$ is not below $S_T^2(1)$ under the sharp order.

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