

On the local moduli of squareness

by

ANTONIO J. GUIRAO (Murcia)

Abstract. We introduce the notions of pointwise modulus of squareness and local modulus of squareness of a normed space X . This answers a question of C. Benítez, K. Przesławski and D. Yost about the definition of a sensible localization of the modulus of squareness. Geometrical properties of the norm of X (Fréchet smoothness, Gâteaux smoothness, local uniform convexity or strict convexity) are characterized in terms of the behaviour of these moduli.

1. Introduction. Let us recall the notion of modulus of squareness, originally defined in [7], where it arose naturally from studying Lipschitz continuous set-valued functions. Given a normed space X , one observes that for any $x, y \in X$ with $\|y\| < 1 < \|x\|$, there is a unique $z = z(x, y)$ in the line segment $[x, y]$ with $\|z\| = 1$. We put

$$\omega(x, y) = \frac{\|x - z(x, y)\|}{\|x\| - 1}$$

and define $\xi = \xi_X : [0, 1) \rightarrow [1, \infty]$ by

$$\xi(\beta) = \sup\{\omega(x, y) : \|y\| \leq \beta < 1 < \|x\|\}.$$

It is shown in [7] that for an inner product space, $\xi(\beta) = \xi_2(\beta) = 1/\sqrt{1 - \beta^2}$, and for any normed space containing $l_1(2)$, $\xi(\beta) = \xi_1(\beta) = (1 + \beta)/(1 - \beta)$. The following theorem [1, Theorem O] puts together all the known properties of this modulus.

THEOREM 1.1. *Let X be any normed space, and ξ its modulus of squareness. Then*

- (a) $\xi(\beta) = \sup\{\xi_M(\beta) : M \subset X, \dim M = 2\}$,
- (b) ξ is strictly increasing and convex,

2000 *Mathematics Subject Classification.* Primary 46B10, 46B20; Secondary 46B03.

Key words and phrases: Banach spaces, locally uniformly rotund norms, Fréchet smooth norms, strict convex norms, Gâteaux smooth norms, modulus of squareness.

Research supported by the grants MTM2005-08379 of MECO (Spain), 00690/PI/04 of Fundación Séneca (CARM, Spain) and AP2003-4453 of MECO (Spain).

- (c) $\xi < \xi_1$ everywhere on $(0, 1)$, unless X contains arbitrarily close copies of $l_1(2)$,
- (d) $\xi' \leq \xi'_1$ almost everywhere on $(0, 1)$,
- (e) $\xi > \xi_2$ everywhere on $(0, 1)$, unless X is an inner product space,
- (f) X is uniformly convex if and only if $\lim_{\beta \rightarrow 1} (1 - \beta)\xi(\beta) = 0$,
- (g) X is uniformly smooth if and only if $\xi'(0) = 0$,
- (h) $\xi_{X^*}(\beta) = 1/\xi^{-1}(1/\beta)$ for $\beta \in [0, 1)$,
- (i) if $\xi(\beta) < 1/(1 - \beta)$ for some β , then X has uniformly normal structure.

The proof of these properties can be found in [1, 7] and also some of them as well as a more geometrical characterization of ξ in [9–11].

Observe in particular that the behaviour of ξ near 1 is related to convexity, and its behaviour near zero is related to smoothness.

The question of the existence of a sensible *localization* of the modulus of squareness was posed in [1]. In order to answer this question we define two new moduli.

From now on and for the sake of clarity, for any norm one vector x , $\lambda > 0$ and y with $\|y\| < 1$, we put

$$\omega_x(\lambda, y) = \omega((1 + \lambda)x, y) \quad \text{and} \quad z_x(\lambda, y) = z((1 + \lambda)x, y).$$

Therefore $\omega_x(\lambda, y) = \|(1 + \lambda)x - z_x(\lambda, y)\|/\lambda$. Moreover, we can deduce that for $y \in \text{span}\{x\}$ and for any $\lambda > 0$, $\omega_x(\lambda, y) = 1$, since $z_x(\lambda, y)$ would be x .

DEFINITION 1.2. For any norm one vectors x, y the *pointwise modulus of squareness at x in direction y* is the function $\xi_{X,x,y} = \xi_{x,y} : [0, 1) \rightarrow [1, \infty)$ defined by

$$\xi_{x,y}(\beta) = \sup\{\omega_x(\lambda, \gamma y) : |\gamma| \leq \beta, \lambda > 0\}.$$

DEFINITION 1.3. For any norm one vector x the *local modulus of squareness at x* is the function $\xi_{X,x} = \xi_x : [0, 1) \rightarrow [1, \infty)$ defined by

$$\xi_x(\beta) = \sup\{\omega_x(\lambda, y) : \|y\| \leq \beta, \lambda > 0\} = \sup_{\|y\|=1} \{\xi_{x,y}(\beta)\}.$$

Observe that for any subspace $M \subset X$ of dimension 2 containing norm one vectors x, y we have $\xi_{x,y} = \xi_{M,x,y}$. For ξ_x we establish an analogue to (a) of Theorem 1.1. Indeed,

$$\xi_x(\beta) = \sup\{\xi_{M,x}(\beta) : x \in M \subset X, \dim M = 2\}.$$

One can see that for any $\beta \in [0, 1)$,

$$\xi(\beta) = \sup\{\xi_x(\beta) : x \in S_X\} = \sup\{\xi_{x,y}(\beta) : x, y \in S_X\}.$$

We shall show how these moduli are related to various geometrical properties of the norm of X . In particular, in Section 3 we recall the notions of Gâteaux smoothness and Fréchet smoothness and show that whether or not a normed space X is Fréchet (resp. Gâteaux) smooth depends on the

behaviour of the local (resp. pointwise) modulus of squareness near zero. In Section 4 we recall the notions of local uniform convexity and strict convexity and show that whether or not X is locally uniformly (resp. strictly) convex depends on the behaviour of the local (resp. pointwise) modulus of squareness near 1. More precisely, we shall establish:

THEOREM 1.4. *Let X be a normed space and x a norm one vector. Then*

- (a) X is Gâteaux smooth at x iff $\xi'_{x,y}(0) = 0$ for all y with $\|y\| = 1$.
- (b) X is Fréchet smooth at x iff $\xi'_x(0) = 0$.
- (c) X is strictly convex at x iff $\lim_{\beta \rightarrow 1} (1 - \beta)\xi_{x,y}(\beta) = 0$ for all y with $\|y\| = 1$.
- (d) X is locally uniformly convex at x iff $\lim_{\beta \rightarrow 1} (1 - \beta)\xi_x(\beta) = 0$.

In the following section we focus on the properties of the ratio $\omega_x(\cdot, \cdot)$.

2. Properties of $\omega_x(\lambda, y)$. By a normed space we mean a pair $(X, \|\cdot\|)$, where X is a linear space and $\|\cdot\|$ is a norm, although we will often write X instead of $(X, \|\cdot\|)$. We set $B_X = \{x \in X : \|x\| \leq 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$.

The following lemma can be found in [1] as part of the proof that ξ is locally Lipschitz continuous.

LEMMA 2.1. *Let X be a normed space and $x, y \in S_X$. Then, for any $\lambda > 0$ and $0 \leq \beta < \gamma < 1$,*

$$\omega_x(\lambda, \gamma y) - \omega_x(\lambda, \beta y) \leq \xi_1(\gamma) - \xi_1(\beta).$$

For fixed norm one vectors x, y , the modulus $\xi_{x,y}$ can be expressed in a simpler way:

PROPOSITION 2.2. *Let X be a normed space and x, y two norm one vectors. Then, for all $\beta \in [0, 1)$,*

$$\xi_{x,y}(\beta) = \sup \{ \omega_x(\lambda, \pm \beta y) : \lambda > 0 \}.$$

Proof. It is enough to show that for any fixed $\lambda > 0$ and any $\gamma \leq \beta$ we have $\omega_x(\lambda, \beta y) \geq \omega_x(\lambda, \gamma y)$. We use the following result which can be found in [3, 4, 8].

LEMMA 2.3. *Let X be a two-dimensional normed space and let K_1, K_2 be closed convex subsets of X with nonempty interior. If $K_1 \subset K_2$ then $r(K_1) \leq r(K_2)$, where $r(K_i)$ denotes the length of the circumference of K_i , $i = 1, 2$.*

This lemma can be applied to the triangles: K_1 with vertices the origin, $z_x(\lambda, \gamma y)$ and $(1+\lambda)x$, and K_2 with vertices the origin, $z_x(\lambda, \beta y)$ and $(1+\lambda)x$.

Therefore

$$\begin{aligned} r(K_1) &= \|(1 + \lambda)x\| + \|z_x(\lambda, \gamma y)\| + \|(1 + \lambda)x - z_x(\lambda, \gamma y)\| \\ &\leq \|(1 + \lambda)x\| + \|z_x(\lambda, \beta y)\| + \|(1 + \lambda)x - z_x(\lambda, \beta y)\| = r(K_2). \end{aligned}$$

Simplifying and dividing by λ , we obtain the desired inequality. ■

PROPOSITION 2.4. *Let X be a normed space. If x, y are norm one vectors and $0 \leq \beta < \gamma < 1$, then*

$$(2.1) \quad \xi_{x,y}(\gamma) - \xi_{x,y}(\beta) \leq \xi_1(\gamma) - \xi_1(\beta),$$

$$(2.2) \quad \xi_x(\gamma) - \xi_x(\beta) \leq \xi_1(\gamma) - \xi_1(\beta).$$

In particular, $\xi_{x,y}$ and ξ_x are locally Lipschitz continuous functions.

Proof. From Lemma 2.1 we deduce that $\omega_x(\lambda, \gamma y) - \xi_{x,y}(\beta) \leq \xi_1(\gamma) - \xi_1(\beta)$ and, by Proposition 2.2, we obtain inequality (2.1), taking suprema over $\lambda > 0$. Inequality (2.2) follows similarly from (2.1), on taking suprema over $y \in S_X$. ■

Trying to simplify the expression for $\xi_{x,y}$ obtained in Proposition 2.2, one can study the behaviour of the function $\omega_x(\cdot, y)$ for fixed $x \in S_X$ and $y \in \mathring{B}_X$. The next useful result is evident.

PROPOSITION 2.5. *Let X be a normed space and $x \in S_X$. Then*

$$1 \leq \omega_x(\lambda) := \sup\{\omega_x(\lambda, y) : y \in \mathring{B}_X\} \leq 1 + 2/\lambda.$$

We now prove that the limit of the function $\omega_x(\lambda, y)$ when λ goes to zero always exists and we compute it.

Recall that for a normed space X and $x, y \in X \setminus \{0\}$, one can define the right derivative of the norm at x in direction y as the limit

$$N_+(x, y) = \lim_{\lambda \searrow 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}.$$

PROPOSITION 2.6. *Let X be any normed space, $x \in S_X$, and $y \in X$ with $\|y\| < 1$. Then*

$$\lim_{\lambda \searrow 0} \omega_x(\lambda, y) = \frac{\|x - y\|}{1 - N_+(x, y)}.$$

In order to prove this result we need to introduce some notation.

Fix a normed space X , $x \in S_X$ and $y \in \mathring{B}_X$ with $y \notin \text{span}\{x\}$. We denote by $z'(\lambda)$ the unique vector which lies in $\text{span}\{z_x(\lambda, y)\}$ and on the ray which starts at x and has direction y , that is,

$$z'(\lambda) = \{x + \mu y : \mu \geq 0\} \cap \text{span}\{z_x(\lambda, y)\}.$$

We can write $z'(\lambda) = x + \mu(\lambda)y$ for some $\mu(\lambda) \geq 0$. Denote by f_λ a continuous functional on X satisfying $f_\lambda(x) = f_\lambda(z_x(\lambda, y)) = 1$. We can also write $z_x(\lambda, y) = (1 + \lambda)x + \nu(\lambda)(y - (1 + \lambda)x)$ for some $\nu(\lambda) \in [0, 1]$.

LEMMA 2.7. *Let X be a normed space, $x \in S_X$ and $y \in \mathring{B}_X$ such that $y \notin \text{span}\{x\}$. Then*

- (a) $\lim_{\lambda \searrow 0} z_x(\lambda, y) = x$.
- (b) $\lim_{\lambda \searrow 0} \mu(\lambda) = 0$.
- (c) $\lim_{\lambda \searrow 0} f_\lambda(y) = N_+(x, y)$.

Proof of Lemma 2.7. For (a) it is enough to show that $\nu(\lambda)$ tends to zero as $\lambda \rightarrow 0$. First, observe that $\varphi(t) = \|(1 + \lambda)x + t(y - (1 + \lambda)x)\|$ is a convex function satisfying $\varphi(1) = \|y\|$ and $\varphi(0) = 1 + \lambda$. Therefore $\varphi(t) \leq (1 + \lambda) + t(\|y\| - (1 + \lambda))$ for $t \in [0, 1]$. Secondly, since $z_x(\lambda, y) \in S_X$, we have $\varphi(\nu(\lambda)) = 1$, that is, $1 \leq (1 + \lambda) + \nu(\lambda)(\|y\| - (1 + \lambda))$. Finally, since $\nu(\lambda) \in [0, 1]$, we obtain $\lim_{\lambda \searrow 0} \nu(\lambda) = 0$ and (a) is proved.

For (b), observe that $z_x(\lambda, y) = (1 + \lambda)(1 - \nu(\lambda))x + \nu(\lambda)y$. Since $z'(\lambda)$ lies in $\text{span}\{z_x(\lambda, y)\}$, there exists $\alpha(\lambda) \in \mathbb{R}$ such that

$$x + \mu(\lambda)y = z'(\lambda) = \alpha(\lambda)z_x(\lambda, y),$$

from which $\alpha(\lambda) = (1 + \lambda)^{-1}(1 - \nu(\lambda))^{-1}$ and then

$$\mu(\lambda) = \nu(\lambda)/[(1 + \lambda)(1 - \nu(\lambda))].$$

Since $\nu(\lambda)$ converges to 0 as $\lambda \rightarrow 0$, (b) is proved.

In order to show (c), observe that, by (b), we have

$$N_+(x, y) = \lim_{\lambda \searrow 0} \frac{\|x + \mu(\lambda)y\| - \|x\|}{\mu(\lambda)} = \lim_{\lambda \searrow 0} \frac{\|z'(\lambda)\| - \|x\|}{\mu(\lambda)}.$$

Since $z'(\lambda) \in \text{span}\{z\}$, $\|z'(\lambda)\| = f_\lambda(z'(\lambda))$. Hence, as $f_\lambda(x) = \|x\|$,

$$N_+(x, y) = \lim_{\lambda \searrow 0} \frac{f_\lambda(z'(\lambda)) - f_\lambda(x)}{\mu(\lambda)} = \lim_{\lambda \searrow 0} \frac{\mu(\lambda)f_\lambda(y)}{\mu(\lambda)} = \lim_{\lambda \searrow 0} f_\lambda(y). \quad \blacksquare$$

Proof of Proposition 2.6. First of all, if $y \in \text{span}\{x\}$ then $1 - N_+(x, y) = \|x - y\|$, and since $\omega_x(\lambda, y) = 1$, this case is clear. So, assume that $y \notin \text{span}\{x\}$ and consider the unique vector $w(\lambda)$ satisfying the conditions $f_\lambda(w(\lambda)) = 1$ and $w(\lambda) \in \{\mu((1 + \lambda)x - y) : \mu \geq 0\}$. One can easily see, by comparing similar triangles, that $\omega_x(\lambda, y) = \|w(\lambda)\|$. Since $f_\lambda(w(\lambda)) = 1$, it is clear that

$$w(\lambda) = (1 + \lambda - f_\lambda(y))^{-1}[(1 + \lambda)x - y],$$

that is,

$$\omega_x(\lambda, y) = \frac{\|(1 + \lambda)x - y\|}{1 + \lambda - f_\lambda(y)}.$$

Using the continuity of the norm and item (c) of the previous lemma we obtain the desired equality. \blacksquare

REMARK 2.8. However, this last fact does not help to compute $\xi_{x,y}(\beta)$, since the function $\omega_x(\cdot, y)$ is neither convex nor monotonic as the following example shows.

EXAMPLE 2.9. For any $0 < \varepsilon < 1/2$, consider in \mathbb{R}^2 the norm defined by

$$\|x\| = \max\{(1 - \varepsilon)^{-1}\|x\|_\infty, \|x\|_1\},$$

and the vectors $x = (1 - \varepsilon, 0)$ and $y = (\varepsilon, 1 - \varepsilon)$. Fix $\beta \geq 1 - \varepsilon$. Here is the graph of the function $\omega_x(\cdot, \beta y)$ for $\varepsilon = 0.2$ and $\beta = 0.88$.

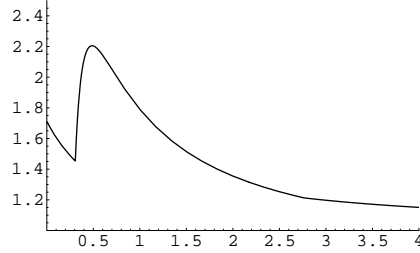


Fig. 1

3. On differentiability and localized squareness moduli. Throughout this section X will be a normed space endowed with the norm $\|\cdot\|$. The collection of support functionals for a norm one vector x is defined as

$$\mathcal{D}(x) = \{f \in X^* : \|f\| = 1, f(x) = \|x\| = 1\}.$$

We recall that the *modulus of smoothness* of a normed space is the function $\varrho : [0, \infty) \rightarrow \mathbb{R}^+$ defined by

$$\varrho(\beta) = \sup\{(\|x + \beta y\| + \|x - \beta y\|)/2 - 1 : \|x\| = \|y\| = 1\}.$$

Localizations of this modulus are the *local modulus of smoothness*, defined for any $x \in S_X$ and all $\beta \in [0, \infty)$ by

$$\varrho_x(\beta) = \sup\{(\|x + \beta y\| + \|x - \beta y\|)/2 - 1 : \|y\| = 1\},$$

and the *pointwise modulus of smoothness*, defined for any norm one vectors x, y and all $\beta \in [0, \infty)$ by

$$\varrho_{x,y}(\beta) = (\|x + \beta y\| + \|x - \beta y\|)/2 - 1.$$

Recall that a normed space is: *Gâteaux smooth at $x \in S_X$ in direction $y \in S_X$* iff $\varrho_{x,y}(\beta)/\beta \rightarrow 0$ as $\beta \rightarrow 0$; *Gâteaux smooth at $x \in S_X$* iff it is Gâteaux smooth at x in every direction $y \in S_X$; *Gâteaux smooth* iff it is Gâteaux smooth at any $x \in S_X$; *Fréchet smooth at $x \in S_X$* iff $\varrho_x(\beta)/\beta \rightarrow 0$ as $\beta \rightarrow 0$; and *Fréchet smooth* iff it is Fréchet smooth at any $x \in S_X$.

For any norm one vectors x, y , we define the function $\varepsilon_{x,y} : [0, \infty) \rightarrow [0, \infty)$ by the formula

$$\varepsilon_{x,y}(\beta) = \sup \left\{ \frac{\|x + \beta w\| - \|x\|}{\beta} - f(w) : w \in Y, f \in \mathcal{D}_Y(x) \right\},$$

where $Y = \text{span}\{x, y\}$ and $\mathcal{D}_Y(x) = \{f|_Y : f \in \mathcal{D}(x)\}$. One can observe that this function is increasing and that the space is Gâteaux smooth at x in direction y if and only if $\varepsilon_{x,y}(\beta) \rightarrow 0$ as $\beta \rightarrow 0$. Let us show a relation between $\varepsilon_{x,y}$ and the pointwise modulus of squareness $\xi_{x,y}$.

PROPOSITION 3.1. *For any norm one vectors x, y and all $\beta \in [0, 1)$,*

$$\xi_{x,y}(\beta) \leq 1 + \frac{2\beta}{(1-\beta)^2} \varepsilon_{x,y} \left(\frac{2\beta}{1-\beta} \right).$$

Proof. Fix $x, y \in S_X$, $\lambda > 0$, $\beta \in [0, 1)$ and a linear functional $f \in \mathcal{D}_Y(x)$. Then there exists $z_0 \in [\beta y, (1+\lambda)x]$ such that $f(z_0) = 1$. Pick a vector u such that $f(u) = 0$ and $z_0 \in [u, (1+\lambda)x]$. It follows that there exists $\mu \geq 0$ such that $u = (1-\mu)(1+\lambda)x + \mu\beta y$ and, since $f(u) = 0$, that $\mu = (1+\lambda)/(1+\lambda-\beta f(y))$. Thus,

$$\|u\| \leq \frac{(1+\lambda)\beta}{1+\lambda-\beta f(y)} (|f(y)| + 1) \leq \frac{2\beta}{1-\beta}.$$

As $z_0 \in [u, (1+\lambda)x]$, there exists $\alpha \in (0, 1)$ such that $z_0 = (1-\alpha)(1+\lambda)x + \alpha u$. Using the fact that $f(z_0) = 1$, it is easily seen that $\alpha = \lambda/(1+\lambda)$. Therefore

$$(3.1) \quad \frac{\|z_0 - x\|}{\lambda} = \frac{\|u\|}{1+\lambda} \leq \|u\| \leq \frac{2\beta}{1-\beta},$$

$$(3.2) \quad \|z_0 - x\| = \frac{\lambda}{1+\lambda} \|u\| \leq \|u\| \leq \frac{2\beta}{1-\beta}.$$

Observe now that, from the definition of $\varepsilon_{x,y}$, it follows that

$$\|(1+\lambda)x - z_0\| - \|\lambda x\| \leq \|x - z_0\| \varepsilon_{x,y} (\|x - z_0\|/\lambda).$$

Dividing by λ and using (3.1) one obtains the inequality

$$(3.3) \quad \frac{\|(1+\lambda)x - z_0\|}{\lambda} \leq 1 + \frac{2\beta}{1-\beta} \varepsilon_{x,y} \left(\frac{2\beta}{1-\beta} \right).$$

Now, put $z = z_x(\lambda, \beta y)$ and denote by ξ_X the modulus of squareness of X . One can easily see that $\|z - z_0\| \leq (\|z_0\| - 1)\xi_X(\beta)$ and $\|z_0\| - 1 \leq \|x - z_0\| \varepsilon_{x,y}(\|x - z_0\|)$. Putting both together, and using (3.1), (3.2) and $\xi_X \leq \xi_1$, one has

$$(3.4) \quad \frac{\|z - z_0\|}{\lambda} \leq \xi_1(\beta) \left(\frac{2\beta}{1-\beta} \right) \varepsilon_{x,y} \left(\frac{2\beta}{1-\beta} \right).$$

Finally, since

$$\omega_x(\lambda, \beta y) \leq \frac{\|(1 + \lambda)x - z_0\|}{\lambda} + \frac{\|z - z_0\|}{\lambda},$$

using (3.3) and (3.4) one obtains

$$\omega_x(\lambda, \beta y) \leq 1 + \frac{2\beta}{1 - \beta} \varepsilon_{x,y} \left(\frac{2\beta}{1 - \beta} \right) (1 + \xi_1(\beta)),$$

which, on taking suprema over $\lambda > 0$, finishes the proof. ■

Now we establish a relation between the pointwise modulus of squareness $\xi_{x,y}$ and the pointwise modulus of smoothness $\varrho_{x,y}$.

PROPOSITION 3.2. *For any norm one vectors x, y and for every $\beta \in [0, 1)$,*

$$(3.5) \quad \varrho_{x,y}(\beta) \leq \xi_{x,y}(\beta) - 1,$$

$$(3.6) \quad \varrho_x(\beta) \leq \xi_x(\beta) - 1.$$

Proof. Observe that the second inequality follows from the first on taking suprema over $y \in S_X$. Therefore we just have to show (3.5). Fix norm one vectors x, y . For a fixed $\beta \in [0, 1)$ and $\lambda > 0$, we set

$$\begin{aligned} y_1 &= y_1(\lambda, \beta y) = -(1 + \lambda)\beta y, & y_2 &= y_2(\lambda, \beta y) = (1 + \lambda)\beta y, \\ x' &= (1 + \lambda)x, & z_i &= (1 - \alpha_i)x' + \alpha_i y_i, \end{aligned}$$

where $\alpha_i \in [0, 1]$ for $i = 1, 2$.

On one hand, $1 = \|z_i\| \geq f(z_i)$ for any $f \in \mathcal{D}(x)$. Therefore $\alpha_i \geq \lambda/(1 + \lambda - f(y_i))$. On the other hand, $\|x' - y_i\| = (1 + \lambda)\|x \pm \beta y\|$. Since, for $\lambda < (1 - \beta)/\beta$,

$$\frac{\alpha_i(\lambda)\|x' - y_i\|}{\lambda} = \omega_x(\lambda, \pm(1 + \lambda)\beta y) \leq \xi_{x,y}((1 + \lambda)\beta),$$

we have

$$\|x' - y_1\| + \|x' - y_2\| \leq \xi_{x,y}((1 + \lambda)\beta) \left(\frac{\lambda}{\alpha_1} + \frac{\lambda}{\alpha_2} \right).$$

Since $\alpha_i \geq \lambda/(1 + \lambda - f(y_i))$ we deduce that

$$\begin{aligned} \|x' - y_1\| + \|x' - y_2\| &\leq \xi_{x,y}((1 + \lambda)\beta)(2 + 2\lambda - (f(y_1) + f(y_2))) \\ &= \xi_{x,y}((1 + \lambda)\beta)(2 + 2\lambda) = 2\xi_{x,y}((1 + \lambda)\beta)(1 + \lambda), \end{aligned}$$

and therefore

$$\|x + \beta y\| + \|x - \beta y\| \leq \frac{\|x' - y_1\| + \|x' - y_2\|}{1 + \lambda} \leq 2\xi_{x,y}((1 + \lambda)\beta),$$

which means that

$$\varrho_{x,y}(\beta) \leq \xi_{x,y}((1 + \lambda)\beta) - 1.$$

Since this is true for $\lambda < (1 - \beta)/\beta$, we can let λ tend to 0 and, by the continuity of $\xi_{x,y}$, we obtain the desired inequality. ■

THEOREM 3.3. *Let ξ_x and $\xi_{x,y}$ be the localized squareness moduli of X . Then*

- (a) X is Gâteaux smooth at $x \in S_X$ in direction $y \in S_X$ if and only if $\xi'_{x,y}(0) = 0$.
- (b) X is Gâteaux smooth at $x \in S_X$ if and only if $\xi'_{x,y}(0) = 0$ for all $y \in S_X$.
- (c) X is Gâteaux smooth if and only if $\xi'_{x,y}(0) = 0$ for all $x, y \in S_X$.
- (d) X is Fréchet smooth at $x \in S_X$ if and only if $\xi'_x(0) = 0$.
- (e) X is Fréchet smooth if and only if $\xi'_x(0) = 0$ for all $x \in S_X$.

Proof. (a) First, by inequality (3.5) of Proposition 3.2, it is straightforward that if $\xi'_{x,y}(0) = 0$ then $\varrho_{x,y}(\beta)/\beta$ tends to 0 as $\beta \rightarrow 0$, i.e. the norm is differentiable at x in direction y .

Secondly, assume that X is Gâteaux smooth at x in direction y . If x and y are linearly dependent the result is trivial. Suppose then that x and y are linearly independent; then applying Proposition 3.1 one has

$$\frac{\xi_{x,y}(\beta) - 1}{\beta} \leq \frac{2}{(1 - \beta)^2} \varepsilon_{x,y} \left(\frac{2\beta}{1 - \beta} \right).$$

Since the norm of X is Gâteaux smooth at x in direction y , we have $\varepsilon_{x,y}(t) \rightarrow 0$ as $t \rightarrow 0$. This implies that $\xi'_{x,y}(0) = 0$.

(b) This follows from (a) since for convex functions the existence of all directional derivatives at x implies Gâteaux smoothness at x .

(c) Evident from (b).

(d) On one hand, by inequality (3.6) of Proposition 3.2, it is clear that if $\xi'_x(0) = 0$ then $\varrho_x(\beta)/\beta$ tends to 0 as $\beta \rightarrow 0$, i.e. the space is Fréchet smooth at x .

On the other hand, if we assume that X is Fréchet smooth at x , then applying Proposition 3.1, for any $y \in S_X$ we have

$$\frac{\xi_{x,y}(\beta) - 1}{\beta} \leq \frac{2}{(1 - \beta)^2} \varepsilon_{x,y} \left(\frac{2\beta}{1 - \beta} \right).$$

Taking suprema over $y \in S_X$ we obtain

$$\frac{\xi_x(\beta) - 1}{\beta} \leq \frac{2}{(1 - \beta)^2} \sup_{y \in S_X} \left\{ \varepsilon_{x,y} \left(\frac{2\beta}{1 - \beta} \right) \right\}.$$

Since the space is Fréchet smooth at x , the right-hand side tends to 0 as $\beta \rightarrow 0$. Therefore $\xi'_x(0) = 0$.

(e) This follows from (d). ■

4. On convexity and localized squareness moduli. This section is devoted to showing a relation between the behaviour of the localized moduli of squareness near 1 and the convexity properties of a normed space X . In the first subsection the local modulus of squareness ξ_x is related to local uniform convexity, and in the second subsection the pointwise modulus of squareness $\xi_{x,y}$ is related to strict convexity.

4.1. Local uniform convexity. Fix a normed space X and $x \in S_X$. The space X is said to be *locally uniformly convex at x* if its *local modulus of convexity*

$$\delta_x(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|y\| = 1, \|x-y\| \geq \varepsilon \right\}$$

is strictly positive for each $\varepsilon > 0$. The number $\varepsilon_0(x) = \sup\{\varepsilon : \delta_x(\varepsilon) = 0\}$ will be called the *characteristic of convexity at x* . Obviously, X is locally uniformly convex at x if and only if $\varepsilon_0(x) = 0$.

One calls $D(x, \beta) = \text{co}(\{x\} \cup \beta B_X)$ the *drop* of βB_X with respect to the point x , and $R(x, \beta) = D(x, \beta) \setminus \beta B_X$ the *residue*. In [1] the authors observe that X is locally uniformly convex at x iff $\text{diam } R(x, \beta) \rightarrow 0$ as $\beta \rightarrow 0$.

Recall that the *radius* of a set A relative to a point x is defined by $\text{rad}(x, A) = \sup_{a \in A} \|x - a\|$. It is clear that $\text{diam}(A)/2 \leq \text{rad}(x, A) \leq \text{diam}(A)$ whenever $x \in A$. For $\|x\| = 1$ and $0 < \beta < 1$, Kadets [6] defined the set $G(x, \beta) = \{y : [y, z] \subset B_X \setminus \beta \overset{\circ}{B}_X\}$, and noted that X is locally uniformly convex at x iff $\text{rad}(x, G(x, \beta)) \rightarrow 0$ as $\beta \rightarrow 1$. Moreover, it is known that the function $\epsilon(x, \beta) = \text{rad}(x, G(x, \beta))$ is uniformly continuous on the set $S_X \times [0, r]$ for all $r < 1$ and that ϵ is continuous at $(x, 1)$ if the norm is locally uniformly convex at $x \in S_X$ (see [2, 5]).

It is also well known that the norm is locally uniformly convex at x if and only if whenever a sequence $\{x_n\}_n$ satisfies

$$\lim_{n \rightarrow \infty} (2(\|x\|^2 + \|x_n\|^2) - \|x + x_n\|^2) = 0,$$

then $\lim_n \|x_n - x\| = 0$. This can be shown easily by using the local modulus of convexity defined above. Finally, we say that the norm of X is locally uniformly convex if it is locally uniformly convex at all $x \in S_X$.

LEMMA 4.1. *If a normed space is locally uniformly convex at $x \in S_X$, then*

$$\lim_{\lambda \rightarrow 0} \sup_{y \in \overset{\circ}{B}_X} \|x - z_x(\lambda, y)\| = 0.$$

Proof. Observe that for any $\lambda > 0$ and y with $\|y\| < 1$ all points of the segment $[(1 + \lambda)x, z_x(\lambda, y)]$ different from $z_x(\lambda, y)$ are outside the closed unit ball. Indeed, the function $f(\alpha) = \|\alpha(1 + \lambda)x + (1 - \alpha)z(\lambda, y)\|$ satisfies $f(0) = 1$ and there exists $\alpha_0 < 0$ such that $f(\alpha_0) = \|y\| < 1$. Since f is

convex we obtain $f(\alpha) > 1$ whenever $\alpha > 0$. In particular,

$$f(1/2) = \frac{1+\lambda}{2} \left\| x + \frac{z_x(\lambda, y)}{1+\lambda} \right\| > 1.$$

Therefore,

$$\begin{aligned} 0 \leq 2\|x\|^2 + 2 \left\| \frac{z_x(\lambda, y)}{1+\lambda} \right\|^2 - \left\| x + \frac{z_x(\lambda, y)}{1+\lambda} \right\|^2 &< 2 + \frac{1}{(1+\lambda)^2} - \frac{4}{(1+\lambda)^2} \\ &= 2 - \frac{2}{(1+\lambda)^2} \end{aligned}$$

where the right hand side tends to 0 uniformly over all $y \in \mathring{B}_X$ and, since the space is locally uniformly convex at x , $z_x(\lambda, y)$ converges to x uniformly in $y \in \mathring{B}_X$. ■

THEOREM 4.2. *For any normed space X and for any $x \in S_X$, the following are equivalent:*

- (a) X is locally uniformly convex at x .
- (b) $\text{diam } G(x, \beta) \rightarrow 0$ as $\beta \rightarrow 1$.
- (c) $\text{diam } R(x, \beta) \rightarrow 0$ as $\beta \rightarrow 1$.
- (d) $\limsup_{\beta \rightarrow 1} (1 - \beta)\xi_x(\beta) = 0$.
- (e) $\liminf_{\beta \rightarrow 1} (1 - \beta)\xi_x(\beta) = 0$.

Moreover, $\liminf_{\beta \rightarrow 1} (1 - \beta)\xi_x(\beta) \geq \varepsilon_0(x)$.

Proof. The equivalence between (a), (b) and (c) is known. We claim that for all $0 \leq \beta < 1$,

$$(4.1) \quad \varepsilon_0(x) - 1 + \beta \leq (1 - \beta)\xi_x(\beta).$$

Letting $\beta \rightarrow 1$ proves the last assertion and (e) \Rightarrow (a).

Inequality (4.1) is trivial if $\varepsilon_0(x) = 0$, so suppose that X is not locally uniformly convex at x . This means that, given any $\lambda > 0$, we can find a norm one vector y , at distance at least $\varepsilon_0(x)$ from x , and such that for all $\gamma, \mu \geq 0$,

$$(1 + \lambda^2)\|\gamma x + \mu y\| \geq \gamma + \mu.$$

Set $x' = (1 + \lambda)x$ and $y' = \beta y$, so that $\|x' - y'\| \geq \varepsilon_0(x) - \lambda - (1 - \beta)$. Then $z = z_x(\lambda, y') = (1 - \alpha)x' + \alpha y'$ must satisfy

$$1 = \|z\| \geq \frac{1 + \lambda - \alpha(1 + \lambda - \beta)}{1 + \lambda^2} \quad \text{and so} \quad \alpha \geq \frac{\lambda - \lambda^2}{1 + \lambda - \beta}.$$

But then

$$\frac{\|x' - z\|}{\lambda} = \frac{\alpha\|x' - y'\|}{\lambda} \geq \frac{(1 - \lambda)(\varepsilon_0(x) - \lambda - (1 - \beta))}{1 + \lambda - \beta}.$$

Letting $\lambda \rightarrow 0$, we see that $\xi_x(\beta) \geq (\varepsilon_0(x) - 1 + \beta)/(1 - \beta)$, which is (4.1).

It is obvious that (d) implies (e), so it only remains to show (a) \Rightarrow (d). Pick sequences $\{\beta_n\}_n$ tending to 1, $\{\delta_n\}_n$ tending to 0, $\lambda_n > 0$ and vectors $y_n \in \beta_n B_X$ such that

$$\xi_x(\beta_n) < \omega_x(\lambda_n, y_n) + \delta_n.$$

We have to distinguish two cases:

(a) If $\liminf_n \lambda_n > 0$, Lemma 2.5 shows that $M = \sup_n \{\omega_x(\lambda_n)\} < \infty$ and so

$$\xi_x(\beta_n) < \omega_x(\lambda_n, y_n) + \delta_n \leq \omega_x(\lambda_n) + \delta_n \leq M + \delta_n.$$

Therefore,

$$\limsup_{n \rightarrow \infty} (1 - \beta_n) \xi_x(\beta_n) \leq \lim_{n \rightarrow \infty} (1 - \beta_n)(M + \delta_n) = 0.$$

(b) If $\liminf_n \lambda_n = 0$, then we can assume, passing to a subsequence, that $\lambda_n \rightarrow 0$. If necessary we can choose y'_n in such a way that $\|y'_n\| = \beta_n$ and $y'_n \in [y_n, (1 + \lambda_n)x] \cap G(z_x(\lambda_n, y_n), \beta_n)$. Set

$$z_n = z_x(\lambda_n, y_n) = \alpha_n(1 + \lambda_n)x + (1 - \alpha_n)y'_n.$$

Then $1 = \|z_n\| \leq \alpha_n(1 + \lambda_n) + (1 - \alpha_n)\beta_n$, from which it follows that $(1 - \alpha_n)(1 - \beta_n) \leq \alpha_n \lambda_n$ and

$$\begin{aligned} (1 - \alpha_n)(1 - \beta_n)\omega_x(\lambda_n, y'_n) &\leq \alpha_n \|(1 + \lambda_n)x - z_n\| = (1 - \alpha_n)\|y'_n - z_n\| \\ &\leq (1 - \alpha_n) \text{rad}(z_n, G(z_n, \beta)). \end{aligned}$$

That is, $(1 - \beta_n)\omega_x(\lambda_n, y_n) = (1 - \beta_n)\omega_x(\lambda_n, y'_n) \leq \epsilon(z_n, \beta_n)$. Lemma 4.1 tells us that z_n tends to x and therefore, since $\epsilon(\cdot, \cdot)$ is continuous at $(x, 1)$, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} (1 - \beta_n) \xi_x(\beta_n) &\leq \limsup_{n \rightarrow \infty} (1 - \beta_n) \omega_x(\lambda_n, y_n) \\ &\leq \lim_{n \rightarrow \infty} \epsilon(z_n, \beta_n) = \epsilon(x, 1) = 0, \end{aligned}$$

which is what we wanted to show. ■

This proposition yields a new characterization of local uniform convexity.

COROLLARY 4.3. *For any normed space X the following are equivalent :*

- (a) X is locally uniformly convex.
- (b) $\text{diam } G(x, \beta) \rightarrow 0$ as $\beta \rightarrow 1$ for all $x \in S_X$.
- (c) $\text{diam } R(x, \beta) \rightarrow 0$ as $\beta \rightarrow 1$ for all $x \in S_X$.
- (d) $\limsup_{\beta \rightarrow 1} (1 - \beta) \xi_x(\beta) = 0$ for all $x \in S_X$.
- (e) $\liminf_{\beta \rightarrow 1} (1 - \beta) \xi_x(\beta) = 0$ for all $x \in S_X$.

4.2. Strict convexity. Let X be a normed space and $x, w \in S_X$. The norm of X is said to be *strictly convex at x in direction w* if there is no proper segment included in the unit sphere starting at x with direction w . Similarly, it is said to be *strictly convex at x* if there is no proper segment included in

the unit sphere starting at x in any direction. X is said to be *strictly convex* if it is strictly convex at all its norm one vectors. We define $\varepsilon_0(x, w)$ to be the supremum of $\varepsilon > 0$ such that the segment $[x, x + \varepsilon w]$ or $[x, x - \varepsilon w]$ lies on the unit sphere. We also define $C_x^w = \{y \in S_X : \exists \lambda \in \mathbb{R}, y = x + \lambda w\}$.

PROPOSITION 4.4. *Let X be a normed space and x, w two norm one vectors. If $\liminf_{\beta \rightarrow 1} (1 - \beta)\xi_{x,y}(\beta) = 0$ for all $y \in C_x^w$, then X is strictly convex at x in direction w . Moreover,*

$$\sup_{y \in C_x^w} \liminf_{\beta \rightarrow 1} (1 - \beta)\xi_{x,y}(\beta) \geq \varepsilon_0(x, w).$$

Proof. Assume that X is not strictly convex at x in direction w . This means that $\varepsilon_0(x, w) > 0$, and that for any $\varepsilon_0(x, w) > \delta > 0$ there exists $y \in C_x^w$ such that $\|y - x\| \geq \varepsilon_0(x, w) - \delta$. Write $z = z_x(\lambda, \beta y)$. There exists $\alpha \in [0, 1]$ such that $z = (1 - \alpha)(1 + \lambda)x + \alpha\beta y$. Let us compute α . Fix $f \in \mathcal{D}(x)$ such that $f([x, y]) = 1$. We have $1 = f(z) = (1 - \alpha)(1 + \lambda) + \alpha\beta$. Therefore $\alpha = \lambda/(1 + \lambda - \beta)$.

On the other hand,

$$\|(1 + \lambda)x - \beta y\| \geq \|x - y\| - \|\lambda x + (1 - \beta)y\| \geq \varepsilon_0(x, w) - \delta - \lambda - (1 - \beta).$$

Therefore,

$$\xi_{x,y}(\beta) \geq \omega_x(\lambda, \beta y) = \alpha \frac{\|(1 + \lambda)x - \beta y\|}{\lambda} \geq \frac{\varepsilon_0(x, w) - \delta - \lambda - (1 - \beta)}{1 + \lambda - \beta}.$$

Letting $\lambda \rightarrow 0$, we obtain $(1 - \beta)\xi_{x,y}(\beta) \geq \varepsilon_0(x, w) - \delta - (1 - \beta)$. Therefore

$$\liminf_{\beta \rightarrow 0} (1 - \beta)\xi_{x,y}(\beta) \geq \varepsilon_0(x, w) - \delta.$$

This implies that $\liminf_{\beta \rightarrow 0} (1 - \beta)\xi_{x,y}(\beta) > 0$, which shows the first and, whenever $\varepsilon_0(x, w) > 0$, the second assertion of the theorem. The proof is finished, since the second assertion is clear when $\varepsilon_0(x, w) = 0$. ■

THEOREM 4.5. *For any normed space X and for any $x \in S_X$ the following are equivalent:*

- (a) X is strictly convex at x .
- (b) $\limsup_{\beta \rightarrow 1} (1 - \beta)\xi_{x,y}(\beta) = 0$ for all $y \in S_X$.
- (c) $\liminf_{\beta \rightarrow 1} (1 - \beta)\xi_{x,y}(\beta) = 0$ for all $y \in S_X$.

Proof. The implication (b) \Rightarrow (c) is evident. The implication (c) \Rightarrow (a) follows from Proposition 4.4. In order to see (a) \Rightarrow (b), fix $y \in S_X$, and pick $\{\beta_n\}_n$ tending to 1, $\{\delta_n\}_n$ tending to 0, $\lambda_n > 0$ and vectors $y_n = \gamma_n y \in \beta_n B_X$ such that

$$\xi_{x,y}(\beta_n) < \omega_x(\lambda_n, y_n) + \delta_n.$$

We have to distinguish two cases:

(a) If $\liminf_n \lambda_n > 0$, then Lemma 2.5 shows that $M = \sup_n \{\omega_x(\lambda_n)\} < \infty$ and so

$$\xi_{x,y}(\beta_n) < \omega_x(\lambda_n, y_n) + \delta_n \leq \omega_x(\lambda_n) + \delta_n \leq M + \delta_n.$$

Therefore,

$$\limsup_{n \rightarrow \infty} (1 - \beta_n) \xi_{x,y}(\beta_n) \leq \lim_{n \rightarrow \infty} (1 - \beta_n)(M + \delta_n) = 0.$$

(b) If $\liminf_n \lambda_n = 0$, we can assume, passing to a subsequence, that $\lambda_n \rightarrow 0$. If necessary we can choose y'_n such that $\|y'_n\| = \beta_n$ and $y'_n \in [y_n, (1 + \lambda_n)x] \cap G_Y(z_x(\lambda_n, y_n), \beta_n)$, where $Y = \text{span}\{x, y\}$. Write $z_n = z_x(\lambda_n, y_n) = \alpha_n(1 + \lambda_n)x + (1 - \alpha_n)y'_n$. Then $1 = \|z_n\| \leq \alpha_n(1 + \lambda_n) + (1 - \alpha_n)\beta_n$, from which it follows that $(1 - \alpha_n)(1 - \beta_n) \leq \alpha_n \lambda_n$ and

$$\begin{aligned} (1 - \alpha_n)(1 - \beta_n)\omega_x(\lambda_n, y'_n) &\leq \alpha_n \|(1 + \lambda_n)x - z_n\| = (1 - \alpha_n)\|y'_n - z_n\| \\ &\leq (1 - \alpha_n) \text{rad}(z_n, G_Y(z_n, \beta_n)). \end{aligned}$$

That is, $(1 - \beta_n)\omega_x(\lambda_n, y_n) = (1 - \beta_n)\omega_x(\lambda_n, y'_n) \leq \epsilon_Y(z_n, \beta_n)$. Since Y is locally uniformly convex at x , Lemma 4.1 tells us that z_n tends to x and therefore, since $\epsilon_Y(\cdot, \cdot)$ is continuous at $(x, 1)$, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} (1 - \beta_n) \xi_{x,y}(\beta_n) &\leq \limsup_{n \rightarrow \infty} (1 - \beta_n) \omega_x(\lambda_n, y_n) \\ &\leq \lim_{n \rightarrow \infty} \epsilon_Y(z_n, \beta_n) = \epsilon_Y(x, 1) = 0, \end{aligned}$$

which is what we wanted to show. ■

From this theorem one can easily deduce the following one.

THEOREM 4.6. *For any normed space X the following are equivalent:*

- (a) X is strictly convex.
- (b) $\limsup_{\beta \rightarrow 1} (1 - \beta) \xi_{x,y}(\beta) = 0$ for all $x, y \in S_X$.
- (c) $\liminf_{\beta \rightarrow 1} (1 - \beta) \xi_{x,y}(\beta) = 0$ for all $x, y \in S_X$.

Acknowledgments. The author wishes to express his gratitude to Professor R. Deville for helpful comments and suggestions which improved the content of this paper.

References

- [1] C. Benítez, K. Przesławski, and D. Yost, *A universal modulus for normed spaces*, *Studia Math.* 127 (1998), 21–46.
- [2] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, *Monograf. Mat.* 58, PWN–Polish Sci. Publ., Warszawa, 1975.
- [3] H. Busemann, *The Geometry of Geodesics*, Academic Press, New York, 1955.
- [4] J. Gao and K. Lau, *On two classes of Banach spaces with uniform normal structure*, *Studia Math.* 99 (1991), 41–56.
- [5] A. J. Guirao, *Topological classification of Banach spaces*, Univ. of Murcia.

- [6] M. I. Kadec, *A proof of the topological equivalence of all separable infinite-dimensional Banach spaces*, Funktsional. Anal. i Prilozhen. 1 (1967), no. 1, 61–70 (in Russian).
- [7] K. Przesławski and D. Yost, *Lipschitz retracts, selectors, and extensions*, Michigan Math. J. 42 (1995), 555–571.
- [8] J. J. Schäffer, *Geometry of Spheres in Normed Spaces*, Lecture Notes in Pure Appl. Math. 20, Dekker, New York, 1976.
- [9] I. Şerb, *On the behaviour of the tangential modulus of a Banach space. I*, Rev. Anal. Numér. Théor. Approx. 24 (1995), 241–248.
- [10] —, *On the behaviour of the tangential modulus of a Banach space. II*, Mathematica 38(61) (1996), 199–207.
- [11] —, *A Day–Nordlander theorem for the tangential modulus of a normed space*, J. Math. Anal. Appl. 209 (1997), 381–391.

Departamento de Matemáticas
Universidad de Murcia
30100 Espinardo (Murcia), Spain
E-mail: ajguirao@um.es

Received December 22, 2006
Revised version November 8, 2007

(6073)