# Unconditionality for $m$-homogeneous polynomials on $\ell_\infty^n$

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*Dedicated to our good friends Pepe Bonet and Manolo Maestre, on the happy occasion of their 60th birthdays*

**Abstract**

Let $\chi(m,n)$ be the unconditional basis constant of the monomial basis $z^\alpha$, $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = m$, of the Banach space of all $m$-homogeneous polynomials in $n$ complex variables, endowed with the supremum norm on the $n$ dimensional unit polydisc $\mathbb{D}^n$. We prove that the quotient of $\sup_m \sqrt[m]{\sup_m \chi(m,n)}$ with $\sqrt{\frac{n}{\log n}}$ tends to 1 whenever $n$ tends to infinity. This fact reflects a quite precise dependence of $\chi(m,n)$ on the degree $m$ of the polynomials and their number $n$ of variables. Moreover, we give an analogous formula for $m$-linear forms, a reformulation of our results in terms of tensor products, and as an application a solution for a problem on Bohr radii.

## 1 Introduction

Unconditional bases form one of the basic concepts in Banach space theory. A Schauder basis $(e_i)_{i\in I}$ of a (complex) Banach space $X$ is said to be uncon-

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ditional if all series representations $x = \sum_{i=1}^{\infty} \alpha_i e_i$ converge unconditionally. Equivalently, if there is a constant $c > 0$ such that for every finite choice $x_1, \ldots, x_n \in \mathbb{C}$ and of signs $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$ we have

$$\Big\| \sum_{i=1}^{n} \varepsilon_i x_i e_i \Big\| \leq c \Big\| \sum_{i=1}^{n} x_i e_i \Big\| .$$

The best constant $c$ in this inequality is called the *unconditional basis constant of* $(e_i)_{i\in I}$ and denoted by $\chi\big((e_i)_{i\in I}; X\big)$. A continuous function $P : X \to \mathbb{C}$ is an $m$-homogeneous polynomial if there exists an $m$-linear form $L : X \times \cdots \times X \to \mathbb{C}$ such that $P(x) = L(x, \ldots, x)$ for every $x \in X$. We denote by $\mathcal{P}(^m X)$ and $\mathcal{L}(^m X)$ the spaces of $m$-homogeneous polynomials and $m$-linear forms on $X$, respectively, with the norms

$$\|P\| = \sup_{\|x\|\leq 1} |P(x)| \quad \text{and} \quad \|L\| = \sup_{\substack{\|x_i\|\leq 1 \\ i=1,\ldots,m}} |L(x_1, \ldots, x_m)| .$$

We focus on polynomials and multilinear forms on $\ell_\infty^n$ (that is $\mathbb{C}^n$ with the $\|\cdot\|_\infty$-norm).

Both Banach spaces $\mathcal{P}(^m \ell_\infty^n)$ and $\mathcal{L}(^m \ell_\infty^n)$ are finite-dimensional and have natural monomial bases. We write $\{e_1, \ldots, e_n\}$ for the canonical basis of $\ell_\infty^n$ and $\{e_1^*, \ldots, e_n^*\}$ for its dual basis. For each index $\mathbf{i} = (i_1, \ldots, i_m)$ with $1 \leq i_1, \ldots, i_m \leq n$ (we denote the set of all such indices by $\mathcal{M}(m,n)$) we consider $e_{\mathbf{i}}^* \in \mathcal{L}(^m \ell_\infty^n)$ given by

$$e_{\mathbf{i}}^* : x = (x_1, \ldots, x_m) \rightsquigarrow e_{i_1}^*(x_1) \cdots e_{i_m}^*(x_m) .$$

Then the family $(e_{\mathbf{i}}^*)_{\mathbf{i}\in\mathcal{M}(m,n)}$ obviously defines a basis of $\mathcal{L}(^m \ell_\infty^n)$. On the other hand, the monomials are the natural basis of $\mathcal{P}(^m \ell_\infty^n)$. These are defined as follows: Each multi index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n = m$ (we denote by $\Lambda(m,n)$ the corresponding set) defines the monomial $z^\alpha \in \mathcal{P}(^m \ell_\infty^n)$

$$z^\alpha : u = (u_1, \ldots, u_n) \rightsquigarrow u_1^{\alpha_1} \cdots u_n^{\alpha_n} .$$

Schütt started the study of the unconditional basis constants of these two bases in [12], where he proved that

$$\frac{1}{8}\sqrt{n} \leq \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^2 \ell_\infty^n)\big) \leq (1+\sqrt{2})\sqrt{n} .$$

This study was carried over in [5], where as a particular case of a more general result we have that for every $m$ there is a constant $C(m) > 0$ such that for each $n$,

$$\frac{1}{C(m)}\sqrt{n}^{m-1} \leq \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^m\ell_\infty^n)\big) \leq C(m)\sqrt{n}^{m-1}$$

and

$$\frac{1}{C(m)}\sqrt{n}^{m-1} \leq \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big) \leq C(m)\sqrt{n}^{m-1}.$$

These estimates are consequences of more general results that relate the unconditional basis constants of (full or symmetric) tensor products with classical concepts from Banach space theory, such as the Banach-Mazur distance to $\ell_1^n$ or the Gordon-Lewis property. The fact that we are specifically working with $\ell_\infty^n$ played no rôle there. This was taken into account in [6], where the Bohnenblust-Hille inequality, a very particular property of $\ell_\infty^n$, was substantially improved and used to show that there exist a universal constant such that

$$\frac{1}{C^m}\Big(\frac{n}{m}\Big)^{\frac{m-1}{2}} \;\leq\; \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big) \;\leq\; C^m\Big(\frac{n}{m}\Big)^{\frac{m-1}{2}} \qquad \text{if } n \geq m$$

and

$$1 \;\leq\; \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big) \;\leq\; C^m \qquad \text{if } n < m$$

Our aim is to prove a refinement of these inequalities which in a very precise sense links the degree $m$ of the polynomials with their number $n$ of variables.

**Theorem 1.1.** *We have*

$$\lim_{n\to\infty} \frac{\sup_m \sqrt[m]{\chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^m\ell_\infty^n)\big)}}{\sqrt{n}} \;=\; 1 \tag{1.1}$$

*and*

$$\lim_{n\to\infty} \frac{\sup_m \sqrt[m]{\chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)}}{\sqrt{\frac{n}{\log n}}} \;=\; 1\,. \tag{1.2}$$

Note that in the polynomial case there is a log term in $n$ which distinguishes it from the multilinear case. Before we proceed with the proof, let us note that a simple calculation characterizes $\chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^m\ell_\infty^n)\big)$ to be the best constant $c > 0$ such that for every $L \in \mathcal{L}(^m\ell_\infty^n)$

$$\sum_{\mathbf{i}\in\mathcal{M}(m,n)} |L(e_{i_1}, \ldots, e_{i_m})| \leq c\|L\|\,. \tag{1.3}$$

Analogously, $\chi\big((z^\alpha)_\alpha;\mathcal{P}(^m\ell_\infty^n)\big)$ is the best constant $c>0$ such that for every $m$-homogeneous polynomial $P=\sum_{\alpha\in\Lambda(m,n)} c_\alpha z^\alpha$ in $n$ variables

$$\sum_{\alpha\in\Lambda(m,n)} |c_\alpha| \le c\|P\|$$

Formulated in this way, it is plain that $\chi\big((z^\alpha)_\alpha;\mathcal{P}(^m\ell_\infty^n)\big)$ is the Sidon constant of the characters $(z^\alpha)_{\alpha\in\Lambda(m,n)}$ acting on the compact abelian group $\mathbb{T}^n$.

## 2 Proof of the main result

*Proof of* (1.1). We prove that

$$\begin{aligned} 1 &\le \liminf_{n\to\infty} \frac{\sup_m \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}};\mathcal{L}(^m\ell_\infty^n)\big)^{\frac{1}{m}}}{\sqrt{n}} \\ &\le \limsup_{n\to\infty} \frac{\sup_m \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}};\mathcal{L}(^m\ell_\infty^n)\big)^{\frac{1}{m}}}{\sqrt{n}} \le 1\,. \end{aligned} \tag{2.1}$$

To do this we need a classical result due to Bohnenblust and Hille [3, Section 2]: For each $m$ there is a (best) constant $\mathrm{BH}_m^{\mathrm{mult}}\ge 1$ such that for every $L\in\mathcal{L}(^m\ell_\infty^n)$ we have

$$\Big(\sum_{i_1,\dots,i_m=1}^n |L(e_{i_1},\dots,e_{i_m})|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \le \mathrm{BH}_m^{\mathrm{mult}}\,\|L\|\,. \tag{2.2}$$

A recent result from [2] shows that there exist a constant $\kappa>1$ such that for all $m$

$$\mathrm{BH}_m^{\mathrm{mult}} \le \kappa\, m^{\frac{1-\gamma}{2}}\,,$$

where $\gamma$ is the Euler-Mascheroni constant. With this at hand, we start the proof of the upper bound from (2.1). Take $L\in\mathcal{L}(^m\ell_\infty^n)$, then by Hölder's inequality

$$\begin{aligned} &\sum_{i_1,\dots,i_m=1}^n |L(e_{i_1},\dots,e_{i_m})| \\ &\qquad\le \Big(\sum_{i_1,\dots,i_m=1}^n |L(e_{i_1},\dots,e_{i_m})|^{\frac{m+1}{2m}}\Big)^{\frac{2m}{m+1}} |\mathcal{M}(m,n)|^{\frac{m-1}{2m}} \\ &\qquad\qquad\le \kappa\, m^{\frac{1-\gamma}{2}} n^{\frac{m-1}{2}} \|L\|\,. \end{aligned}$$

This, in view of (1.3), gives that for every $n$ and $m$ we have

$$\chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}};\mathcal{L}(^m\ell_\infty^n)\big)\le \kappa\, m^{\frac{1-\gamma}{2}} n^{\frac{m-1}{2}}\,,$$

and hence

$$\chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}};\mathcal{L}(^m\ell_\infty^n)\big)^{\frac{1}{m}}\le \frac{\kappa^{\frac{1}{m}}\, m^{\frac{1-\gamma}{2m}}}{n^{\frac{1}{2m}}} n^{\frac{1}{2}}\,.$$

Fix now some $\varepsilon>0$ and choose $m_0$ such that

$$\sup_{m\ge m_0}\frac{\kappa^{\frac{1}{m}}\, m^{\frac{1-\gamma}{2m}}}{n^{\frac{1}{2m}}}\le \sup_{m\ge m_0}\kappa^{\frac{1}{m}}\, m^{\frac{1-\gamma}{2m}}\le 1+\varepsilon\,.$$

But for each fixed $m$ the sequence $\big(\kappa^{\frac{1}{m}}\, m^{\frac{1-\gamma}{2m}} n^{\frac{-1}{2m}}\big)_n$ tends to 0, hence we can find $n_0$ such that for all $n\ge n_0$

$$\sup_{m<m_0}\frac{\kappa^{\frac{1}{m}}\, m^{\frac{1-\gamma}{2m}}}{n^{\frac{1}{2m}}}\le 1\,.$$

Then for all $n\ge n_0$ we have

$$\begin{aligned}
&\frac{\sup_m \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}};\mathcal{L}(^m\ell_\infty^n)\big)^{\frac{1}{m}}}{\sqrt{n}}\\
&\le \max\left\{\frac{\sup_{m<m_0} \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}};\mathcal{L}(^m\ell_\infty^n)\big)^{\frac{1}{m}}}{\sqrt{n}}\,,\ \frac{\sup_{m\ge m_0} \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}};\mathcal{L}(^m\ell_\infty^n)\big)^{\frac{1}{m}}}{\sqrt{n}}\right\}\\
&\le 1+\varepsilon\,.
\end{aligned}$$

This shows the right estimate in (2.1). To prove the left estimate in (2.1), by the Chevet type inequality from [1, Theorem 4.4] there is an absolute constant $C>0$ such that for each choice of $m,n$ there are signs $\varepsilon_{\mathbf{i}}=\pm 1$ with $\mathbf{i}\in\mathcal{M}(m,n)$ for which

$$\begin{aligned}
n^m&=\Big(\sup_{z\in B_{\ell_\infty^n}}\sum_{i=1}^n |z_i|\Big)^m=\Big\|\sum_{\mathbf{i}\in\mathcal{M}(m,n)} e_{\mathbf{i}}^*\Big\|_{\mathcal{L}(^m\ell_\infty^n)}\\
&=\Big\|\sum_{\mathbf{i}\in\mathcal{M}(m,n)} \varepsilon_{\mathbf{i}}\varepsilon_{\mathbf{i}} e_{\mathbf{i}}^*\Big\|_{\mathcal{L}(^m\ell_\infty^n)}\\
&\le \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}};\mathcal{L}(^m\ell_\infty^n)\big)\Big\|\sum_{\mathbf{i}\in\mathcal{M}(m,n)} \varepsilon_{\mathbf{i}} e_{\mathbf{i}}^*\Big\|_{\mathcal{L}(^m\ell_\infty^n)}\\
&\le \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}};\mathcal{L}(^m\ell_\infty^n)\big) Cm(\log n)^{\frac{3}{2}} n^{\frac{m+1}{2}}\,.
\end{aligned}\tag{2.3}$$

As a consequence we have

$$\sup_m \frac{1}{C^{\frac{1}{m}} m^{\frac{1}{m}} n^{\frac{1}{2m}} (\log n)^{\frac{3}{2m}}}\le \frac{\sup_m \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}};\mathcal{L}(^m\ell_\infty^n)\big)^{\frac{1}{m}}}{\sqrt{n}}\,,$$

and hence for $m = n$

$$\Big(\frac{1}{Cn^{\frac{1}{2}}(\log n)^{\frac{3}{2}}}\Big)^{\frac{1}{n}} \leq \frac{\sup_m \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^m\ell_\infty^n)\big)^{\frac{1}{m}}}{\sqrt{n}}\,.$$

But if $n$ tends to $\infty$, then we obtain the remaining left estimate in (2.1). $\square$

*Proof of* (1.2). The basic idea to estimate $\chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)$ is essentially the same, but technically more demanding. Again we split the proof in two steps, and check the following upper and lower bounds

(2.4)

$$1 \leq \liminf_{n\to\infty} \frac{\sup_m \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)^{\frac{1}{m}}}{\sqrt{\frac{n}{\log n}}} \leq \limsup_{n\to\infty} \frac{\sup_m \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)^{\frac{1}{m}}}{\sqrt{\frac{n}{\log n}}} \leq 1\,.$$

To begin with, we fix some $\varepsilon > 0$, and want to show that

$$\limsup_{n\to\infty} \frac{\sup_m \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)^{\frac{1}{m}}}{\sqrt{\frac{n}{\log n}}} \leq 1 + \varepsilon\,. \tag{2.5}$$

One of the key tools is going to be again the Bohnenblust-Hille inequality, this time in its polynomial form. From (2.2) it is easy to prove, using the polarization formula (this is done in [3, Section 3]), that for each $m$ there exists a (best) constant $\mathrm{BH}_m^{\mathrm{pol}} \geq 1$ such that for every $m$-homogeneous polynomial $P = \sum_{\alpha\in\Lambda(m,n)} c_\alpha z^\alpha$ we have

$$\Big(\sum_{\alpha\in\Lambda(m,n)} |c_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \leq \mathrm{BH}_m^{\mathrm{pol}}\, \|P\|\,.$$

Again, a good control of the growth of $\mathrm{BH}_m^{\mathrm{pol}}$ is going to be crucial. In [6] it was shown that for each $\delta > 0$ there is a constant $c(\delta) \geq 1$ such that $\mathrm{BH}_m^{\mathrm{pol}} \leq c(\delta)(\sqrt{2}+\delta)^m$ for every $m$, and in [2] that here $\sqrt{2}$ even can be replaced by 1. Using Hölder's inequality we get that for every $n, m$ and every polynomial $P = \sum_{\alpha\in\Lambda(m,n)} c_\alpha z^\alpha$

$$\sum_{\alpha\in\Lambda(m,n)} |c_\alpha| \leq \mathrm{BH}_m^{\mathrm{pol}}\, |\Lambda(m,n)|^{\frac{m-1}{2m}} \sup_{z\in\mathbb{D}^n} \Big|\sum_{\alpha\in\Lambda(m,n)} c_\alpha(f) z^\alpha\Big|\,.$$

It is a well known fact that

$$|\Lambda(m,n)| = \binom{n+m-1}{m} \leq e^m \left(1 + \frac{n}{m}\right)^m\,,$$

hence for every $n$

$$\sup_m \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big) \leq \sup_m \left[\mathrm{BH}_m^{\mathrm{pol}}\, e^{\frac{m-1}{2}} \left(1 + \frac{n}{m}\right)^{\frac{m-1}{2}}\right]\,.$$

With this, in order to prove (2.5) it is enough to show that there is some $n_0$ such that for all $n \geq n_0$ and every $m$

$$\left[\mathrm{BH}_m^{\mathrm{pol}}\, e^{\frac{m-1}{2}} \left(1+\frac{n}{m}\right)^{\frac{m-1}{2}} \left(\frac{\log n}{n}\right)^{\frac{m}{2}}\right] \leq (1+\varepsilon)^m . \tag{2.6}$$

Our strategy is going to be the following. First find a proper $n_0$ and then consider three cases for $m$, showing that in each one of these cases (2.6) holds. Let us note first that, since $\lim_m \left(\frac{(1+\varepsilon)^m}{m}\right)^{\frac{1}{m-1}} = 1+\varepsilon$, we can choose $m_1 = m_1(\varepsilon)$ so that for every $m \geq m_1$

$$1+\frac{\varepsilon}{2} \leq \left(\frac{(1+\varepsilon)^m}{m}\right)^{\frac{1}{m-1}} . \tag{2.7}$$

As already quoted above, for each $\delta > 0$ there exists a constant $c(\delta) \geq 1$ such that for every $m$

$$\mathrm{BH}_m^{\mathrm{pol}} \leq c(\delta)(1+\delta)^m .$$

Choose $\delta > 0$ so that $1+\delta < (1+\varepsilon)^{1/4}$ and $m_2 = m_2(\varepsilon)$ such that $c(\delta)^{1/m} < (1+\varepsilon)^{1/4}$ for all $m \geq m_2$. Then we have

$$\sup_{m_2 \leq m} \mathrm{BH}_m^{\mathrm{pol}} \leq (1+\varepsilon)^{\frac{m}{2}} . \tag{2.8}$$

We fix $m_0 = \max\{m_1, m_2\}$. Let us now take $n_1 = n_1(\varepsilon)$ such that for all $n \geq n_1$

$$1+\frac{1}{\sqrt{n}} < 1+\frac{\varepsilon}{2} , \tag{2.9}$$

and $n_2 = n_2(\varepsilon)$ such that for all $n \geq n_2$

$$\frac{\sqrt{2e\log n}}{n^{\frac{1}{4}}} \leq \sqrt{1+\varepsilon} . \tag{2.10}$$

Since for each fixed $m$ the sequence $\left(\left(1+\frac{n}{m}\right)^{\frac{m-1}{2}} \left(\frac{\log n}{n}\right)^{\frac{m}{2}}\right)_n$ obviously tends to zero, there clearly is some $n_3 = n_3(\varepsilon)$ such that for all $n \geq n_3$

$$\sup_{m \leq m_0} \left[\mathrm{BH}_m^{\mathrm{pol}}\, e^{\frac{m-1}{2}} \left(1+\frac{n}{m}\right)^{\frac{m-1}{2}} \left(\frac{\log n}{n}\right)^{\frac{m}{2}}\right] \leq 1 . \tag{2.11}$$

We set now $n_0 = \max\{n_1, n_2, n_3, (m_0+1)^2\}$. Observe that, although one may think that $n_0$ depends on $\varepsilon$ and $m_0$, we have that $m_0$ only depends on $\varepsilon$; hence actually $n_0 = n_0(\varepsilon)$.

Now in order to prove (2.6) we fix $n \geq n_0$ and choose $m \geq 2$. We consider three different cases for $m$: either $m \leq m_0$, or $m_0 \leq m < \sqrt{n}$, or $\sqrt{n} \leq m$. Observe that (2.11) already gives that (2.6) holds in the first case $m \leq m_0$. For the remaining two cases let us note first that by (2.7) and (2.9) we have for every $m_0 \leq m$

$$1 + \frac{1}{\sqrt{n}} \leq \left( \frac{(1+\varepsilon)^m}{m} \right)^{\frac{1}{m-1}} .$$

Then, if $m_0 \leq m < \sqrt{n}$, a straightforward calculation gives

(2.12)

$$\left(1 + \frac{n}{m}\right)^{\frac{m-1}{2}} \leq \left(\frac{\sqrt{n} + n}{m}\right)^{\frac{m-1}{2}} \leq \frac{(1+\varepsilon)^{\frac{m}{2}} n^{\frac{m-1}{2}}}{\sqrt{m}} \frac{1}{m^{\frac{m-1}{2}}} = \left(1+\varepsilon\right)^{\frac{m}{2}} \frac{n^{\frac{m}{2}}}{n^{\frac{1}{2}} m^{\frac{m}{2}}} ,$$

and with (2.8) this implies that

$$\begin{aligned} &\mathrm{BH}_m^{\mathrm{pol}}\, e^{\frac{m-1}{2}} \left(1 + \frac{n}{m}\right)^{\frac{m-1}{2}} \left(\frac{\log n}{n}\right)^{\frac{m}{2}} \\ &\qquad \leq (1+\varepsilon)^m e^{\frac{m-1}{2}} \frac{n^{\frac{m}{2}}}{n^{\frac{1}{2}} m^{\frac{m}{2}}} \left(\frac{\log n}{n}\right)^{\frac{m}{2}} \leq (1+\varepsilon)^m \left(\frac{e \log n}{n^{\frac{1}{m}} m}\right)^{\frac{m}{2}} . \end{aligned}$$

Now a simple calculation yields that the function $x \in (0, \infty) \mapsto x n^{\frac{1}{x}}$ has a global minimum $e \log n$ at $x = \log n$. This proves (2.6) in the second case: $m_0 \leq m < \sqrt{n}$ (remember that $n \geq n_0$ was fixed). Finally, for the third case $\sqrt{n} \leq m$ let us note that then trivially

$$\left(1 + \frac{n}{m}\right)^{\frac{m-1}{2}} \leq \left(2\sqrt{n}\right)^{\frac{m}{2}} ,$$

and hence in this last case we get (2.6) using (2.8) and (2.10). This completes the proof of (2.5).

Finally, it remains to show the left inequality in (2.4). The main tool is again of probabilistic nature, and we are going to use the Kahane-Salem-Zygmund inequality [11, Chapter 6, Theorem 4]: There is a universal constant $\mathrm{C}_{\mathrm{KSZ}} > 0$ such that for every scalar family $c_\alpha$, $\alpha \in \Lambda(m,n)$ there exists a choice of signs $\varepsilon_\alpha \in \{1, -1\}$ for $\alpha \in \Lambda(m,n)$ such that, proceeding as in (2.3), we have

$$\sup_{z \in \mathbb{D}^n} \Big| \sum_{\alpha \in \Lambda(m,n)} \varepsilon_\alpha c_\alpha z^\alpha \Big| \leq \mathrm{C}_{\mathrm{KSZ}} \sqrt{n \log m \sum_{\alpha \in \Lambda(m,n)} |c_\alpha|^2} \, .$$

We consider $c_\alpha = \frac{m!}{\alpha!}$, $\alpha \in \Lambda(m,n)$. Then there are $\varepsilon_\alpha \in \{1,-1\}$ such that

$$\sum_{\alpha\in\Lambda(m,n)} \frac{m!}{\alpha!} \le \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big) \sup_{z\in\mathbb{D}^n} \Big| \sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \Big|$$
$$\le \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)\, \mathrm{C}_{\mathrm{KSZ}} \sqrt{n \log m \sum_{\alpha\in\Lambda(m,n)} \Big(\frac{m!}{\alpha!}\Big)^2}\,.$$

By the multi binomial formula we have

$$\sum_{\alpha\in\Lambda(m,n)} \frac{m!}{\alpha!} = n^m \quad \text{and} \quad \sum_{\alpha\in\Lambda(m,n)} \Big(\frac{m!}{\alpha!}\Big)^2 \le m! \sum_{\alpha\in\Lambda(m,n)} \frac{m!}{\alpha!} = m! n^m\,.$$

This gives for all $m,n$

$$\frac{n^{\frac{m-1}{2m}}}{\mathrm{C}_{\mathrm{KSZ}}^{\frac{1}{m}} \big(m! \log m\big)^{\frac{1}{2m}}} \le \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)^{\frac{1}{m}}\,.$$

Recall Stirling's formula $m! \le 2\sqrt{2\pi m}\,\big(\frac{m}{e}\big)^m$ , hence for all $m,n$

$$\frac{n^{\frac{m-1}{2m}}}{\mathrm{C}_{\mathrm{KSZ}}^{\frac{1}{m}} (\log m)^{\frac{1}{2m}} 2^{\frac{1}{2m}} (2\pi m)^{\frac{1}{4m}} \big(\frac{m}{e}\big)^{\frac{1}{2}}} \le \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)^{\frac{1}{m}}\,.$$

Now we put $m = \lceil \log n \rceil$ and divide by $\sqrt{\frac{n}{\log n}}$ to obtain that for all $n$ we have

$$\frac{\sqrt{\frac{\log n}{n}}\, n^{\frac{\log n - 1}{2\log n}}}{\mathrm{C}_{\mathrm{KSZ}}^{\frac{1}{\log n}} \big(\log(1+\log n)\big)^{\frac{1}{2\log n}}\, 2^{\frac{1}{2\log n}}\, (2\pi \log n)^{\frac{1}{4\log n}}\, \big(\frac{1+\log n}{e}\big)^{\frac{1}{2}}}$$
$$\le \frac{\sup_m \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)^{\frac{1}{m}}}{\sqrt{\frac{n}{\log n}}}\,.$$

But, if $n \to \infty$, then the left side of this inequality tends to 1. This clearly gives the left inequality in (2.4), and completes the proof of Theorem 1.1. $\square$

# 3 Two consequences

## 3.1 A tensor product formulation

It is a well known fact that the Banach spaces $\mathcal{L}(^m\ell_\infty^n)$ of $m$-linear forms can be represented as a $m$-fold tensor product, and the Banach space $\mathcal{P}(^m\ell_\infty^n)$ of

$m$-homogeneous polynomials as an $m$-fold symmetric tensor product. More precisely, we have

$$\mathcal{L}({}^m\ell_\infty^n) = \bigotimes_{\varepsilon}^{m} \ell_1^n \quad \text{and} \quad \mathcal{P}({}^m\ell_\infty^n) = \bigotimes_{\varepsilon_s}^{s,m} \ell_1^n \tag{3.1}$$

isometrically as Banach spaces (see e.g. [9, Chapter 1] or [10]); here $\varepsilon$ stands for the injective tensor norm on the tensor product $\bigotimes^m \ell_1^n$, and $\varepsilon_s$ for the symmetric injective tensor norm on the symmetric tensor product $\bigotimes^{s,m} \ell_1^n$. Under the identification from (3.1) the basis $(e_{\mathbf{i}}^*)_{\mathbf{i}\in\mathcal{M}(m,n)}$ of $\mathcal{L}({}^m\ell_\infty^n)$ transfers into the basis of $\bigotimes_\varepsilon^m \ell_1^n$ given by $e_{\mathbf{i}} = (e_{i_1} \otimes \cdots \otimes e_{i_m})$ for $\mathbf{i} \in \mathcal{M}(m,n)$. Analogously, the image of the monomial basis $(z^\alpha)_{\alpha\in\Lambda(m,n)}$ in $\mathcal{P}({}^m\ell_\infty^n)$ is the basis

$$S(e_{\mathbf{j}}) = \frac{1}{m!} \sum_{\sigma\in\Pi_m} (e_{j_{\sigma(1)}} \otimes \cdots \otimes e_{j_{\sigma(m)}}), \quad \mathbf{j} = (j_1, \ldots, j_m) \in \mathcal{J}(m,n)$$

of $\bigotimes_{\varepsilon_s}^{s,m} \ell_1^n$, where $\Pi_m$ stands for all permutations of $\{1, \ldots, m\}$ and

$$\mathcal{J}(m,n) = \{\mathbf{j} \in \mathcal{M}(m,n)\colon 1 \le j_1 \le \cdots \le j_m \le n\} .$$

With this notation Theorem 1.1 has the following immediate translation in terms of tensor products.

**Corollary 3.1.** *We have*

$$\lim_{n\to\infty} \frac{\sup_m \chi\big((e_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)};\ \bigotimes_\varepsilon^m \ell_1^n\big)^{\frac{1}{m}}}{\sqrt{n}} = 1$$

*and*

$$\lim_{n\to\infty} \frac{\sup_m \chi\big((Se_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)};\ \bigotimes_{\varepsilon_s}^{s,m} \ell_1^n\big)^{\frac{1}{m}}}{\sqrt{\frac{n}{\log n}}} = 1 .$$

## 3.2 Bohr radius

The $n$-th Bohr radius $K_n$ is defined to be the supremum over all $0 \le r \le 1$ such that for all holomorphic functions $f : \mathbb{D}^n \to \mathbb{C}$ we have

$$\sup_{z\in r\mathbb{D}^n} \sum_{\alpha\in\mathbb{N}_0^n} \left|\frac{\partial^\alpha f(0)}{\alpha!} z^\alpha\right| \le \sup_{z\in\mathbb{D}^n} \left|\sum_{\alpha\in\mathbb{N}_0^n} \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha\right|,$$

and $K_n^m$, the $m$-th homogeneous Bohr radius, is defined analogously, taking only $m$-homogeneous polynomials $\sum_{\alpha\in\mathbb{N}_0^n, |\alpha|=m} c_\alpha z^\alpha$. These objects have been extensively studied over the last years. By [7, Corollary 2.3] we have

$$\frac{1}{3} \inf_m K_n^m \ \le\ K_n \ \le\ \min\left\{\frac{1}{3}, \inf_m K_n^m\right\} . \tag{3.2}$$

In the special case $n = 1$ obviously $K_1^m = 1$, hence

$$K_1 = \frac{1}{3}\,.$$

This is Bohr's famous power series theorem from [4], and it shows that the factor 1/3 in (3.2) at least for small $n$ is indispensable. But how important is this factor for large $n$ ? Or, to put it in technical terms: Does the following equality

$$\lim_{n\to\infty} \frac{K_n}{\inf_m K_n^m} \;=\; \frac{1}{3}$$

hold ? This question appears explicitly in [8, Problem 4.4]. Let us see with Theorem 1.1 that this is not the case: First of all, from [2] we know that

$$\lim_{n\to\infty} \frac{K_n}{\sqrt{\frac{\log n}{n}}} = 1$$

(improving the inequality $1 \le \liminf_n K_n\sqrt{\frac{n}{\log n}} \le \limsup_n K_n\sqrt{\frac{n}{\log n}} \le \sqrt{2}$ from [6]). On the other hand, straight forward arguments give (see [7, Lemma 2.1]) that for all $n, m$

$$K_n^m = \frac{1}{\sqrt[m]{\chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)}}\,.$$

These two facts, together with Theorem 1.1, readily give our last result which shows that, when $n$ grows, the factor 1/3 looses influence in (3.2).

**Corollary 3.2.** *We have*

$$\lim_{n\to\infty} \frac{K_n}{\inf_m K_n^m} \;=\; 1\,.$$

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