Solving random homogeneous linear second-order differential equations: A full probabilistic description

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Abstract. In this paper a complete probabilistic description for the solution of random homogeneous linear second-order differential equations via the computation of its two first probability density functions is given. As a consequence, all unidimensional and two-dimensional statistical moments can be straightforwardly determined, in particular, mean, variance and covariance functions, as well as the first-order conditional law. With the aim of providing more generality, in a first step, all involved input parameters are assumed to be statistically dependent random variables having an arbitrary joint probability density function. Secondly, the particular case that just initial conditions are random variables is also analysed. Both problems have common and distinctive feature which are highlighted in our analysis. The study is based on Random Variable Transformation method. As a consequence of our study, the well-known deterministic results are nicely generalized. Several illustrative examples are included.

Mathematics Subject Classification (2010). 60H35, 60H10, 37H10.

Keywords. Random Variable Transformation method, first and second probability density functions, random homogeneous linear second-order differential equations.

1. Motivation

The quantification of uncertainty in dynamic models is currently playing an important role in many applied areas. Classical deterministic differential equations, which have demonstrated to be powerful tools for analysing problems that appear in areas such as Physics, Engineering, Chemistry, Epidemiology, etc., need to consider randomness in their formulation in order to account for measurement errors and inherent complexity of problems under

This work was completed with the support of our TEX-pert.

study. It has motivated the development of two main classes of differential equations dealing with uncertainty, namely, random differential equations (r.d.e.'s) and stochastic differential equations (s.d.e.'s). In the latter case, differential equations are forced by an irregular stochastic process, typically driven by a Wiener process. Solving s.d.e.'s requires the use of a special stochastic calculus, usually referred to as Itô Calculus, whose cornerstone is the Itô Lemma [1]. Whereas r.d.e.'s are those in which random effects are directly manifested in its inputs parameters (initial/boundary conditions, source term and coefficients). These inputs are assumed to satisfy regularity properties such as continuity, differentiability, etc., in some adequate stochastic sense such as mean square calculus [2, 3]. In [3] one can find an updated overview of the state of the art of r.d.e.'s. A major advantage of considering r.d.e.'s is that a wide range of probabilistic distributions can be assigned to its inputs including Exponential, Gaussian, Beta, etc, distributions. As a consequence, r.d.e.'s provide great flexibility in dealing with real models. In dealing with r.d.e.'s the main efforts have focussed on extending the deterministic theory to the random framework. This includes both the development of analytic and numerical methods for solving r.d.e.'s [4–6]. However, it is important to point out that in the random context besides computing the solution stochastic process (s.p.), say $Z(t)$, it is also of great interest the determination of its main statistical properties. Most of the contributions focus on the computation of the mean, $\mu_Z(t) = \mathbb{E}[Z(t)]$, and the variance, $\sigma_Z^2(t) = \mathbb{V}[Z(t)]$, functions of the solution s.p. However, a more convenient goal is the determination of its first probability density function (1-p.d.f.), $f_1(z,t)$, since from it one can easily compute not just these two first statistical moments,

$$\mathbb{E}[Z(t)] = \int_{-\infty}^{\infty} z f_1(z,t)\,\mathrm{d}z, \quad \mathbb{V}[Z(t)] = \int_{-\infty}^{\infty} (z - \mathbb{E}[Z(t)])^2 f_1(z,t)\,\mathrm{d}z, \tag{1.1}$$

but also the one-dimensional statistical moment of any order

$$\mathbb{E}[(Z(t))^k] = \int_{-\infty}^{\infty} z^k f_1(z,t)\,\mathrm{d}z, \quad k = 0, 1, 2, 3, \ldots$$

The 1-p.d.f. characterizes, from a probabilistic point of view, the solution s.p. $Z(t)$ at every time instant t. In general, more challenging is the determination of the rest of the so-called *fidis* (finite dimensional distributions), i.e., the n-dimensional p.d.f.'s of the solution s.p. for $n \geq 2$, because it usually involves complex developments and computations. These higher p.d.f.'s account for important probabilistic information. For instance, the 2-p.d.f., $f_2(z_1, t_1; z_2, t_2)$, provides a complete probabilistic description of the solution s.p. at every arbitrary pair of times, say t_1 and t_2. In particular, the 2-p.d.f. allows us to compute the correlation function

$$\Gamma_Z(t_1,t_2) = \mathbb{E}[Z(t_1)Z(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_1 z_2 f_2(z_1,t_1;z_2,t_2)\,\mathrm{d}z_1\mathrm{d}z_2, \tag{1.2}$$

which is an important measure of linear statistical interdependence between $Z(t_1)$ and $Z(t_2)$. It is straightforward the determination of covariance function, $C_Z(t_1,t_2)$, from correlation. All these deterministic functions play an important role in the theory of s.p.'s. In addition, from the two first p.d.f.'s, one can easily compute the conditional p.d.f. as well

$$f(z_2, t_2|z_1, t_1) = \frac{f_2(z_1, t_1; z_2, t_2)}{f_1(z_1, t_1)}. \tag{1.3}$$

It is important to stress that in the relevant case where the solution s.p. is Gaussian, the two first p.d.f.'s fully characterize the solution from a probabilistic standpoint.
The aim of this paper is the computation of the two first p.d.f.'s of the solution s.p. to an important class of r.d.e.'s under very general assumptions.

In [7], the 1-p.d.f. of the solution s.p. of random homogeneous linear second-order difference equations was determined by taking advantage of Random Variable Transformation (RVT) method. The aim of contribution [7] is twofold, first, providing a complete probabilistic description of the solution of that simple but significant class of dynamic discrete problems and, secondly, generalizing their deterministic counterpart. Motivated by the important role that ordinary differential equations play both in theory and in applications (Chemistry, Economics, Engineering, Epidemiology, etc.), our goal in this paper is extending the analysis provided in [7] to random homogeneous linear second-order differential equations. As a significant difference with respect to the study presented in [7], it is important to underline that now the 2-p.d.f. of the solution s.p. will be computed as well. In addition, the two first p.d.f.'s will be fully specified in a particular case where the solution s.p. can become Gaussian. In this manner a probabilistic description of the solution of these two important classes of discrete and continuous dynamical models will be completed. We point out that on purpose, hereinafter we will follow a similar structure in the presentation given in [7] in order to facilitate the comparison of results established regarding the 1-p.d.f. in both papers.

Let us consider the random homogeneous linear second-order differential equation

$$\ddot{Z}(t) + A_1\dot{Z}(t) + A_2 Z(t) = 0,\ t > 0, \qquad Z(0) = X_0,\ \dot{Z}(0) = X_1, \tag{1.4}$$

where input parameters X_0, X_1, A_1 and A_2 are assumed to be dependent absolutely continuous random variables (r.v.'s) defined on a common probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. In the following,

$$\begin{aligned} D_{X_0} &= \{\, x_0 = X_0(\omega), \omega \in \Omega :\ x_{0,1} \le x_0 \le x_{0,2}\}\,, \\ D_{X_1} &= \{\, x_1 = X_1(\omega), \omega \in \Omega :\ x_{1,1} \le x_1 \le x_{1,2}\}\,, \\ D_{A_1} &= \{\, a_1 = A_1(\omega), \omega \in \Omega :\ a_{1,1} \le a_1 \le a_{1,2}\}\,, \\ D_{A_2} &= \{\, a_2 = A_2(\omega), \omega \in \Omega :\ a_{2,1} \le a_2 \le a_{2,2}\}\,, \end{aligned}$$

will denote the domains of X_0, X_1, A_1 and A_2, respectively, where the joint p.d.f., $f_{X_0,X_1,A_1,A_2}(x_0, x_1, a_1, a_2)$, is defined. As we have indicated previously our main goal is to determine the 1-p.d.f., $f_1(z, t)$, and the 2-p.d.f., $f_2(z_1, t_1; z_2, t_2)$, of the solution s.p. $Z(t)$ of the initial value problem (IVP) (1.4). As we shall see later, this will be done by applying RVT technique. In dealing with the computation of the 1-p.d.f. via RVT method within the context of r.d.e.'s, it is worth pointing out that some significant contributions are [8–14]. Most of these contributions usually assume that uncertainty enters via a single r.v. having a specific probability distribution which facilitates the corresponding analysis. Regarding the application of RVT method to account for the 2-p.d.f., to the best of our knowledge, it is has only been addressed by assuming specific standard distributions for random inputs. Some interesting contributions dealing with random differential equations include [15]. RVT technique has also been applied in combination with other techniques like Taylor series expansion to compute approximations to the p.d.f.'s of the reliability and performability indices in the context of Markov reliability and reward models [16]. Additionally to RVT method, over the last few years polynomial chaos technique has demonstrated to be a very powerful approach to deal with a wide range of randomness in differential equations [17, 18]. This approach concentrates on computing the main statistical moments associated to the solution s.p., namely the expectation and the variance, instead of the 1 and 2-p.d.f.'s.

This paper is organized as follows. In Section 2 the 1-p.d.f. and 2-p.d.f. of the solution of IVP (1.4) is determined. This is done assuming that all coefficients and initial conditions are absolutely continuous r.v.'s with arbitrary joint p.d.f. For convenience, the study is split in two steps by considering the real or complex character of the so-called characteristic roots. In Section 3, the particular case where just initial conditions are r.v.'s with a joint p.d.f. is addressed. In Section 4 some illustrative examples are exhibited. The closing section contains our main conclusions.

2. Computing the 1-p.d.f. and 2-p.d.f. of the solution stochastic process

This section is addressed to determine both, the 1-p.d.f., $f_1(z,t)$, and the 2-p.d.f., $f_2(z_1,t_1;z_2,t_2)$, of the solution to the IVP (1.4) using the RVT technique. This technique has numerous versions which are adapted to different contexts [2, 19]. Throughout the exposition we will apply the version stated in Th.1 of [7] including its notation.

Analogously as it also happens in the deterministic theory, a closed-form representation to the solution s.p. $Z(t)$ of IVP (1.4) can be given depending of the real or complex nature of the expressions

$$\alpha_1(\omega) = \frac{-A_1(\omega) + \sqrt{\Delta(\omega)}}{2}, \quad \alpha_2(\omega) = \frac{-A_1(\omega) - \sqrt{\Delta(\omega)}}{2}, \quad \omega \in \Omega, \quad (2.1)$$

where

$$\Delta(\omega) = (A_1(\omega))^2 - 4A_2(\omega). \quad (2.2)$$

$\alpha_1(\omega)$ and $\alpha_2(\omega)$ are the zeros of the so-called characteristic equation

$$\alpha^2 + A_1(\omega)\alpha + A_2(\omega) = 0, \quad (2.3)$$

associated to r.d.e. of IVP (1.4). The real or complex character of these random roots is obviously delineated by the following events $\mathcal{E}_i$, $1 \leq i \leq 3$,

$$\left\{ \begin{array}{lcl} p_1 & = & \mathbb{P}\left[\mathcal{E}_1 = \{\omega \in \Omega : \Delta(\omega) > 0\}\right], \\ p_2 & = & \mathbb{P}\left[\mathcal{E}_2 = \{\omega \in \Omega : \Delta(\omega) < 0\}\right], \\ p_3 & = & \mathbb{P}\left[\mathcal{E}_3 = \{\omega \in \Omega : \Delta(\omega) = 0\}\right], \end{array} \right. \tag{2.4}$$

where $\Delta(\omega)$ is defined in (2.2) and p_i, $1 \leq i \leq 3$, denote their likelihoods, respectively. Notice that these key events play here the same role they play in dealing with the random discrete counterpart of problem (1.4) (see expression (4) in [7]). Since $A_1 = A_1(\omega)$ and $A_2 = A_2(\omega)$ are assumed to be absolutely continuous r.v.'s, notice that $p_3 = 0$. Notice that the case where $0 < p_1, p_2 < 1$ with $p_1 + p_2 = 1$ constitutes the genuine random analysis, otherwise our approach leads to classical results. Based on the same arguments exhibited in [7], the computation of the 1-p.d.f. $f_1(z,t)$ will be split in two pieces, $f_{1\mathrm{R}}(z,t)$ and $f_{1\mathrm{C}}(z,t)$. Once the pieces $f_{1\mathrm{R}}(z,t)$ and $f_{1\mathrm{C}}(z,t)$, corresponding to the contributions of real and imaginary roots, have been computed (see subsequent expressions (2.10) and (2.15), respectively), the complete 1-p.d.f., $f_1(z,t)$, of the solution s.p. of IVP (1.4) will be determined as follows

$$f_1(z,t) = f_{1\mathrm{R}}(z,t) + f_{1\mathrm{C}}(z,t). \tag{2.5}$$

Notice that, $f_{1\mathrm{R}}(z,t)$ and $f_{1\mathrm{C}}(z,t)$, are assumed to be non-null for every z such that $z = Z(t)(\omega)$, with $\omega \in \mathcal{E}_i$, $i = 1, 2$, respectively. Moreover,

$$\int_{\mathbb{R}} f_1(z,t)\,\mathrm{d}z = \int_{\mathbb{R}} f_{1\mathrm{R}}(z,t)\,\mathrm{d}z + \int_{\mathbb{R}} f_{1\mathrm{C}}(z,t)\,\mathrm{d}z = p_1 + p_2 = 1.$$

Analogously, the 2-p.d.f. can be expressed as

$$f_2(z_1,t_1;z_2,t_2) = f_{2\mathrm{R}}(z_1,t_1;z_2,t_2) + f_{2\mathrm{C}}(z_1,t_1;z_2,t_2), \tag{2.6}$$

where $f_{2\mathrm{R}}(z_1,t_1;z_2,t_2)$ and $f_{2\mathrm{C}}(z_1,t_1;z_2,t_2)$ are given by (2.12) and (2.17), respectively.

Remark 2.1. Despite the notation adopted for $f_{1\mathrm{R}}(z,t)$ and $f_{1\mathrm{C}}(z,t)$ in (2.5), notice that they are not p.d.f.'s because they are not normalized (their integrals are, respectively, p_1 and p_2).

2.1. Real and distinct random roots

Throughout this section the probability p_1 defined in (2.4) will be assumed to be positive in order to guarantee the existence of characteristic roots with real realizations. When the characteristic roots are real and distinct, the solution of the IVP (1.4) can be written as follows

$$Z(t) = h_{\mathrm{R}}(t)X_0 + g_{\mathrm{R}}(t)X_1, \qquad \left\{ \begin{array}{lcl} g_{\mathrm{R}}(t) & = & \dfrac{\mathrm{e}^{\alpha_1 t} - \mathrm{e}^{\alpha_2 t}}{\alpha_1 - \alpha_2}, \\ h_{\mathrm{R}}(t) & = & \dfrac{\alpha_1 \mathrm{e}^{\alpha_2 t} - \alpha_2 \mathrm{e}^{\alpha_1 t}}{\alpha_1 - \alpha_2}, \end{array} \right. \tag{2.7}$$

being $\alpha_1 = \alpha_1(\omega)$ and $\alpha_2 = \alpha_2(\omega)$ the expressions defined by (2.1).

2.1.1. Computing the piece $f_{1\mathrm{R}}(z,t)$ of the 1-p.d.f. Let us fix t and let us apply Th.1 of [7] with $n=4$ in order to determine the piece $f_{1\mathrm{R}}(z,t)$ of the total p.d.f. $f_1(z,t)$ of r.v. $Z=Z(t)$ using the following identification

$$\begin{array}{ll} \mathbf{V}=(X_0,X_1,A_1,A_2), & f_{\mathbf{V}}(\mathbf{v})=f_{X_0,X_1,A_1,A_2}(x_0,x_1,a_1,a_2), \\ \mathbf{W}=(W_1,W_2,W_3,W_4)=\mathbf{r}(\mathbf{V}), & \mathbf{s}(\mathbf{W})=\mathbf{r}^{-1}(\mathbf{V}), \end{array}$$

$$\left\{ \begin{array}{lllll} W_1 & = & r_1(\mathbf{V}) & = & h_{\mathrm{R}}(t)X_0+g_{\mathrm{R}}(t)X_1, \\ W_2 & = & r_2(\mathbf{V}) & = & X_1, \\ W_3 & = & r_3(\mathbf{V}) & = & A_1, \\ W_4 & = & r_4(\mathbf{V}) & = & A_2, \end{array} \right.$$
$$\Rightarrow \left\{ \begin{array}{lllll} X_0 & = & s_1(\mathbf{W}) & = & \dfrac{W_1-\tilde{g}_{\mathrm{R}}(t)W_2}{\tilde{h}_{\mathrm{R}}(t)}, \\ X_1 & = & s_2(\mathbf{W}) & = & W_2, \\ A_1 & = & s_3(\mathbf{W}) & = & W_3, \\ A_2 & = & s_4(\mathbf{W}) & = & W_4, \end{array} \right. \tag{2.8}$$

being

$$\tilde{g}_{\mathrm{R}}(t)=\frac{\mathrm{e}^{\tilde{\alpha}_1 t}-\mathrm{e}^{\tilde{\alpha}_2 t}}{\tilde{\alpha}_1-\tilde{\alpha}_2}, \quad \tilde{h}_{\mathrm{R}}(t)=\frac{\tilde{\alpha}_1\mathrm{e}^{\tilde{\alpha}_2 t}-\tilde{\alpha}_2\mathrm{e}^{\tilde{\alpha}_1 t}}{\tilde{\alpha}_1-\tilde{\alpha}_2},$$

and

$$\begin{array}{lllll} \tilde{\alpha}_1 & = & \tilde{\alpha}_1(\omega) & = & \dfrac{-W_3+\sqrt{(W_3)^2-4W_4}}{2}, \\ \tilde{\alpha}_2 & = & \tilde{\alpha}_2(\omega) & = & \dfrac{-W_3-\sqrt{(W_3)^2-4W_4}}{2}. \end{array} \tag{2.9}$$

Taking into account that $\tilde{\alpha}_1(\omega)\neq\tilde{\alpha}_2(\omega)$ with probability 1 (w.p. 1), it is easy to check that the Jacobian is nonzero

$$J_4=\det\left(\frac{\partial \mathbf{v}}{\partial \mathbf{w}}\right)=\frac{1}{\tilde{h}_{\mathrm{R}}(t)}=\frac{\tilde{\alpha}_1-\tilde{\alpha}_2}{\tilde{\alpha}_1\mathrm{e}^{\tilde{\alpha}_2 t}-\tilde{\alpha}_2\mathrm{e}^{\tilde{\alpha}_1 t}}\neq 0, \qquad \text{w.p. } 1.$$

Therefore, by applying Th.1 of [7] the joint p.d.f. of the random vector $\mathbf{W}=(W_1,W_2,W_3,W_4)$ is given by

$$f_{\mathbf{W}}(\mathbf{w})=\frac{f_{X_0,X_1,A_1,A_2}\left(\dfrac{w_1-\tilde{g}_{\mathrm{R}}(t)w_2}{\tilde{h}_{\mathrm{R}}(t)},w_2,w_3,w_4\right)}{\left|\tilde{h}_{\mathrm{R}}(t)\right|}, \quad w_{1,i}\leq w_i\leq w_{2,i},\ 1\leq i\leq 4.$$

Finally, taking into account that $Z = W_1$ and the definition of the event $\mathcal{E}_1$, one gets

$$
\begin{aligned}
& f_{1\mathrm{R}}(z,t) \\
&= \int_{w_{2,1}}^{w_{2,2}} \int_{w_{3,1}}^{w_{3,2}} \int_{\min\left[w_{4,1};\frac{(w_3)^2}{4}\right]}^{\min\left[w_{4,2};\frac{(w_3)^2}{4}\right]} f_{W_1,W_2,W_3,W_4}(w_1,w_2,w_3,w_4)\,\mathrm{d}w_4\mathrm{d}w_3\mathrm{d}w_2 \\
&= \int_{x_{1,1}}^{x_{1,2}} \int_{a_{1,1}}^{a_{1,2}} \int_{\min\left[a_{2,1};\frac{(a_1)^2}{4}\right]}^{\min\left[a_{2,2};\frac{(a_1)^2}{4}\right]} \frac{f_{X_0,X_1,A_1,A_2}\left(\frac{z-g_{\mathrm{R}}(t)x_1}{h_{\mathrm{R}}(t)}, x_1, a_1, a_2\right)}{|h_{\mathrm{R}}(t)|}\,\mathrm{d}a_2\mathrm{d}a_1\mathrm{d}x_1,
\end{aligned}
\tag{2.10}
$$

being $g_{\mathrm{R}}(t), h_{\mathrm{R}}(t)$ and α_1, α_2, defined by (2.7) and (2.1), respectively.

2.1.2. Computing the piece $f_{2\mathrm{R}}(z_1,t_1;z_2,t_2)$ of the 2-p.d.f. In the subsequent, we will still take advantage of RVT method to determine the piece $f_{2\mathrm{R}}(z_1,t_1;z_2,t_2)$ of the total 2-p.d.f. $f_2(z_1,t_1;z_2,t_2)$ of the solution s.p. of IVP (1.4). With this aim, let us fix times t_1 and t_2. Then, we apply Th.1 of [7] with $n = 4$ to determine the joint p.d.f. of r.v.'s $Z_1 = Z(t_1)$ and $Z_2 = Z(t_2)$ considering the mapping $\mathbf{r}(\mathbf{V})$ as defined in (2.8) except for components $W_1 = r_1(\mathbf{V})$ and $W_2 = r_2(\mathbf{V})$, which now, for convenience, are defined by

$$
\left\{
\begin{array}{lllll}
W_1 & = & r_1(\mathbf{V}) & = & h_{\mathrm{R}}(t_1)X_0 + g_{\mathrm{R}}(t_1)X_1, \\
W_2 & = & r_2(\mathbf{V}) & = & h_{\mathrm{R}}(t_2)X_0 + g_{\mathrm{R}}(t_2)X_1,
\end{array}
\right.
\tag{2.11}
$$

where $g_{\mathrm{R}}(t)$ and $h_{\mathrm{R}}(t)$ are defined by (2.7). Then, the inverse $s(\mathbf{W})$ is given by (2.8) except for the components $X_0 = s_1(\mathbf{W})$ and $X_1 = s_2(\mathbf{W})$, which now result as

$$
s_1(\mathbf{W}) = \frac{W_1\tilde{g}_{\mathrm{R}}(t_2) - W_2\tilde{g}_{\mathrm{R}}(t_1)}{\tilde{g}_{\mathrm{R}}(t_2)\tilde{h}_{\mathrm{R}}(t_1) - \tilde{g}_{\mathrm{R}}(t_1)\tilde{h}_{\mathrm{R}}(t_2)},
$$

$$
s_2(\mathbf{W}) = \frac{W_2\tilde{h}_{\mathrm{R}}(t_1) - W_1\tilde{h}_{\mathrm{R}}(t_2)}{\tilde{g}_{\mathrm{R}}(t_2)\tilde{h}_{\mathrm{R}}(t_1) - \tilde{g}_{\mathrm{R}}(t_1)\tilde{h}_{\mathrm{R}}(t_2)}.
$$

Notice that $s_1(\mathbf{W})$ and $s_2(\mathbf{W})$ are well-defined because

$$
\tilde{g}_{\mathrm{R}}(t_2)\tilde{h}_{\mathrm{R}}(t_1) - \tilde{g}_{\mathrm{R}}(t_1)\tilde{h}_{\mathrm{R}}(t_2) = \frac{e^{\tilde{\alpha}_1 t_1 + \tilde{\alpha}_2 t_2} - e^{\tilde{\alpha}_1 t_2 + \tilde{\alpha}_2 t_1}}{\alpha_1 - \alpha_2} \neq 0, \quad t_1 \neq t_2.
$$

As a consequence, the Jacobian of inverse transformation is different from zero

$$
J_4 = \det\left(\frac{\partial \mathbf{v}}{\partial \mathbf{w}}\right) = \frac{1}{\tilde{g}_{\mathrm{R}}(t_2)\tilde{h}_{\mathrm{R}}(t_1) - \tilde{g}_{\mathrm{R}}(t_1)\tilde{h}_{\mathrm{R}}(t_2)} \neq 0, \qquad \text{w.p. } 1.
$$

Therefore, the joint p.d.f., $f_{\mathbf{W}}(\mathbf{w})$, of random vector $\mathbf{W} = (W_1, W_2, W_3, W_4)$ can be obtained by applying Th.1 of [7]. Taking into account that $Z_1 = W_1$, $Z_2 = W_2$ and the definition of event $\mathcal{E}_1$ (see (2.4)), the joint p.d.f. of r.v.'s $Z_1 = Z(t_1)$ and $Z_2 = Z(t_2)$ can be computed, and hence the piece $f_{2\mathrm{R}}(z_1, t_1; z_2, t_2)$ of the total 2-p.d.f. of $Z(t)$ is determined. This yields

$$f_{2\mathrm{R}}(z_1, t_1; z_2, t_2)$$

$$= \int_{a_{1,1}}^{a_{1,2}} \int_{\min\left[a_{2,1}; \frac{(a_1)^2}{4}\right]}^{\min\left[a_{2,2}; \frac{(a_1)^2}{4}\right]} f_{X_0,X_1,A_1,A_2} \left(\frac{z_1 g_{\mathrm{R}}(t_2) - z_2 g_{\mathrm{R}}(t_1)}{g_{\mathrm{R}}(t_2)h_{\mathrm{R}}(t_1) - g_{\mathrm{R}}(t_1)h_{\mathrm{R}}(t_2)}, \right.$$

$$\left. \frac{z_2 h_{\mathrm{R}}(t_1) - z_1 h_{\mathrm{R}}(t_2)}{g_{\mathrm{R}}(t_2)h_{\mathrm{R}}(t_1) - g_{\mathrm{R}}(t_1)h_{\mathrm{R}}(t_2)}, a_1, a_2 \right) \frac{1}{|g_{\mathrm{R}}(t_2)h_{\mathrm{R}}(t_1) - g_{\mathrm{R}}(t_1)h_{\mathrm{R}}(t_2)|} \, \mathrm{d}a_2 \mathrm{d}a_1, \tag{2.12}$$

being $g_{\mathrm{R}}(t), h_{\mathrm{R}}(t)$ and α_1, α_2, defined by (2.7) and (2.1), respectively.

According to (1.3), the piece of the total conditional p.d.f. $f_{2\mathrm{R}}(z_2, t_2|z_1, t_1)$ corresponding to event $\mathcal{E}_1$ is determined from expressions (2.10) and (2.12). We do not explicit the expression because its writing is cumbersome.

2.2. Complex random roots

In order to guarantee the existence of complex roots of the characteristic equation (2.3), let us now assume that $p_2 > 0$. For convenience, hereinafter

$$\left.\begin{array}{l} \mathrm{Re}(\alpha_1(\omega)) = -\dfrac{A_1(\omega)}{2}, \\[2ex] \mathrm{Im}(\alpha_1(\omega)) = \dfrac{\sqrt{-\Delta(\omega)}}{2}, \quad \Delta(\omega) = (A_1(\omega))^2 - 4A_2(\omega), \end{array}\right. \qquad \omega \in \Omega,$$

will denote the real and imaginary parts of the random root $\alpha_1 = \alpha_1(\omega)$, respectively. In this manner, when the characteristic roots are complex, the solution of the IVP (1.4) can be represented as follows

$$Z(t) = h_{\mathrm{C}}(t)X_0 + g_{\mathrm{C}}(t)X_1, \tag{2.13}$$

where

$$\left\{ \begin{array}{rcl} g_{\mathrm{C}}(t) & = & \dfrac{\mathrm{e}^{\mathrm{Re}(\alpha_1)t}}{\mathrm{Im}(\alpha_1)} \sin\left(\mathrm{Im}(\alpha_1)t\right), \\[2ex] h_{\mathrm{C}}(t) & = & \mathrm{e}^{\mathrm{Re}(\alpha_1)t} \left[\cos\left(\mathrm{Im}(\alpha_1)t\right) - \dfrac{\mathrm{Re}(\alpha_1)}{\mathrm{Im}(\alpha_1)} \sin\left(\mathrm{Im}(\alpha_1)t\right) \right]. \end{array} \right. \tag{2.14}$$

2.2.1. Computing the piece $f_{1\mathrm{C}}(z, t)$ of the 1-p.d.f. Next, we will take advantage of this closed-form expression for the solution of (1.4) together with the application of Th.1. of [7] to determine the piece $f_{1\mathrm{C}}(z, t)$ associated to the event $\mathcal{E}_2$ that contributes to determine the 1-p.d.f. $f_1(z, t)$. Let us define the mapping r_1 and its inverse s_1 as

$$W_1 = r_1(\mathbf{V}) = h_{\mathrm{C}}(t)X_0 + g_{\mathrm{C}}(t)X_1, \quad X_0 = s_1(\mathbf{W}) = \frac{W_1 - \tilde{g}_{\mathrm{C}}(t)W_2}{\tilde{h}_{\mathrm{C}}(t)},$$

being $\tilde{g}_{\mathrm{C}}(t)$ and $\tilde{h}_{\mathrm{C}}(t)$ the expressions resulting after substituting α_1 by $\tilde{\alpha}_1$ in (2.14) (see (2.9)). We keep the rest of the mappings r_2, r_3, r_4, s_2, s_3 and

s_4 as they were defined in (2.8). Notice that now the Jacobian is given by

$$J_4 = \frac{1}{\tilde{h}_{\mathrm{C}}(t)} = \frac{\mathrm{e}^{-\operatorname{Re}(\tilde{\alpha}_1)t}}{\left|\cos\left(\operatorname{Im}(\tilde{\alpha}_1)t\right) - \frac{\operatorname{Re}(\tilde{\alpha}_1)}{\operatorname{Im}(\tilde{\alpha}_1)}\sin\left(\operatorname{Im}(\tilde{\alpha}_1)t\right)\right|} \neq 0, \qquad \text{w.p. 1.}$$

Hence, by applying Th.1. of [7] the joint p.d.f. of the random vector $\mathbf{W} = (W_1, W_2, W_3, W_4)$ is

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{f_{X_0,X_1,A_1,A_2}\left(\dfrac{w_1 - \tilde{g}_{\mathrm{C}}(t)w_2}{\tilde{h}_{\mathrm{C}}(t)}, w_2, w_3, w_4\right)}{\left|\tilde{h}_{\mathrm{C}}(t)\right|}, \quad w_{1,i} \leq w_i \leq w_{2,i},\ 1 \leq i \leq 4.$$

Finally, taking into account that $Z = W_1$ one gets

$$f_{1\mathrm{C}}(z,t) = \int_{x_{1,1}}^{x_{1,2}} \int_{a_{1,1}}^{a_{1,2}} \int_{\max\left[a_{2,1};\frac{(a_1)^2}{4}\right]}^{\max\left[a_{2,2};\frac{(a_1)^2}{4}\right]} \frac{f_{X_0,X_1,A_1,A_2}\left(\frac{z - g_{\mathrm{C}}(t)x_1}{h_{\mathrm{C}}(t)}, x_1, a_1, a_2\right)}{|h_{\mathrm{C}}(t)|} \,\mathrm{d}a_2\mathrm{d}a_1\mathrm{d}x_1, \tag{2.15}$$

being $g_{\mathrm{C}}(t)$, $h_{\mathrm{C}}(t)$, and α_1, α_2 defined by (2.14) and (2.1), respectively.

2.2.2. Computing the piece $f_{2\mathrm{C}}(z_1, t_1; z_2, t_2)$ of the 2-p.d.f. The development exhibited in Section 2.1.2 to deal with the computation of the piece of the 2-p.d.f. in the case where both characteristic roots are real and distinct can be properly adapted to the complex case. In order to apply Th.1 of [7] to compute the joint p.d.f. of r.v.'s $Z_1 = Z(t_1)$ and $Z_2 = Z(t_2)$, let us consider the mapping $\mathbf{r}(\mathbf{V})$ as the one defined in (2.8) but now taking the components $W_1 = r_1(\mathbf{V})$ and $W_2 = r_2(\mathbf{V})$ as follows

$$\left\{\begin{array}{lllll} W_1 & = & r_1(\mathbf{V}) & = & h_{\mathrm{C}}(t_1)X_0 + g_{\mathrm{C}}(t_1)X_1, \\ W_2 & = & r_2(\mathbf{V}) & = & h_{\mathrm{C}}(t_2)X_0 + g_{\mathrm{C}}(t_2)X_1, \end{array}\right. \tag{2.16}$$

being $g_{\mathrm{C}}(t)$ and $h_{\mathrm{C}}(t)$ the functions defined by (2.14). Again, the inverse mapping, $s(\mathbf{W})$, is given by (2.8) except for the components $X_0 = s_1(\mathbf{W})$ and $X_1 = s_2(\mathbf{W})$, which now are given by

$$s_1(\mathbf{W}) = \frac{W_1\tilde{g}_{\mathrm{C}}(t_2) - W_2\tilde{g}_{\mathrm{C}}(t_1)}{\tilde{g}_{\mathrm{C}}(t_2)\tilde{h}_{\mathrm{C}}(t_1) - \tilde{g}_{\mathrm{C}}(t_1)\tilde{h}_{\mathrm{C}}(t_2)},$$

$$s_2(\mathbf{W}) = \frac{W_2\tilde{h}_{\mathrm{C}}(t_1) - W_1\tilde{h}_{\mathrm{C}}(t_2)}{\tilde{g}_{\mathrm{C}}(t_2)\tilde{h}_{\mathrm{C}}(t_1) - \tilde{g}_{\mathrm{C}}(t_1)\tilde{h}_{\mathrm{C}}(t_2)}.$$

Since $t_1 \neq t_2$ and we are in the complex case, one gets

$$\tilde{g}_{\mathrm{C}}(t_2)\tilde{h}_{\mathrm{C}}(t_1) - \tilde{g}_{\mathrm{C}}(t_1)\tilde{h}_{\mathrm{C}}(t_2) = \frac{\mathrm{e}^{\operatorname{Re}(\alpha_1)(t_1+t_2)}}{\operatorname{Im}(\alpha_1)}\sin\left(\operatorname{Im}(\alpha_1)(t_1 - t_2)\right) \neq 0.$$

As a consequence, the Jacobian of inverse transformation is well-defined and different from zero

$$J_4 = \det\left(\frac{\partial \mathbf{v}}{\partial \mathbf{w}}\right) = \frac{1}{\tilde{g}_{\mathrm{C}}(t_2)\tilde{h}_{\mathrm{C}}(t_1) - \tilde{g}_{\mathrm{C}}(t_1)\tilde{h}_{\mathrm{C}}(t_2)} \neq 0, \qquad \text{w.p. 1.}$$

Following an analogous development as the one showed to compute $f_{2\mathrm{R}}(z_1,t_1;z_2,t_2)$, one obtains

$$
\begin{aligned}
&f_{2\mathrm{C}}(z_1,t_1;z_2,t_2)\\
&=\int_{a_{1,1}}^{a_{1,2}}\int_{\max\left[a_{2,1};\frac{(a_1)^2}{4}\right]}^{\max\left[a_{2,2};\frac{(a_1)^2}{4}\right]} f_{X_0,X_1,A_1,A_2}\left(\frac{z_1 g_{\mathrm{C}}(t_2)-z_2 g_{\mathrm{C}}(t_1)}{g_{\mathrm{C}}(t_2)h_{\mathrm{C}}(t_1)-g_{\mathrm{C}}(t_1)h_{\mathrm{C}}(t_2)},\right.\\
&\left.\frac{z_2 h_{\mathrm{C}}(t_1)-z_1 h_{\mathrm{C}}(t_2)}{g_{\mathrm{C}}(t_2)h_{\mathrm{C}}(t_1)-g_{\mathrm{C}}(t_1)h_{\mathrm{C}}(t_2)},a_1,a_2\right)\frac{1}{|g_{\mathrm{C}}(t_2)h_{\mathrm{C}}(t_1)-g_{\mathrm{C}}(t_1)h_{\mathrm{C}}(t_2)|}\,\mathrm{d}a_2\mathrm{d}a_1,
\end{aligned}
\tag{2.17}
$$

being $g_{\mathrm{C}}(t), h_{\mathrm{C}}(t)$ and α_1,α_2, defined by (2.14) and (2.1), respectively.

3. An important particular case: When initial conditions are random variables

In this section we deal with the computation of the two first p.d.f.'s of the solution s.p. of (1.4) in the particular case that initial conditions X_0 and X_1 are r.v.'s with joint p.d.f. $f_{X_0,X_1}(x_0,x_1)$ and, coefficients A_1 and A_2 are deterministic constants. For the sake of clarity in the presentation, we will distinguish the deterministic nature of coefficients by using lower-case letters, i.e., $A_1 \to a_1$ and $A_2 \to a_2$ thereafter. We consider the analysis of this particular case by two main reasons. First, since coefficients a_1 and a_2 are deterministic, the description of both p.d.f.'s does not require the consideration of any probabilistic event of type $\mathcal{E}_i$, $1 \le i \le 3$, introduced in (2.4), and as a consequence, the representation of both p.d.f.'s does not need to be separated as we did in (2.5) and (2.6). Secondly, now a new representation of the solution s.p. of (1.4) must be considered in the case that both characteristic roots are real and identical. Notice that this case has not been considered in the general case addressed in Section 2 because its correspondent counterpart, given by event $\mathcal{E}_3$, had null probability. Additionally, it is important to point out that important properties, such as Gaussianity, can be inherited by the solution s.p. under specific hypotheses upon random initial conditions X_0 and X_1. This latter issue will be discussed later.

3.1. Computing the two first p.d.f.'s

The expression of the 1-p.d.f. in the cases where the characteristic roots, α_1,α_2, are real and distinct, or complex constants can be straightforwardly deduced from the development glossed in Section 2. Nevertheless, for the sake of completeness, we will specify them below. Hereinafter, we just indicate the minor adjustments we have had to make in order to adapt RVT method to the present context.

In the case that α_1,α_2 are real and distinct ($\Delta > 0$), mapping $\mathbf{r}$ and its inverse $\mathbf{s}$ are just defined by the two first components given in (2.8). This yields

$$f_1(z,t) = \int_{x_{1,1}}^{x_{1,2}} \frac{f_{X_0,X_1}\left(\frac{z-g_{\mathrm{R}}(t)x_1}{h_{\mathrm{R}}(t)}, x_1\right)}{|h_{\mathrm{R}}(t)|}\,\mathrm{d}x_1, \tag{3.1}$$

being $g_{\mathrm{R}}(t), h_{\mathrm{R}}(t)$ defined by (2.7) and, α_1, α_2 the (deterministic) values given by (2.1) but setting the correct deterministic notation $A_1 \to a_1$ and $A_2 \to a_2$. Whereas, the 2-p.d.f. is given by

$$f_2(z_1,t_1;z_2,t_2) = \frac{1}{|g_{\mathrm{R}}(t_2)h_{\mathrm{R}}(t_1) - g_{\mathrm{R}}(t_1)h_{\mathrm{R}}(t_2)|}$$

$$\times f_{X_0,X_1}\left(\frac{z_1 g_{\mathrm{R}}(t_2) - z_2 g_{\mathrm{R}}(t_1)}{g_{\mathrm{R}}(t_2)h_{\mathrm{R}}(t_1) - g_{\mathrm{R}}(t_1)h_{\mathrm{R}}(t_2)}, \frac{z_2 h_{\mathrm{R}}(t_1) - z_1 h_{\mathrm{R}}(t_2)}{g_{\mathrm{R}}(t_2)h_{\mathrm{R}}(t_1) - g_{\mathrm{R}}(t_1)h_{\mathrm{R}}(t_2)}\right). \tag{3.2}$$

In contrast to what happened with the case that coefficients A_1 and A_2 were r.v.'s, now $f_{1\mathrm{R}}(z,t)$ and $f_{2\mathrm{R}}(z_1,t_1;z_2,t_2)$ are the total 1-p.d.f., $f_1(z,t)$, and 2-p.d.f., $f_2(z_1,t_1;z_2,t_2)$, respectively.
In the case that α_1, α_2 are complex the 1-p.d.f. and the 2-p.d.f. are given, respectively, by

$$f_1(z,t) = \int_{x_{1,1}}^{x_{1,2}} \frac{f_{X_0,X_1}\left(\frac{z-g_{\mathrm{I}}(t)x_1}{h_{\mathrm{C}}(t)}, x_1\right)}{|h_{\mathrm{C}}(t)|}\,\mathrm{d}x_1, \tag{3.3}$$

and

$$f_2(z_1,t_1;z_2,t_2) = \frac{1}{|g_{\mathrm{C}}(t_2)h_{\mathrm{C}}(t_1) - g_{\mathrm{C}}(t_1)h_{\mathrm{C}}(t_2)|}$$

$$\times f_{X_0,X_1}\left(\frac{z_1 g_{\mathrm{C}}(t_2) - z_2 g_{\mathrm{C}}(t_1)}{g_{\mathrm{C}}(t_2)h_{\mathrm{C}}(t_1) - g_{\mathrm{C}}(t_1)h_{\mathrm{C}}(t_2)}, \frac{z_2 h_{\mathrm{C}}(t_1) - z_1 h_{\mathrm{C}}(t_2)}{g_{\mathrm{C}}(t_2)h_{\mathrm{C}}(t_1) - g_{\mathrm{C}}(t_1)h_{\mathrm{C}}(t_2)}\right). \tag{3.4}$$

Let us now assume that both characteristic roots coincide. Then, it is easy to check that the solution s.p. of (2.1) is given by

$$Z(t) = h_{\mathrm{D}}(t)X_0 + g_{\mathrm{D}}(t)X_1, \quad \left\{ \begin{array}{lcl} g_{\mathrm{D}}(t) & = & t\mathrm{e}^{\alpha t}, \\ h_{\mathrm{D}}(t) & = & (1-\alpha t)\mathrm{e}^{\alpha t}, \end{array} \right. \quad \alpha = -\frac{a_1}{2}. \tag{3.5}$$

In order to compute the 1-p.d.f., let us fix t and define the following mapping $\mathbf{W} = \mathbf{r}(\mathbf{V})$ and its inverse $\mathbf{V} = \mathbf{s}(\mathbf{W})$:

$$\begin{array}{lclcl} W_1 & = & r_1(\mathbf{V}) & = & h_{\mathrm{D}}(t)X_0 + g_{\mathrm{D}}(t)X_1 \quad \Rightarrow \quad X_0 = \dfrac{W_1 - g_{\mathrm{D}}(t)W_2}{h_{\mathrm{D}}(t)}, \\ W_2 & = & r_2(\mathbf{V}) & = & X_1 \quad \Rightarrow \quad X_1 = W_2. \end{array}$$

The Jacobian of inverse mapping is $J_2 = 1/h_{\mathrm{D}}(t) \neq 0$. Then, we first apply Th.1 of [7] to determine the joint p.d.f. of $\mathbf{W} = (W_1, W_2)$ and secondly, we marginalize this p.d.f. to get the 1-p.d.f. of the solution s.p. we are looking for

$$f_1(z,t) = \int_{x_{1,1}}^{x_{1,2}} \frac{f_{X_0,X_1}\left(\frac{z-g_{\mathrm{D}}(t)x_2}{h_{\mathrm{D}}(t)}, x_1\right)}{|h_{\mathrm{D}}(t)|}\,\mathrm{d}x_1. \tag{3.6}$$

Here $g_{\rm D}(t), h_{\rm D}(t)$ and α are given by (3.5). We finally determine the 2-p.d.f. of the solution s.p. of IVP (2.1). For this we compute the joint p.d.f. of r.v.'s $Z_1 = Z(t_1)$ and $Z_2 = Z(t_2)$ by defining the mapping $\mathbf{r}(\mathbf{V})$ whose components $W_1 = r_1(\mathbf{V})$ and $W_2 = r_2(\mathbf{V})$ are

$$\left\{ \begin{array}{lllll} W_1 & = & r_1(\mathbf{V}) & = & h_{\rm D}(t_1)X_0 + g_{\rm D}(t_1)X_1, \\ W_2 & = & r_2(\mathbf{V}) & = & h_{\rm D}(t_2)X_0 + g_{\rm D}(t_2)X_1, \end{array} \right. \tag{3.7}$$

where $g_{\rm D}(t)$ and $h_{\rm D}(t)$ are given by (3.5). The inverse mapping, $s(\mathbf{W})$, of $\mathbf{r}(\mathbf{V})$ is

$$s_1(\mathbf{W}) = \frac{W_1 g_{\rm D}(t_2) - W_2 g_{\rm D}(t_1)}{g_{\rm D}(t_2)h_{\rm D}(t_1) - g_{\rm D}(t_1)h_{\rm D}(t_2)},$$
$$s_2(\mathbf{W}) = \frac{W_2 h_{\rm D}(t_1) - W_1 h_{\rm D}(t_2)}{g_{\rm D}(t_2)h_{\rm D}(t_1) - g_{\rm D}(t_1)h_{\rm D}(t_2)}.$$

Since $t_1 \neq t_2$, one gets

$$g_{\rm D}(t_2)h_{\rm D}(t_1) - g_{\rm D}(t_1)h_{\rm D}(t_2) = (t_2 - t_1)e^{\alpha(t_1+t_2)} \neq 0.$$

Hence, the Jacobian of inverse transformation is well-defined and different from zero

$$J_2 = \det\left(\frac{\partial \mathbf{v}}{\partial \mathbf{w}}\right) = \frac{1}{g_{\rm D}(t_2)h_{\rm D}(t_1) - g_{\rm D}(t_1)h_{\rm D}(t_2)} \neq 0, \qquad \text{w.p. 1}.$$

Therefore, the 2-p.d.f. $f_2(z_1, t_1; z_2, t_2)$ is

$$\begin{aligned} & f_2(z_1, t_1; z_2, t_2) \\ & = f_{X_0,X_1}\left(\frac{z_1 g_{\rm D}(t_2) - z_2 g_{\rm D}(t_1)}{g_{\rm D}(t_2)h_{\rm D}(t_1) - g_{\rm C}(t_1)h_{\rm C}(t_2)},\right. \\ & \left.\frac{z_2 h_{\rm D}(t_1) - z_1 h_{\rm D}(t_2)}{g_{\rm D}(t_2)h_{\rm D}(t_1) - g_{\rm D}(t_1)h_{\rm D}(t_2)}\right) \frac{1}{|g_{\rm D}(t_2)h_{\rm D}(t_1) - g_{\rm D}(t_1)h_{\rm D}(t_2)|}, \end{aligned} \tag{3.8}$$

being $g_{\rm D}(t)$ and $h_{\rm D}(t)$ the deterministic functions defined by (3.5).
In Section 2 we have provided explicit expressions for both the first and the second p.d.f.'s of the solutions s.p.'s of IVP (1.4) in the general case where all inputs (coefficients and initial conditions) are r.v.'s. Whereas in this section the particular case where just initial conditions are r.v.'s has been studied. In both cases, one can continue, at least theoretically, by computing higher *fidis*, but their analytical representations will become cumbersome. Although, the full collection of *fidis* characterizes the solution s.p. of (2.1), there exist important situations where computing the two first p.d.f.'s is enough as when the s.p. is Gaussian. In such cases, only the determination of expectation and covariance functions are required to give a complete probabilistic description of the solution s.p. As it has been indicated in Section 1, these two deterministic functions can be obtained from the two first p.d.f.'s. Regarding the two cases analysed previously, if all inputs are r.v.'s, the solution s.p. $Z(t)$ is not likely going to become Gaussian, in general, due to the complex random transformations involved in its representation (see (2.7) and (2.13)–(2.14)).

With the aim of illustrating a scenario where Gaussian behaviour takes place, below we will assume a particular situation (corresponding to the degenerated Liouvillian case) where just the initial conditions are r.v.'s. Nevertheless, we warn the reader that other complex situations could also lead to the solution s.p., $Z(t)$, be Gaussian. In fact, assuming that X_0 and X_1 are independent Gaussian r.v.'s, say, $X_0 \sim \mathrm{N}(\mu_{X_0}; \sigma^2_{X_0})$ and $X_1 \sim \mathrm{N}(\mu_{X_1}; \sigma^2_{X_1})$, then using well-known properties of Gaussian transformations, $Z(t)$ is Gaussian. Under such hypotheses the determination of the mean and covariance functions can be done directly from (2.7), (2.13) and (3.5). In fact, the expectation function is

$$\mu_Z(t) = h_{\mathrm{C}}(t)\mathbb{E}[X_0] + g_{\mathrm{C}}(t)\mathbb{E}[X_1] = h_{\mathrm{C}}(t)\mu_{X_0} + g_{\mathrm{C}}(t)\mu_{X_1}, \quad i = \mathrm{R}, \mathrm{D}, \mathrm{C}, \tag{3.9}$$

whereas, using the independence between X_0 and X_1, the correlation function can be simplified as

$$C_Z(t_1, t_2) = h_{\mathrm{C}}(t_1)h_{\mathrm{C}}(t_2)\sigma^2_{X_0} + g_{\mathrm{C}}(t_1)g_{\mathrm{C}}(t_2)\sigma^2_{X_1}, \quad i = \mathrm{R}, \mathrm{D}, \mathrm{C}. \tag{3.10}$$

In (3.9) and (3.10), the deterministic functions $h_{\mathrm{C}}(t)$ and $g_{\mathrm{C}}(t)$ are defined by (2.7), (3.5) and (2.14), respectively. They correspond to the cases where characteristic roots are real and distinct (R), real and double (D) and complex (C), respectively.

4. Examples

In this section, we will show two full examples where the main results obtained throughout Sections 2 and 3 are illustrated.

Example 1. In the context of Section 2, we think that the better way to illustrate the theoretical results there established is computing the 1-p.d.f. $f_1(z, t)$ of the solution $Z(t)$ to IVP (1.4) at some values of t. This will be done considering three scenarios depending on the values of the crucial probabilities p_1 and p_2 associated to events $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Therefore, as we did in [7], we will again consider the three illustrative situations:

- Case I ($p_1 \gg p_2$): The event $\mathcal{E}_1$ is more likely than event $\mathcal{E}_2$. In other words, real and distinct roots of the characteristic equation are more probable than imaginary roots entailing that the *probabilistic* contribution of $f_{1\mathrm{R}}(z, t)$ to $f_1(z, t)$ is greater than $f_{1\mathrm{C}}(z, t)$.
- Case II ($p_1 \approx p_2 \approx \frac{1}{2}$): The events $\mathcal{E}_1$ and $\mathcal{E}_2$ are equiprobable, and hence both pieces $f_{1\mathrm{R}}(z, t)$ and $f_{1\mathrm{C}}(z, t)$ have a similar *probabilistic* weight to determine $f_1(z, t)$.
- Case III ($p_1 \ll p_2$): The event $\mathcal{E}_1$ is less likely than $\mathcal{E}_2$. This case can be easily interpreted as the counterpart of Case I.

In order to show graphically the computation of the 1-p.d.f. in each one of the three cases, we will assume a joint Gaussian distribution for the random vector input parameters, i.e., $\eta_i = (X_0, X_1, A_1, A_2)^{\mathrm{T}} \sim \mathrm{N}(\mu_{\eta_i}; \Sigma_{\eta_i})$. To accommodate each one of the three cases, we will take different mean vectors

μ_{η_i}, $i = 1, 2, 3$, and a common covariance matrix $\Sigma_{\eta_i} = \Sigma$. Specifically, we take

$$\mu_{\eta_i} = \begin{cases} (1,1,3,1)^{\mathrm{T}} & \text{if} \quad i = 1 \text{ (Case I)}, \\ (1,1,2,1)^{\mathrm{T}} & \text{if} \quad i = 2 \text{ (Case II)}, \\ (1,1,1,1)^{\mathrm{T}} & \text{if} \quad i = 3 \text{ (Case III)}, \end{cases}$$

$$\Sigma = \begin{pmatrix} 2/50 & 0 & -1/50 & 1/50 \\ 0 & 7/50 & 3/50 & -2/50 \\ -1/50 & 3/50 & 1/4 & 1/50 \\ 1/50 & -2/50 & 1/50 & 1/3 \end{pmatrix}. \tag{4.1}$$

Values of probabilities p_1 and p_2 defined by (2.4), which are associated to Cases I-III, are collected in Table 1. These values have been computed according to the following formulaes also used in the analysis of the example exhibited in [7]

$$p_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{(a_1)^2}{4}} f_{A_1,A_2}(a_1, a_2) \mathrm{d}a_2 \, \mathrm{d}a_1, \tag{4.2}$$

where

$$f_{A_1,A_2}(a_1, a_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_0,X_1,A_1,A_2}(x_0, x_1 a_1, a_2) \mathrm{d}x_0 \, \mathrm{d}x_1 .$$

Here, $f_{X_0,X_1,A_1,A_2}(x_0, x_1 a_1, a_2)$ denotes the joint Gaussian p.d.f. of random vector $\eta_i = (X_0, X_1, A_1, A_2)^{\mathrm{T}}$ with mean μ_{η_i}, $1 \le i \le 3$, corresponding to each one of the Cases I–III and variance-covariance matrix Σ, both defined by (4.1).

In Figure 1, graphical representations for the 1-p.d.f. $f_1(z, t)$ at $t = 0, 1, 2, 3, 4, 5$ in Cases I, II and III are shown. From them, one observes that in all the cases $f_1(z, t)$ tends to have wide tails and a peak about $z_0 = 0$. This is in agreement with Figure 2 where one observes that the standard deviation increases as t does. Based on the well-known conditions $A_1 > 0$ and $A_2 > 0$, that characterize the asymptotic stability of the null solution, $Z(t) = 0$, to the deterministic counterpart of IVP (1.4), now we have computed the following probability p_{s} of the associated key event $\mathcal{S}$

$$p_s = \mathbb{P}[\mathcal{S}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} f_{X_0,X_1,A_1,A_2}(x_0, x_1, a_1, a_2) \mathrm{d}a_1 \, \mathrm{d}a_2 \, \mathrm{d}x_0 \, \mathrm{d}x_1, \tag{4.3}$$

where

$$\mathcal{S} = \{\omega \in \Omega : \, A_1(\omega) > 0, A_2(\omega) > 0\} .$$

Values of p_s in each one of the Cases I–III are shown in Table 1. Notice that these values are close to 1, and thus asymptotic stability of the null solution holds with high likely.

Figure 2 shows the mean, $\mu_Z(t)$, and plus/minus the standard deviation, $\sigma_Z(t)$, of the solution s.p. $Z(t)$ in each one of the Cases I, II and III. These statistical moments have been computed on the basis of the 1-p.d.f. $f_1(z, t)$

Case	p_1	p_2	p_s
I	0.96641	0.03359	0.9999999990134
II	0.521039	0.478961	0.9999683287582
III	0.133076	0.866924	0.97724986822365

TABLE 1. Columns p_1 and $p_2 = 1 - p_1$ collect the values of the probabilities associated to Cases I-III, in the context of Example 1, when $\eta_i = (X_0, X_1, A_1, A_2)^{\mathrm{T}} \sim \mathrm{N}(\mu_{\eta_i}; \Sigma)$ with μ_{η_i}, $i = 1, 2, 3$ and Σ given by (4.1). Values of p_1 have been computed by (4.2). Values of p_s represent the probabilities associated to asymptotic stability according to (4.3) in every case.

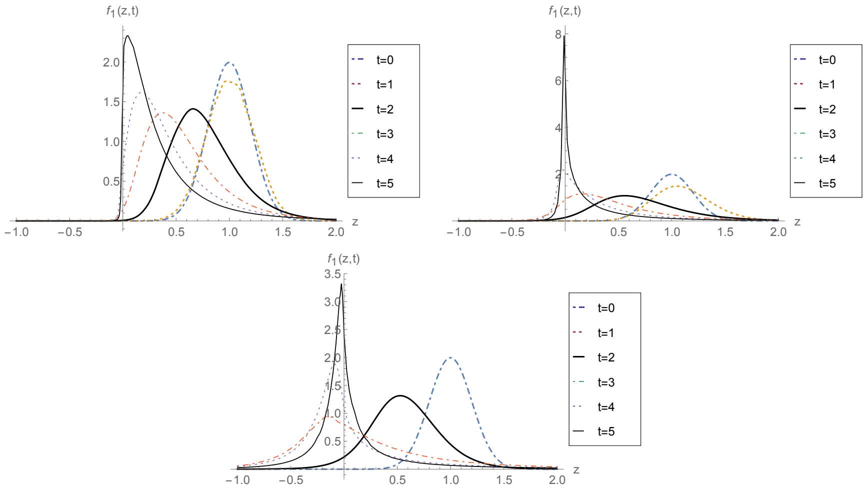

FIGURE 1. Plots of the 1-p.d.f. $f_1(z, t)$ of the solution $Z(t)$ to IVP (1.4) in Case I (top left), Case II (top right) and Case III (bottom) at different values of $t = 0, 1, 2, 3, 4, 5$, in the context of Example 1.

by applying expressions given in (1.1). The representations used for $f_1(z, t)$ in each one of the above cases are, (2.5) together with (2.10) and (2.15).

Finally, in Figure 3 the covariance surface, $C_Z(t_1, t_2)$, of the solution s.p. $Z(t)$ in each one of the Cases I, II and III has been plotted. This important deterministic function of two-variables t_1 and t_2 has been computed on the basis of the 2-p.d.f. $f_2(z_1, t_1; z_2, t_2)$ by applying expression (1.2) together with (2.15) and (2.12), as well as the mean function, previously determined.

Example 2. In this example we illustrate the theoretical development exhibited in Section 3 for the IVP (1.4) with deterministic coefficients a_1 and a_2. For the sake of completeness, we have considered the three possible cases

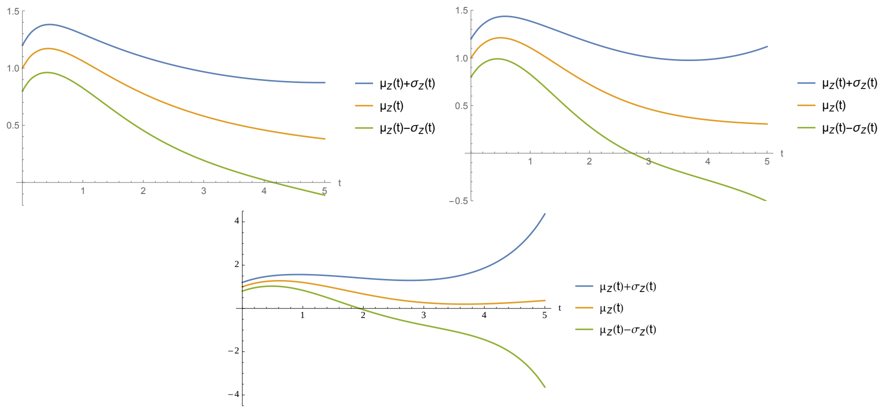

FIGURE 2. Plots of the mean, $\mu_Z(t)$, and plus/minus the standard deviation, $\sigma_Z(t)$, of the solution s.p. $Z(t)$ to IVP (1.4) in Case I (top left), Case II (top right) and Case III (bottom) on the time interval $0 \leq t \leq 5$, in the context of Example 1.

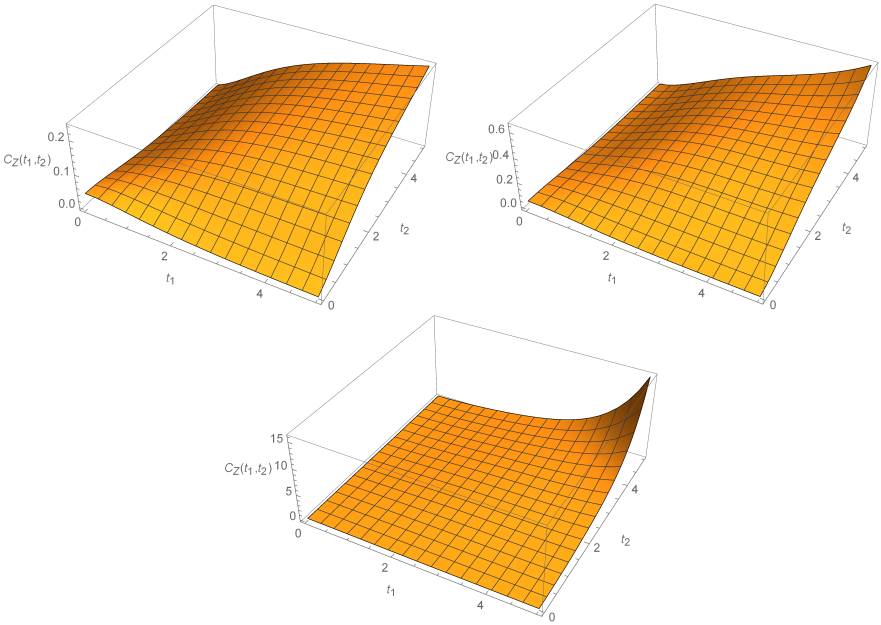

FIGURE 3. Plots of the covariance function, $C_Z(t_1, t_2)$, of the solution s.p. $Z(t)$ to IVP (1.4) in Case I (top left), Case II (top right) and Case III (bottom) on the region $(t_1, t_2) \in [0, 5] \times [0, 5]$, in the context of Example 1.

with respect to the nature of the roots of the characteristic equation. These cases depend upon the values of a_1 and a_2. In Figure 4 we have plotted the expectation plus/minus standard deviation, $\mu_Z(t) \pm \sigma_Z(t)$, and the covariance function, $C_Z(t_1, t_2)$, in each one of the following numerical situations:

- Case I (real and distinct roots): $a_1 = 3$, $a_2 = 1$.
- Case II (real and identical roots): $a_1 = 2$, $a_2 = 1$.
- Case III (complex roots): $a_1 = 1$, $a_2 = 1$.

In all the above cases, the expectation and variance of initial conditions X_0 and X_1 have been taken as $\mu_{X_0} = 2$, $\mu_{X_1} = 1$, and, $\sigma^2_{X_0} = 1/4$ and $\sigma^2_{X_1} = 1/16$, respectively. Computations of $\mu_Z(t) \pm \sigma_Z(t)$ and $C_Z(t_1, t_2)$ have been carried out using (3.9) and (3.10), respectively.

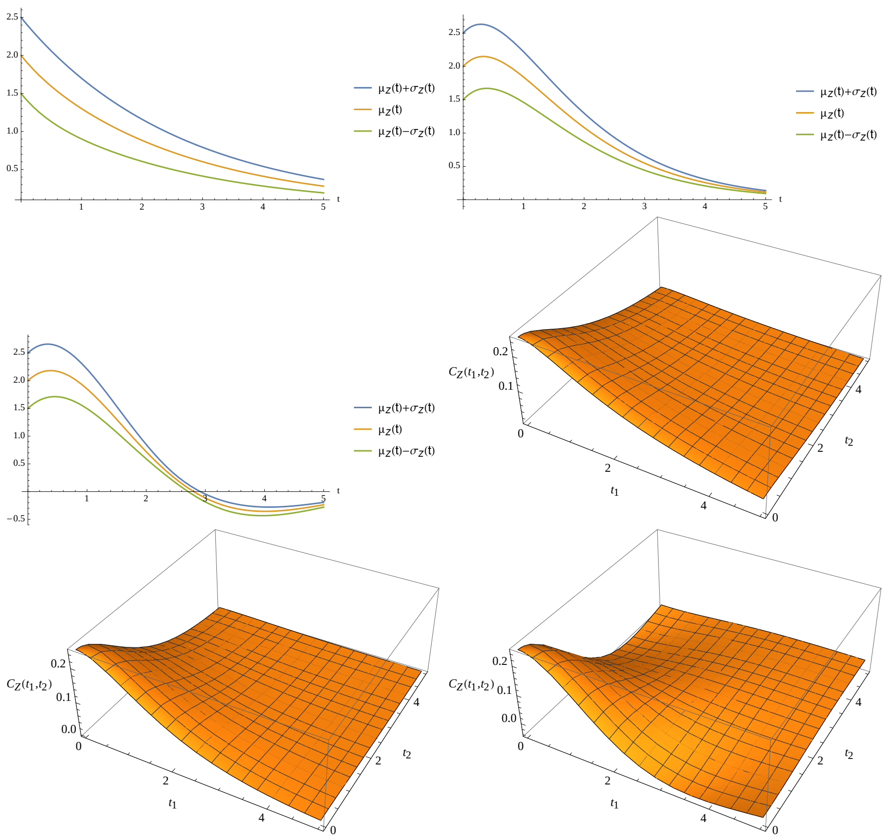

FIGURE 4. Graphical representations of the expectation plus/minus standard deviation, $\mu_Z(t) \pm \sigma_Z(t)$, and covariance function, $C_Z(t_1, t_2)$, of the solution s.p. $Z(t)$ to IVP (1.4) in Case I (top), Case II (middle) and Case III (bottom), in the context of Example 2.

5. Conclusions

The aim of this paper has been to complete the study recently provided in [7] but now dealing with its continuous counterpart, random homogeneous linear second-order differential equations. With regard to this earlier contribution, we point out that the current manuscript enlarges the study presented in [7] because now we have not only determined the first probability density function (1-p.d.f.) of the solution stochastic process of initial value problem (1.4) under very general hypotheses but also its second p.d.f. (2-p.d.f.). This is an important feature because besides providing a characterization of the solution stochastic process at every time instant through the 1-p.d.f., statistical dependence between two different time instant of the solution are also fully characterized. In addition, all one and two-dimensional statistical moments of the solution can be obtained from both p.d.f.'s, in particular, the expectation, the variance and the correlation functions. This is a distinctive feature with respect to other contributions dealing with random differential equations which often concentrate just on computing the mean and variance of the solution.

We think that besides constituting a nice generalization of classical theory, the results established in this paper are expected to be very useful in real applications. Indeed, numerous models are based on these type of differential equations and, in practice, their input parameters (coefficients, source term, and initial/boundary conditions) need to be fixed from physical measurements which usually contain errors. Other times, uncertainty can be attributed because ignorance or complexity of the phenomenon under study. Thereby, many real problems that are modelled by the initial value problem (1.4) may benefit from the theoretical results established in this article.

Acknowledgements

This work has been partially supported by the Spanish M.C.Y.T. grant MTM2013-41765-P.

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