

## ON THE CONTINUITY OF THE GROUP INVERSE

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*Abstract.* Let  $\{A_m\}_{m=1}^\infty$  be a sequence of complex group invertible matrices that converges to  $A$ . We characterize when  $A$  is group invertible and  $\{A_m^\#\}_{m=1}^\infty$  converges to  $A^\#$  in terms of the canonical angles between  $A_m$  and  $A_m^*$ , where  $X^\#$  denotes the group inverse of the matrix  $X$ . We compare this characterization with some known characterizations of the continuity of the Drazin inverse.

### 1. Introduction

Let  $\mathbb{C}_{m,n}$  be the set of  $m \times n$  complex matrices, and let  $A^*$ ,  $\mathcal{R}(A)$ ,  $\mathcal{N}(A)$ , and  $\text{rank}(A)$  denote the conjugate transpose, column space, null space, and rank of  $A \in \mathbb{C}_{m,n}$ , respectively. Furthermore, let  $A^\dagger$  stand for the Moore-Penrose inverse of  $A$ , i.e., the unique matrix satisfying the equations

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad AA^\dagger = (AA^\dagger)^*, \quad A^\dagger A = (A^\dagger A)^*.$$

It can be proved (see e.g. [1, Chapter 4]) that if  $A \in \mathbb{C}_{n,n}$ , then there is at most one matrix  $X \in \mathbb{C}_{n,n}$  such that

$$AXA = A, \quad XAX = X, \quad AX = XA.$$

Such matrix (when it exists) is customarily written as  $A^\#$  and is called the group inverse of  $A$ . A useful characterization of the existence of the group inverse of  $A \in \mathbb{C}_{n,n}$  is that  $A^\#$  exists if and only if  $\text{rank}(A) = \text{rank}(A^2)$  (see e.g. [1, Section 4.4]).

We shall denote the zero matrix in  $\mathbb{C}_{n,m}$  by  $0_{n,m}$ , and when there is no danger of confusion, we will simply write  $0$ . In addition,  $\mathbf{1}_n$  and  $\mathbf{0}_n$  will denote the  $1 \times n$  row vectors all of whose components are 1 and 0, respectively. If  $\mathcal{S}$  is a subspace of  $\mathbb{C}_{n,1}$ , then  $P_{\mathcal{S}}$  and  $\dim \mathcal{S}$  stand for the orthogonal projector onto the subspace  $\mathcal{S}$  and the dimension of  $\mathcal{S}$ , respectively.

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Since we will deal with limits of sequences of matrices, we need a topology. Throughout this paper, we shall use the Euclidean norm in  $\mathbb{C}_{m,n}$ : i.e.,  $\|A\| = \sup\{\|A\mathbf{v}\| : \|\mathbf{v}\| = 1, \mathbf{v} \in \mathbb{C}_{n,1}\}$  for  $A \in \mathbb{C}_{m,n}$ . A well-known result is that the Euclidean norm is unitarily invariant: if  $V \in \mathbb{C}_{m,m}$ ,  $W \in \mathbb{C}_{n,n}$  are unitary and  $A \in \mathbb{C}_{m,n}$ , then  $\|VAW\| = \|A\|$ . Another useful identity is the so-called  $C^*$ -identity:

$$\|A\|^2 = \|AA^*\| = \|A^*A\|, \quad A \in \mathbb{C}_{m,n}. \tag{1}$$

We shall also use the concept of canonical angles (also called principal angles) which will be defined in the next paragraph [10]:

DEFINITION 1. Let  $\mathcal{X}$ ,  $\mathcal{Y}$  be nontrivial subspaces of  $\mathbb{C}_{n,1}$  and let  $r = \min\{\dim \mathcal{X}, \dim \mathcal{Y}\}$ . We define the *canonical angles*  $\theta_1, \dots, \theta_r \in [0, \pi/2]$  between  $\mathcal{X}$  and  $\mathcal{Y}$  by

$$\cos \theta_i = \sigma_i(P_{\mathcal{X}}P_{\mathcal{Y}}), \quad i = 1, \dots, r,$$

where the nonnegative real numbers  $\sigma_1(P_{\mathcal{X}}P_{\mathcal{Y}}), \dots, \sigma_r(P_{\mathcal{X}}P_{\mathcal{Y}})$  are the singular values of  $P_{\mathcal{X}}P_{\mathcal{Y}}$ . We will have in mind the possibility that one canonical angle is repeated.

In [2] it was given the following theorem:

THEOREM 1. Let  $A \in \mathbb{C}_{n,n}$ ,  $r = \text{rank}(A)$ , and let  $\theta_1, \dots, \theta_p$  be the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  belonging to  $]0, \pi/2[$ . Denote by  $x$  and  $y$  the multiplicities of the angles  $0$  and  $\pi/2$  as a canonical angle between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$ , respectively. There exists a unitary matrix  $V \in \mathbb{C}_{n,n}$  such that

$$A = V \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} V^*, \tag{2}$$

where  $M \in \mathbb{C}_{r,r}$  is nonsingular,

$$C = \text{diag}(\mathbf{0}_y, \cos \theta_1, \dots, \cos \theta_p, \mathbf{1}_x),$$

$$S = \begin{bmatrix} \text{diag}(\mathbf{1}_y, \sin \theta_1, \dots, \sin \theta_p) & \mathbf{0}_{p+y, n-(r+p+y)} \\ \mathbf{0}_{x, p+y} & \mathbf{0}_{x, n-(r+p+y)} \end{bmatrix},$$

and  $r = y + p + x$ . Furthermore,  $x$  and  $y + n - r$  are the multiplicities of the singular values  $1$  and  $0$  in  $P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)}$ , respectively.

Observe that the matrices  $C$  and  $S$  of this former result satisfy

$$C^2 + SS^* = I_r. \tag{3}$$

We shall use the Theorem 1 in this work to characterize the continuity of the group inverse.

If a matrix  $A$  is represented as in (2), then we can write explicitly the Moore-Penrose inverse of  $A$  and characterize the group invertibility of  $A$ .

**THEOREM 2.** *Let  $A \in \mathbb{C}_{n,n}$  be represented as in (2), where the meaning of  $C$ ,  $S$  and  $V$  is written in Theorem 1. Then  $\|A\| = \|M\|$ ,*

$$A^\dagger = V \begin{bmatrix} CM^{-1} & 0 \\ S^*M^{-1} & 0 \end{bmatrix} V^*, \quad \|A^\dagger\| = \|M^{-1}\|, \tag{4}$$

$$AA^\dagger = V \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} V^*, \quad A^\dagger A = V \begin{bmatrix} C^2 & CS \\ S^*C & S^*S \end{bmatrix} V^*. \tag{5}$$

Furthermore,  $A$  is group invertible if and only if  $C$  is invertible. In this situation, then

$$A^\# = V \begin{bmatrix} C^{-1}M^{-1} & C^{-1}M^{-1}C^{-1}S \\ 0 & 0 \end{bmatrix} V^*, \quad \|A^\#\| = \|C^{-1}M^{-1}C^{-1}\|. \tag{6}$$

*Proof.* The only statement that was not proved in [2] is “if  $A$  is group invertible, then  $\|A^\#\| = \|C^{-1}M^{-1}C^{-1}\|$ ”. Let us denote  $D = C^{-1}M^{-1}C^{-1}$ . Now, by using the  $C^*$ -identity (1) and the identity (3), we get

$$\|A^\#\|^2 = \|A^\#(A^\#)^*\| = \left\| \begin{bmatrix} DC & DS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} CD^* & 0 \\ S^*D^* & 0 \end{bmatrix} \right\| = \|DD^*\| = \|D\|^2.$$

This proves  $\|A^\#\| = \|D\| = \|C^{-1}M^{-1}C^{-1}\|$ .  $\square$

Observe that if  $A$  has group inverse, by the previous result, none of the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  is  $\pi/2$ .

### 2. Main result

It is well known that the standard inverse and the Moore-Penrose inverse of a matrix are not necessarily continuous functions of the elements of the matrix. The following example shows clearly this behaviour:

**EXAMPLE 1.** Let

$$A_n = \begin{bmatrix} 1/n & 0 \\ 0 & 0 \end{bmatrix}, \quad n \in \mathbb{N}, \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is evident that  $\lim_{n \rightarrow \infty} A_n = A$  and  $\lim_{n \rightarrow \infty} A_n^\dagger \neq A^\dagger$ . This classical example shows that we must impose conditions to assure the continuity of the Moore-Penrose inverse. The following result is known:

**THEOREM 3.** *Let  $\{A_m\}_{m=1}^\infty$  be a sequence of complex  $n \times p$  matrices converging to  $A$ . Then the following affirmations are equivalent:*

- (i)  $\lim_{m \rightarrow \infty} A_m^\dagger = A^\dagger$ .
- (ii) There exists  $m_0 \in \mathbb{N}$  such that  $\text{rank}(A_m) = \text{rank}(A)$  for  $m \geq m_0$ .
- (iii)  $\sup\{\|A_m^\dagger\| : m \in \mathbb{N}\} < \infty$ .

(iv)  $\lim_{m \rightarrow \infty} A_m A_m^\dagger = AA^\dagger$ .

For the proof of (i)  $\iff$  (ii), see [8, 9]. For the remaining equivalences, see [6].

Let us observe that the above Example 1 also shows that  $\lim_{n \rightarrow \infty} A_n = A$  and the existence of  $A^\#$  and  $A_n^\#$  for  $n \in \mathbb{N}$  does not imply  $\lim_{n \rightarrow \infty} A_n^\# = A^\#$ .

In general, the subset of  $\mathbb{C}_{n,n}$  composed of group invertible matrices is not closed in  $\mathbb{C}_{n,n}$  as the following example shows:

EXAMPLE 2. Let

$$B_n = \begin{bmatrix} 1/n & 1 \\ 0 & 0 \end{bmatrix}, n \in \mathbb{N}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since  $\text{rank}(B_n^2) = \text{rank}(B_n)$ , then  $B_n$  are group invertible for any  $n \in \mathbb{N}$ , while  $\text{rank}(B_n) = 1 \neq 0 = \text{rank}(B^2)$ , which shows that  $B$  is not group invertible. Obviously,  $\lim_{n \rightarrow \infty} B_n = B$ .

The decomposition given in Theorem 1 permits illustrate the reason in Example 2, matrix  $B$  is not group invertible. We decompose matrices  $B_n$  and  $B$  as in (2):

$$B_n = \begin{bmatrix} 1/n & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_n c_n & m_n s_n \\ 0 & 0 \end{bmatrix}, m_n = \frac{\sqrt{1+n^2}}{n}, c_n = \frac{1}{\sqrt{1+n^2}}, s_n = \frac{n}{\sqrt{1+n^2}}.$$

and

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m & m s \\ 0 & 0 \end{bmatrix}, m = 1, c = 0, s = 1.$$

If  $\theta_n$  is the unique canonical angle between  $\mathcal{R}(B_n)$  and  $\mathcal{R}(B_n^*)$  (there is only one canonical angle because  $\text{rank}(B_n) = 1$ ), then  $\tan \theta_n = s_n/c_n = n$ . Hence  $\theta_n = \arctan n$ . Note that  $\lim_{n \rightarrow \infty} \theta_n = \pi/2$ . Recall that when a canonical angle between  $\mathcal{R}(X)$  and  $\mathcal{R}(X^*)$  is  $\pi/2$ , then  $X$  has no group inverse (Theorem 2) and thus, the only canonical angle between  $\mathcal{R}(B)$  and  $\mathcal{R}(B^*)$  is  $\pi/2$ .

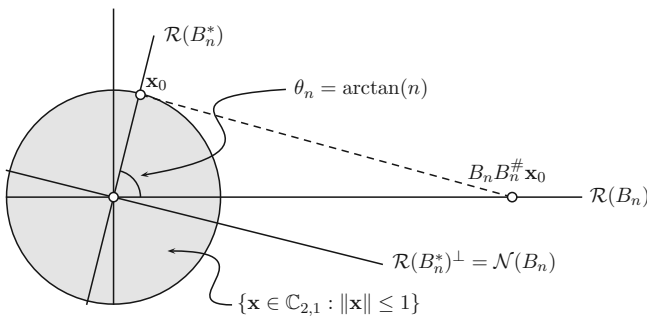


Figure 1: The geometry of the Example 2. In this figure, the vector  $\mathbf{x}_0$  maximizes  $\|B_n B_n^\# \mathbf{x}\|$  with the restriction  $\|\mathbf{x}\| \leq 1$ : this vector satisfies  $\|B_n B_n^\# \mathbf{x}_0\| = \|B_n B_n^\#\|$ .

Also, the study of  $B_n B_n^\#$  lights the behavior of the non group invertibility of  $B$ . By using Theorem 2, we get

$$B_n B_n^\# = \begin{bmatrix} m_n c_n & m_n s_n \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_n^{-1} m_n^{-1} & c_n^{-1} m_n^{-1} c_n^{-1} s_n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & s_n/c_n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & n \\ 0 & 0 \end{bmatrix},$$

which clearly shows that the sequence  $\{B_n B_n^\#\}_{n=1}^\infty$  is not bounded. This example is pictured in Fig. 1. Let us recall that if  $X$  is any group invertible matrix, then  $XX^\#$  is the projector onto  $\mathcal{R}(X)$  along  $\mathcal{N}(X)$ .

Besides the canonical angles between two subspaces of  $\mathbb{C}_{n,1}$ , there is another measure of the separation of two subspaces [5, Sect. 4.4]

DEFINITION 2. Let  $\mathcal{X}$  and  $\mathcal{Y}$  be two subspaces of  $\mathbb{C}_{n,1}$ . We define the gap between  $\mathcal{X}$  and  $\mathcal{Y}$  by

$$\widehat{\delta}(\mathcal{X}, \mathcal{Y}) = \max\{\delta(\mathcal{X}, \mathcal{Y}), \delta(\mathcal{Y}, \mathcal{X})\},$$

where  $\delta(\mathcal{X}, \mathcal{Y}) = \sup\{\text{dist}(u, \mathcal{Y}) : u \in \mathcal{X}, \|u\| = 1\}$ .

In the following result we shall find  $\widehat{\delta}(\mathcal{R}(A), \mathcal{R}(A^*))$  in terms of the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  when  $A \in \mathbb{C}_{n,n}$ . In addition, also  $\|AA^\#\|$  will be computed when  $A$  is a group invertible matrix.

THEOREM 4. Let  $A \in \mathbb{C}_{n,n}$ . Then  $\widehat{\delta}(\mathcal{R}(A), \mathcal{R}(A^*)) = \sin \psi$ , where  $\psi$  is the greatest canonical angle between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$ . If in addition,  $A$  is group invertible, then  $\|AA^\#\| = 1/\cos \psi$ .

*Proof.* Let us represent  $A$  as in Theorem 1 and denote  $r = \text{rank}(A)$ . It is known that if  $\mathcal{X}$  and  $\mathcal{Y}$  are two subspaces of  $\mathbb{C}_{n,1}$ , then  $\widehat{\delta}(\mathcal{X}, \mathcal{Y}) = \max\{\|I_n - P_{\mathcal{Y}} P_{\mathcal{X}}\|, \|I_n - P_{\mathcal{X}} P_{\mathcal{Y}}\|\}$  (see e.g., [4, Lemma 4.1.2]). By using  $P_{\mathcal{R}(A)} = AA^\dagger$ ,  $P_{\mathcal{R}(A^*)} = A^\dagger A$  and the equalities (5) we have

$$(I_n - P_{\mathcal{R}(A)}) P_{\mathcal{R}(A^*)} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} C^2 & CS \\ S^*C & S^*S \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ S^*C & S^*S \end{bmatrix},$$

and now we use (1) and (3):

$$\|(I_n - P_{\mathcal{R}(A)}) P_{\mathcal{R}(A^*)}\|^2 = \left\| \begin{bmatrix} 0 & 0 \\ S^*C & S^*S \end{bmatrix} \begin{bmatrix} 0 & CS \\ 0 & S^*S \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 & 0 \\ 0 & S^*S \end{bmatrix} \right\| = \|S\|^2.$$

In a similar way we have

$$(I_n - P_{\mathcal{R}(A^*)}) P_{\mathcal{R}(A)} = \begin{bmatrix} SS^* & -CS \\ -S^*C & -S^*S \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} SS^* & 0 \\ -S^*C & 0 \end{bmatrix}$$

and

$$\|(I_n - P_{\mathcal{R}(A^*)}) P_{\mathcal{R}(A)}\|^2 = \left\| \begin{bmatrix} SS^* & -CS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} SS^* & 0 \\ -S^*C & 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} SS^* & 0 \\ 0 & 0 \end{bmatrix} \right\| = \|S\|^2$$

because  $(SS^*)^2 + CSS^*C = (I_r - C^2)^2 + C(I_r - C^2)C = I_r - C^2 = SS^*$ . And therefore,  $\widehat{\delta}(\mathcal{R}(A), \mathcal{R}(A^*)) = \|S\| = \sin \psi$ .

In the rest of the proof, let us assume that  $A$  is group invertible. We shall use (1) to compute  $\|AA^\# \|$ : We get

$$AA^\# = V \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C^{-1}M^{-1} & C^{-1}M^{-1}C^{-1}S \\ 0 & 0 \end{bmatrix} V^* = V \begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix} V^* \tag{7}$$

and

$$(AA^\#)(AA^\#)^* = V \begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r & 0 \\ S^*C^{-1} & 0 \end{bmatrix} V^* = V \begin{bmatrix} I_r + C^{-1}SS^*C^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*.$$

In view of (3) we have  $I_r + C^{-1}SS^*C^{-1} = I_r + C^{-1}(I_r - C^2)C^{-1} = C^{-2}$ . Thus, if  $0 \leq \theta_1 \leq \dots \leq \theta_r < \pi/2$  are the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$ , then

$$\|AA^\# \|^2 = \|(AA^\#)(AA^\#)^*\| = \|C^{-2}\| = \max\{1/\cos^2 \theta_i, i = 1, \dots, r\} = 1/\cos^2 \theta_r.$$

This proves the theorem.  $\square$

The former Theorem 4 implies the following corollary which clarifies one hypothesis of the main result of the paper, Theorem 5.

**COROLLARY 1.** *Let  $\{A_m\}_{m=1}^\infty$  be a sequence of complex  $n \times n$  group invertible matrices. Let  $\psi_m$  be the greatest canonical angle between  $\mathcal{R}(A_m)$  and  $\mathcal{R}(A_m^*)$ . Then the following conditions are equivalent*

- (i)  $\{A_m A_m^\#\}_{m=1}^\infty$  is a bounded sequence.
- (ii) There exist  $\psi \in [0, \pi/2[$  and  $m_0 \in \mathbb{N}$  such that  $\psi_m \leq \psi$  for all  $m \geq m_0$ .
- (iii) There exist  $s < 1$  and  $m_0 \in \mathbb{N}$  such that  $\widehat{\delta}(\mathcal{R}(A_m), \mathcal{R}(A_m^*)) \leq s$  for all  $m \geq m_0$ .

Here comes the main result of the paper which is justified in some manner by Examples 1 and 2:

**THEOREM 5.** *Let  $\{A_m\}_{m=1}^\infty$  be a sequence of complex  $n \times n$  group invertible matrices converging to  $A$ . Let  $\psi_m$  be the greatest canonical angle between  $\mathcal{R}(A_m)$  and  $\mathcal{R}(A_m^*)$ . The following affirmations are equivalent*

- (i)  $A$  is group invertible and  $\lim_{m \rightarrow \infty} A_m^\# = A^\#$ .
- (ii)  $\{A_m^\#\}_{m=1}^\infty$  is a bounded sequence.
- (iii)  $\lim_{m \rightarrow \infty} A_m^\dagger = A^\dagger$  and there exist  $\psi \in [0, \pi/2[$  and  $m_0 \in \mathbb{N}$  such that  $\psi_m \leq \psi$  for all  $m \geq m_0$ .
- (iv)  $A$  is group invertible and  $\lim_{m \rightarrow \infty} A_m A_m^\# = AA^\#$ .

*Proof.*

(i)  $\Rightarrow$  (ii) is evident since  $\{A_m^\# \}_{m=1}^\infty$  is convergent, in particular is bounded.

(ii)  $\Rightarrow$  (iii): Since  $\{A_m\}_{m=1}^\infty$  and  $\{A_m^\#\}_{m=1}^\infty$  are bounded sequences, then the sequence  $\{A_m A_m^\#\}_{m=1}^\infty$  is bounded. By Corollary 1 there exist  $\psi \in [0, \pi/2[$  and  $m_0 \in \mathbb{N}$  such that  $\psi_m \leq \psi$  for all  $m \geq m_0$ .

Let  $r_m = \text{rank}(A_m)$ . By Theorem 1, we can represent  $A_m$  as follows

$$A_m = V_m \begin{bmatrix} M_m C_m & M_m S_m \\ 0 & 0 \end{bmatrix} V_m^*, \tag{8}$$

where  $V_m \in \mathbb{C}_{n,n}$  are unitary,  $M_m \in \mathbb{C}_{r_m, r_m}$  are nonsingular, and  $C_m \in \mathbb{C}_{r_m, r_m}$ ,  $S_m \in \mathbb{C}_{r_m, n-r_m}$  are real matrices such that the meanings of all these matrices are the same as those of  $V$ ,  $M$ ,  $C$  and  $S$ , respectively, in Theorem 1. Observe that  $\|C_m\| \leq 1$ . By Theorem 2, one has that every  $C_m$  is nonsingular,  $\|A_m^\dagger\| = \|M_m^{-1}\|$  and  $\|A_m^\#\| = \|C_m^{-1} M_m^{-1} C_m^{-1}\|$ . Thus

$$\|A_m^\dagger\| = \|M_m^{-1}\| = \|C_m C_m^{-1} M_m^{-1} C_m^{-1} C_m\| \leq \|C_m\| \|C_m^{-1} M_m^{-1} C_m^{-1}\| \|C_m\| \leq \|A_m^\#\|.$$

By hypothesis,  $\{A_m^\#\}_{m=1}^\infty$  is bounded, and thus  $\{A_m^\dagger\}_{m=1}^\infty$  is bounded. By Theorem 3 one has  $\lim_{m \rightarrow \infty} A_m^\dagger = A^\dagger$ .

(iii)  $\Rightarrow$  (i): Let  $r$  be the rank of  $A$ . Since  $\lim_{m \rightarrow \infty} A_m^\dagger = A^\dagger$ , by Theorem 3, there exists  $m_0 \in \mathbb{N}$  such that  $\text{rank}(A_m) = r$  for  $m \geq m_0$ . From now on, we will take  $m \geq m_0$ . By Theorem 1, we can represent  $A_m$  as in (8), where  $V_m \in \mathbb{C}_{n,n}$  are unitary,  $M_m \in \mathbb{C}_{r,r}$  are nonsingular, and  $C_m, S_m$  are real matrices such that the meanings of all these matrices are the same as those of  $V$ ,  $M$ ,  $C$  and  $S$ , respectively, in Theorem 1.

Let us denote  $F_m = C_m^{-1} M_m^{-1} C_m^{-1}$ , which is nonsingular. By using the left identity of (6) one has

$$A_m^\# = V_m \begin{bmatrix} C_m^{-1} M_m^{-1} & C_m^{-1} M_m^{-1} C_m^{-1} S_m \\ 0 & 0 \end{bmatrix} V_m^* = V_m \begin{bmatrix} F_m C_m & F_m S_m \\ 0 & 0 \end{bmatrix} V_m^*,$$

By applying the left identity of (4) one obtains

$$(A_m^\#)^\dagger = V_m \begin{bmatrix} C_m F_m^{-1} & 0 \\ S_m^* F_m^{-1} & 0 \end{bmatrix} V_m^* = V_m \begin{bmatrix} C_m^2 M_m C_m & 0 \\ S_m^* C_m M_m C_m & 0 \end{bmatrix} V_m^*. \tag{9}$$

On the other hand, by employing the representations of  $A_m A_m^\dagger$  and  $A_m^\dagger A_m$  given in (5) one has

$$\begin{aligned} (A_m^\dagger A_m) A_m (A_m A_m^\dagger) &= V_m \begin{bmatrix} C_m^2 & C_m S_m \\ S_m^* C_m & S_m^* S_m \end{bmatrix} \begin{bmatrix} M_m C_m & M_m S_m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} V_m^* \\ &= V_m \begin{bmatrix} C_m^2 M_m C_m & 0 \\ S_m^* C_m M_m C_m & 0 \end{bmatrix} V_m^*. \end{aligned} \tag{10}$$

Since  $\{A_m\}_{m=1}^\infty$  and  $\{A_m^\dagger\}_{m=1}^\infty$  are convergent sequences by hypothesis, the expressions (9)–(10) show that  $\{(A_m^\#)^\dagger\}_{m=1}^\infty$  is a convergent sequence.

For  $m \in \mathbb{N}$ , let  $B_m = (A_m^\#)^\dagger$ . We have proved that  $\{B_m\}_{m=1}^\infty$  is a convergent sequence. Evidently, one has  $B_m^\dagger = A_m^\#$ . By hypothesis and the right equalities of (4) and (6), one has

$$\|A_m^\#\| = \|C_m^{-1}M_m^{-1}C_m^{-1}\| \leq \|C_m^{-1}\|^2\|M_m^{-1}\| = \frac{\|A_m^\dagger\|}{\cos^2 \psi_m} < \frac{\|A_m^\dagger\|}{\cos^2 \psi}. \tag{11}$$

The sequence  $\{A_m^\dagger\}_{m=1}^\infty$  is bounded because by hypothesis, this sequence is convergent. Therefore (11) proves that  $\{A_m^\#\}_{m=1}^\infty$  is bounded. Thus,  $\{B_m^\dagger\}_{m=1}^\infty$  is a bounded sequence. By Theorem 3 we have  $\{B_m^\dagger\}_{m=1}^\infty$  converges to some matrix, say  $R$ , in other words,  $\lim_{m \rightarrow \infty} A_m^\# = R$ . Now (recall that  $\{A_m\}_{m=1}^\infty$  converges to  $A$ ), by letting  $m \rightarrow \infty$  in the equalities

$$A_m = A_m A_m^\# A_m, \quad A_m^\# = A_m^\# A_m A_m^\#, \quad A_m A_m^\# = A_m^\# A_m$$

we obtain, respectively,

$$A = ARA, \quad R = RAR, \quad AR = RA,$$

from which we get that  $A$  is group invertible and  $A^\# = R = \lim_{m \rightarrow \infty} A_m^\#$ .

(i)  $\Rightarrow$  (iv): Evident.

(iv)  $\Rightarrow$  (iii): By hypothesis, the sequence  $\{A_m A_m^\#\}_{m=1}^\infty$  is convergent, hence bounded. By Corollary 1, there exist  $\psi \in [0, \pi/2[$  and  $m_0 \in \mathbb{N}$  such that  $\psi_m \leq \psi$  for all  $m \geq m_0$ .

Next, we shall prove  $\lim_{m \rightarrow \infty} A_m^\dagger = A^\dagger$ . Let  $r_m = \text{rank}(A_m)$ . By Theorem 1, we can represent matrices  $A_m$  as in (8), where the meanings of  $V_m \in \mathbb{C}_{n,n}$ ,  $M_m, C_m \in \mathbb{C}_{r_m, r_m}$ ,  $S_m \in \mathbb{C}_{r_m, n-r_m}$  are the same as those of matrices  $V, M, C, S$  in Theorem 1, respectively. By a similar computation as in (7) we have

$$A_m^\# A_m = V_m \begin{bmatrix} I_{r_m} & C_m^{-1} S_m \\ 0 & 0 \end{bmatrix} V_m^* = V_m \begin{bmatrix} C_m^{-1} C_m & C_m^{-1} S_m \\ 0 & 0 \end{bmatrix} V_m^*.$$

By the left equality of (4) applied to  $A_m^\# A_m$  we get

$$(A_m^\# A_m)^\dagger = V_m \begin{bmatrix} C_m^2 & 0 \\ S_m^* C_m & 0 \end{bmatrix} V_m^*.$$

Now we use the  $C^*$ -identity to get an expression of  $\|(A_m^\# A_m)^\dagger\|$ :

$$\begin{aligned} \|(A_m^\# A_m)^\dagger\|^2 &= \left\| \left[ (A_m^\# A_m)^\dagger \right] \left[ (A_m^\# A_m)^\dagger \right]^* \right\| = \left\| \begin{bmatrix} C_m^2 & C_m S_m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_m^2 & 0 \\ S_m^* C_m & 0 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} C_m^4 + C_m S_m S_m^* C_m & 0 \\ 0 & 0 \end{bmatrix} \right\| = \|C_m^4 + C_m S_m S_m^* C_m\|. \end{aligned} \tag{12}$$

By taking into account (3) we get  $C_m^4 + C_m S_m S_m^* C_m = C_m (C_m^2 + S_m S_m^*) C_m = C_m^2$ . Hence (12) reduces to  $\|(A_m^\# A_m)^\dagger\| = \|C_m\|$  because  $C_m$  is a diagonal and real matrix. Furthermore, since  $C_m$  is a diagonal matrix whose elements are the cosines of some angles, then  $\|C_m\| \leq 1$ .



We summarize what we have at our disposal. By hypothesis,  $\{A_m A_m^\# \}_{m=1}^\infty$  converges to  $AA^\#$ , and in the previous paragraph we proved that  $\{(A_m A_m^\#)^\dagger \}_{m=1}^\infty$  is a bounded sequence. By Theorem 3, there exists  $m_0 \in \mathbb{N}$  such that  $\text{rank}(A_m A_m^\#) = \text{rank}(AA^\#)$  for all  $m \geq m_0$ . By taking into account that  $A_m A_m^\#$  is the projector onto  $\mathcal{R}(A_m)$  along  $\mathcal{N}(A_m)$  we have  $\text{rank}(A_m A_m^\#) = \text{rank}(A_m)$  for any  $m \geq m_0$ , and analogously, we get  $\text{rank}(AA^\#) = \text{rank}(A)$ . Thus, we can assure  $\text{rank}(A_m) = \text{rank}(A)$  for all  $m \geq m_0$ . Finally, by Theorem 3 we get  $\lim_{m \rightarrow \infty} A_m^\dagger = A^\dagger$ .  $\square$

REMARKS.

1. Observe that the proof of the former result (see equations (9)–(10)) distills that for any matrix  $A \in \mathbb{C}_{n,n}$  group invertible,

$$(A^\#)^\dagger = (A^\dagger A)A(AA^\dagger).$$

2. The two conditions in item (iii) of Theorem 5 are independent as can be seen in the examples 1 and 2 given in the first section: In Example 1, matrices  $A_n$  satisfy  $A_n A_n^\dagger = A_n^\dagger A_n = \text{diag}(1, 0)$ , and by using the definition of the canonical angles, we have that the greatest canonical angle between  $\mathcal{R}(A_n)$  and  $\mathcal{R}(A_n^*)$  is 0; however  $\lim_{m \rightarrow \infty} A_m^\dagger$  does not exist. In Example 2, as we showed, the greatest canonical angle between  $\mathcal{R}(B_n)$  and  $\mathcal{R}(B_n^*)$  is  $\arctan n$ ; however  $\text{rank}(B_m) = \text{rank}(B) = 1$ , which yields (by Theorem 3) that  $\lim_{m \rightarrow \infty} B_m^\dagger = B^\dagger$ .

It is worth comparing Theorem 3 with the continuity of the Drazin inverse, since the group inverse is a particular case of the Drazin inverse. Let us review some known facts of the Drazin inverse (see [1, Chapter 4] or [3, Chapter 7] for a deeper study): Let  $A \in \mathbb{C}_{n,n}$ . The smallest positive integer  $k$  for which  $\text{rank}(A^k) = \text{rank}(A^{k+1})$  holds is called the *index* of  $A$ , and denoted  $\text{Ind}(A)$ . Furthermore, for every  $A \in \mathbb{C}_{n,n}$ , there is a unique matrix denoted by  $A^D$  that satisfies

$$A^k A^D A = A^k, \quad A^D A A^D = A^D, \quad A A^D = A^D A.$$

This matrix  $A^D$  is called the *Drazin inverse* of  $A$ . Evidently, the group inverse is a particular case of the Drazin inverse. Since the continuity of the Drazin inverse has been studied by some authors, it is convenient to compare such studies with our Theorem 3.

S.L. Campbell and C.D. Meyer were the first authors in characterizing the Drazin inverse. They defined the *core rank* of  $A \in \mathbb{C}_{n,n}$  as the rank of  $A^k$ , where  $k = \text{Ind}(A)$ . See [3, Theorem 10.7.1] for the proof of the following result:

**THEOREM 6.** *Let  $\{A_m\}_{m=1}^\infty$  a sequence in  $\mathbb{C}_{n,n}$  that converges to  $A$ . Then the sequence  $\{A_m^D\}_{m=1}^\infty$  converges to  $A^D$  if and only if there exists  $m_0 \in \mathbb{N}$  such that the core rank of  $A_m$  is equal to the core rank of  $A$  for  $m \geq m_0$ .*

As is easily seen, in this characterization, no appeal to the canonical angles appears.

Another study for the continuity of the Drazin inverse was made in [7]. In such paper, the following result was proved (we slightly change the statement because the original setting of this work is a Banach algebra).

**THEOREM 7.** *Let  $\{A_m\}_{m=1}^{\infty}$  a sequence in  $\mathbb{C}_{n,n}$  that converges to  $A$ . Then  $\{A_m^D\}_{m=1}^{\infty}$  converges to  $A^D$  if and only if  $\sup\{\|A_m^D\| : m \in \mathbb{N}\} < \infty$ .*

We shall see how this result implies (ii)  $\Rightarrow$  (i) of Theorem 5: Let  $A_m$  be group invertible matrices for any  $m \in \mathbb{N}$  such that  $\lim_{m \rightarrow \infty} A_m = A$  and  $\sup\{\|A_m^{\#}\| : m \in \mathbb{N}\} < \infty$ . By Theorem 7 we get  $\lim_{m \rightarrow \infty} A_m^{\#} = A^D$ . By making  $m \rightarrow \infty$  in the equality  $A_m A_m^{\#} A_m = A_m$  we get  $AA^D A = A$ . This implies that  $A$  is group invertible and  $A^D = A^{\#}$ .

However, let us remark that the proof of Theorem 5 presents a unified approach by using the canonical angles.

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