On an efficient k-step iterative method for nonlinear equations

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Abstract

This paper is devoted to the construction and analysis of an efficient k-step iterative method for nonlinear equations. The main advantage of this method is that it does not need to evaluate any high order Fréchet derivative. Moreover, all the k-step have the same matrix, in particular only one LU decomposition is required in each iteration. We study the convergence order, the efficiency and the dynamics in order to motivate the proposed family. We prove, using some recurrence relations, a semilocal convergence result in Banach spaces. Finally, a numerical application related to nonlinear conservative systems is presented.

Keywords: Nonlinear equations, iterative methods, efficiency, order of convergence, dynamics, semilocal convergence.
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1 Introduction

It is well-known that finding exact solutions of nonlinear equations $F(x) = 0$ is a common problem appearing in science and engineering. This problem is difficult and we then usually use iterative methods to approximate the solutions of $F(x) = 0$. Let $F : D \subset \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be a nonlinear function $F(x) \equiv (F_1(x), F_2(x), \ldots, F_m(x))$ with $F_i : D \subseteq \mathbb{R}^m \to \mathbb{R}$, $i = 1, 2, \ldots, m$, $x = (x_1, x_2, \ldots, x_m)$ and D a nonempty open convex domain in $\mathbb{R}^m$. We usually apply iterative methods of the form

$$x_{n+1} = \Phi(x_n), \quad n \geq 0, \tag{1}$$

starting with a given initial approximation x_0 of the root α, where Φ is a function defined on a closed subset Ω of $\mathbb{R}^m$ that maps Ω into itself.

The choice of a method for solving $F(x) = 0$ usually depends on its efficiency, which links the speed of convergence (order of convergence) of the method to its computational cost. Two classic measurements of the efficiency, in the sense defined by Traub [14] and Ostrowski [11], are the efficiency index (EI) and the computational efficiency (CE), which are respectively defined by

$$EI = \rho^{1/a} \qquad \text{and} \qquad CE = \rho^{1/p}, \tag{2}$$

where ρ is the local order of convergence of the method, a represents the number of the evaluations of functions necessary to apply the method and p is the number of operations (products and quotients) that are needed to compute each iteration of the method.

For one-point iterative methods without memory, it is known that the order of convergence ρ is a natural number and the methods depend explicitly of the first $\rho - 1$ derivatives of the function involved in the equation. On the other hand, the computational cost increases as it is necessary to calculate the successive derivatives of the function involved in the algorithm of a method.

In this paper, we are interested in numerical methods that avoid the expensive computation of the derivatives of the function F at each step. We propose an efficient k-step Newton-type iterative method. The main advantage of this method is that it does not need to evaluate any high order Fréchet derivative, having the same matrix in each k-step, in particular only one LU decomposition is required in each iteration. This type of method appears in many applications where the authors heuristically choose a given number of steps with frozen derivatives (see for instance this incomplete list of Refs. [5, 6, 7, 9, 10]. We study the order, the efficiency and the dynamics in order to motivate the proposed family in sections 2 and 3. In section 4, we prove, using recurrence relations, a semilocal convergence result and include a numerical application related to nonlinear conservative systems.

2 Motivation

Clearly, we can improve (2) by increasing the local order of convergence with a minimum computational cost. Following this idea, a well-known result that improves the efficiency index and adapted to $\mathbb{R}^m$ states [14]:

Theorem 1 *If, under suitable conditions, iterative method (1) has order of convergence ρ, then the iterative method defined by*

$$\begin{cases} x_0 \in D, \\ y_n = \Phi(x_n), \\ x_{n+1} = y_n - [F'(x_n)]^{-1} F(y_n), \end{cases} \tag{3}$$

has order of convergence at least $\rho + 1$. □

Notice that using an additional evaluation of the vectorial function F in (3), the order of convergence of (1) has been increased in one unity.

In the previous theorem, there is a particular situation in which the order of convergence increases two units. We present this situation in the following result.

Theorem 2 *Let us suppose that the errors in sequences x_n and y_n are respectively $e_n = x_n - \alpha$ and $\tilde{e}_n = y_n - \alpha = B_1 e_n + B_2 e_n^2 + O(e_n^3)$, where $B_1, B_2 \in \mathscr{L}(\mathbb{R}^m)$. If $B_1 = 2$ is a constant linear operator, then the local order of convergence of the iterative method defined in (3) is at least 3. More precisely, if $\Gamma = [F'(\alpha)]^{-1}$ exists in a neighborhood of α, then*

$$e_{n+1} = -2\,(A_3 + A_2 B_2)\, e_n^3 \; + \; O(e_n^4),$$

where $A_i = \Gamma \frac{F^{(i)}(\alpha)}{i!} \in \mathscr{L}(\mathbb{R}^m)$, $i = 1, 2$.

Proof: We consider the Taylor's development of $F(x_n)$ around the solution α:

$$F(x_n) = \Gamma^{-1}[e_n + A_2 e_n^2 + A_3 e_n^3] + O(e_n^4)$$

where $A_i = \Gamma \frac{F^{(i)}(\alpha)}{i!} \in \mathscr{L}(\mathbb{R}^m)$, $i = 1, 2 \ldots$ so we have:

$$F'(x_n) = \Gamma^{-1}[I + 2A_2 e_n + 3A_3 e_n^2] + O(e_n^3),$$

By assuming $\tilde{e}_n = y_n - \alpha = B_1 e_n + B_2 e_n^2 + B_3 e_n^3 + O(e_n^4)$ we have the following Taylor's series:

$$F(y_n) = \Gamma^{-1}[B_1 e_n + \left(B_1^2 A_2 + B_2\right) e_n^2 + \left(B_1^3 A_3 + 2B_1 A_2 B_2 + B_3\right) e_n^3] + O(e_n^4)$$

and the corresponding development for the inverse operator $[F'(x_n)]^{-1}$ over $F(z_n)$ give us:

$$\begin{aligned} [F'(x_n)]^{-1} F(y_n) = {} & B_1 e_n + \left(-2A_2 B_1 + A_2 B_1^2 + B_2\right) e_n^2 \\ & + \left(\left(4A_2^2 - 3A_3\right) B_1 + A_3 B_1^3 + 2A_2 B_1 B_2 - 2A_2 \left(A_2 B_1^2 + B_2\right) + B_3\right) e_n^3 + O(e_n^4), \end{aligned}$$

then,

$$e_{n+1} = y_n - [F'(x_n)]^{-1}F(y_n) = -A_2\,(-2+B_1)\,B_1 e_n^2 \\ + \left(2A_2^2\,(-2+B_1)\,B_1 - A_3 B_1\left(-3+B_1^2\right) - 2A_2\,(-1+B_1)\,B_2\right) e_n^3 + O(e_n^4),$$

so, if $B_1 = 2$ the error equation is of third order:

$$e_{n+1} = -2\,(A_3 + A_2 B_2)\, e_n^3 + O(e_n^4).$$

□

If we consider the iterative process:

$$\begin{cases} x_0 \in D, \\ x_{n+1} = x_n + \Gamma_n F(x_n), n \geqslant 0, \end{cases} \tag{4}$$

then, it is easy to obtain that $e_{n+1} = 2e_n + O(e_n^2)$. So, applying the previous theorem it follows that the iterative process:

$$\begin{cases} x_0 \in D, \\ y_n = x_n + \Gamma_n F(x_n), \\ x_{n+1} = y_n - \Gamma_n F(y_n), \quad n \geqslant 0, \end{cases} \tag{5}$$

has order of convergence three. In [2], the authors prove that this iterative process (5) seems to have simpler dynamics that if we consider the different modifications shown by means of Newton's method [3]. This fact tells us that, from a numerical point of view, the implementation of this iterative process (5) is more favourable than if we consider "k-step Newton's method with frozen derivative", given by Traub's method [14]. So, we consider (5) as the source of our study. Now, If we apply many steps frozen the derivative, applying theorem 1 we obtain that the iterative process given by

$$\begin{cases} x_0 \in D, \\ x_n^{(1)} = x_n^{(0)} + \Gamma_n F(x_n^{(0)}), \\ x_n^{(2)} = x_n^{(1)} - \Gamma_n F(x_n^{(1)}), \\ \quad\vdots \\ x_n^{(k-1)} = x_n^{(k-2)} - \Gamma_n F(x_n^{(k-2)}), \\ x_n^{(k)} = x_n^{(k-1)} - \Gamma_n F(x_n^{(k-1)}), \quad n \geqslant 0, \end{cases} \tag{6}$$

where $\Gamma_n = F'(x_n)^{-1}$, $x_n = x_n^{(0)}$ and $x_{n+1} = x_n^{(k)}$, has order of convergence $k+1$, with $k \geq 2$.

In each iteration , we only need to compute a LU decomposition since the matrix in the k-steps is the same. Moreover, the method only use first order derivatives.

Our aim is, first of all, to analyze the competitiveness of the method in function of k by performing a study of the efficiency and their dynamics. Finally, we establish the conditions for completing a result of semilocal convergence that is particularly interesting for the fact of dealing with a k-steps iterative method.

3 Efficiency analysis and dynamics

In order to compare different methods, we use the efficiency index and the computational efficiency, (2). Notice that in the proposed multi-step method, (6), we only perform a new function evaluation in each step, so the value of EI in function of k, the steps preformed, and n, the size problem, is:

$$EI(k,m) = (k+1)^{\dfrac{1}{m^2+km}}$$

The computational efficiency is given by the number of products and quotients that we need for solving k linear systems with the same matrix of coefficient, by using LU factorization, so we have:

$$EC(k,n) = (k+1)^{\dfrac{m^3+3km^2+m}{3}}$$

where n is the size of each system.

We compare different methods by taking values for k from 2 to 7 and also Newton's method and considering problems of different sizes, m from 2 to 26.

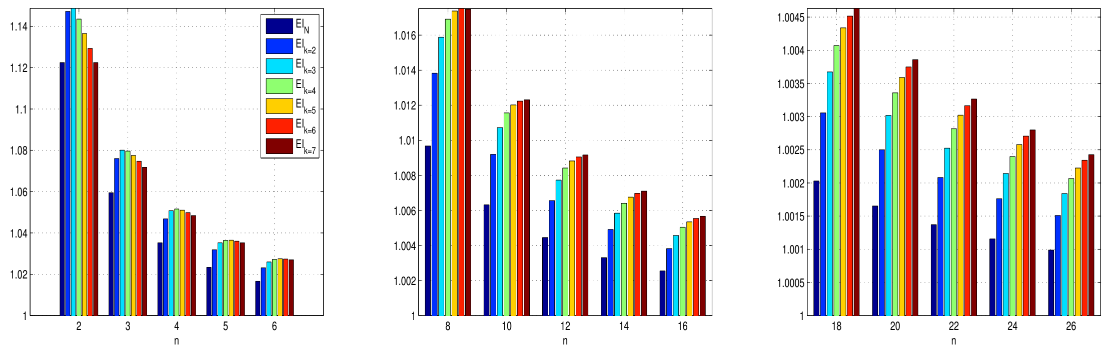

Figure 1: Efficiency index for $k = 2:7$ and $2 \leq m \leq 26$

In figure 1 we observe that the method defined by (6) has always better efficiency index than Newton's method. Specifically for small values of m the most efficient methods are the corresponding to order four but as the system size grows clearly can be noted that the method performs better the more steps it does, although we note that the largest increase for the efficiency occurs for methods of third and fourth order of convergence.

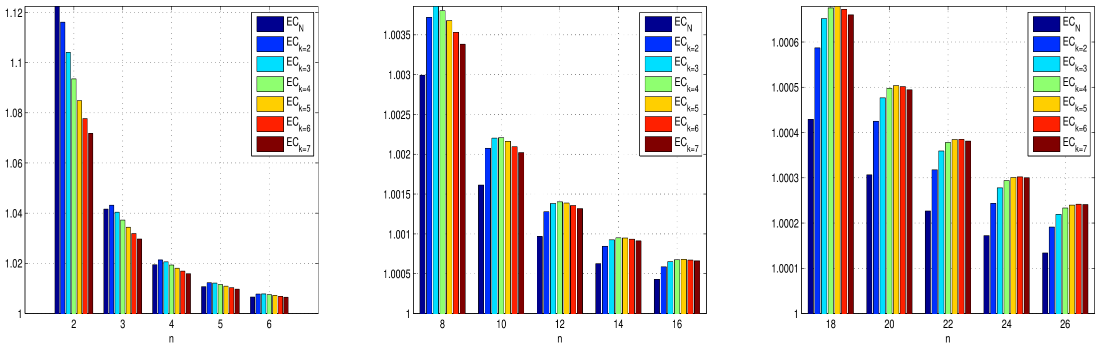

Figure 2: Computational efficiency for $k = 2:7$ and $2 \leq m \leq 34$

In figure 2 we show the computational efficiency. As can be observed in the graphics only for problems of size $m = 2$ Newton's method has better efficiency than the multi-step methods. However, for medium size problems $3 \leq m \leq 6$ the best computational efficiency correspond to the two steps method, that is of order three. But it is very interesting to point out that for large problems methods with 4, 5 and 6 steps are the most efficient.

3.1 Some dynamical pictures

We plot the attraction basins that the methods generate when they are applied to extract radicals. The attraction basins clarify the structures of the universal Julia sets associated with the corresponding iterative methods.This allows us to observe graphically the dynamical behavior of the rational maps.

We apply the k-step methods with different orders of convergence: 3, 4, 5, 6 and 7, to obtain the three roots of the polynomial $p(z) = z^3 - 1$, and we paint their attraction basins. We consider a square containing the roots and we choose these points as initial guesses. In all the cases, we use a tolerance 10^{-4} and a maximum of 80 iterations. We assign a color to each attraction basin of the roots. If we do not obtain the desired tolerance with the fixed iterations, we do not continue and we decide that the iterative method starting at each initial guess does not converge to any root and assign black color to those points. As we observe in the following figures, the iterative functions have three forward invariant Fatou components which are super-attracting where the iterates converge to the corresponding roots. In the next section we present conditions to ensure convergence, in particular we find some balls included in the Fatou components.

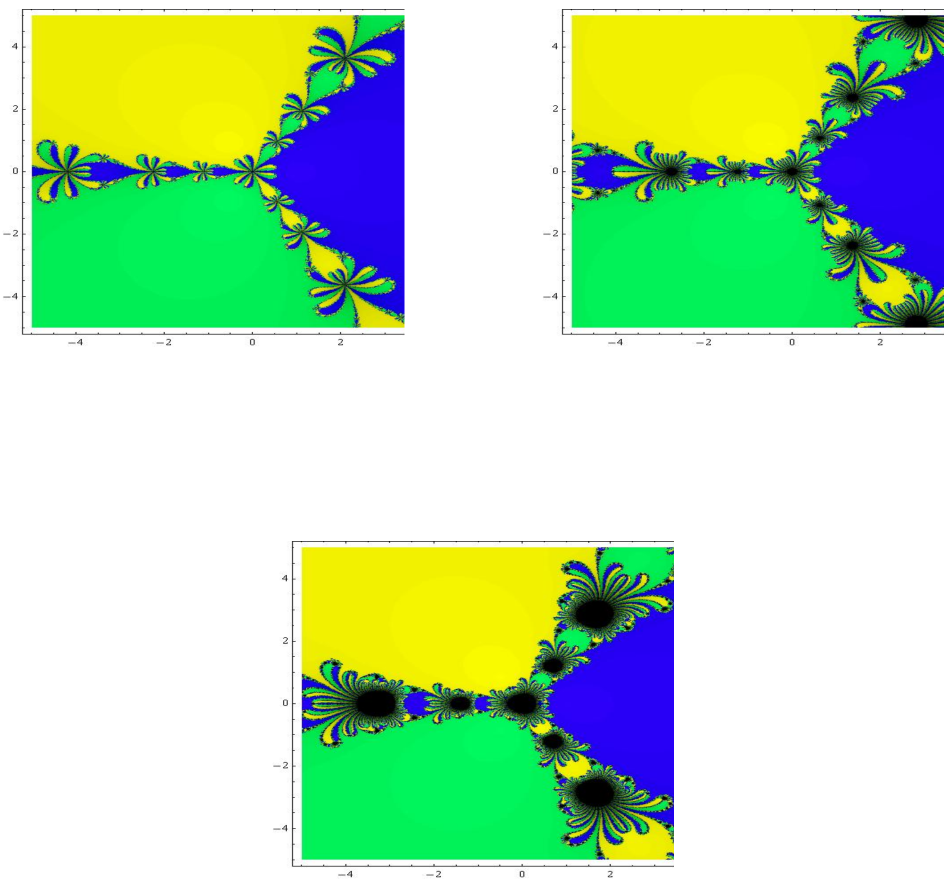

Figure 3: Julia and Fatou components for orders 3, 4 and 5.

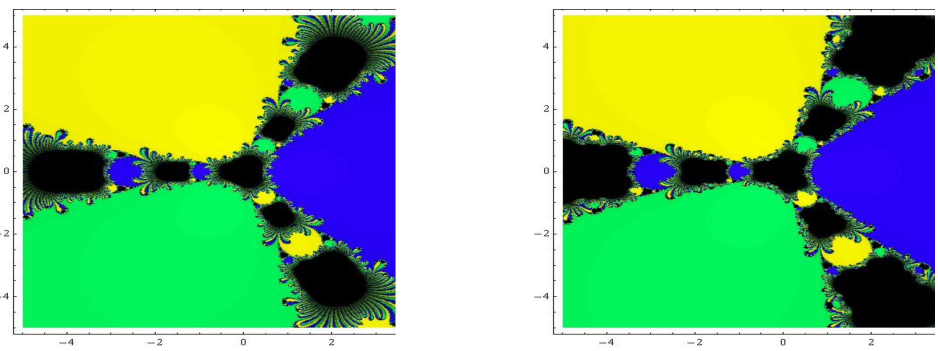

Figure 4: Julia and Fatou components for orders 6 and 7.

4 Semilocal convergence

Let us assume that $\Gamma_0 = F'(x_0)^{-1}$ exists for some $x_0 \in D$ and the following conditions are satisfied:

(C_1) $\quad \|\Gamma_0\| \leq \beta$ and $\|\Gamma_0 F(x_0)\| \leq \eta$

(C_2) $\quad \|F'(x) - F'(y)\| \leq K\|x - y\|, x, y \in D.$

Taking into account these previous conditions we obtain the following result:

Lemma 3 *Suppose that there exits R, with $R \in \mathbb{R}_+$ such as $B(x_0, R\eta) \subset \Omega$ and $\beta K R\eta < 1$, then for each $x_n \in B(x_0, R\eta)$, $\Gamma_n = F'(x_n)^{-1}$ exits and*

$$\|\Gamma_n\| \leq \frac{\beta}{1 - \beta K R\eta}$$

Proof:
In order to get the existence of Γ_n we write:

$$\|I - \Gamma_0 F'(x_n)\| \leq \|\Gamma_0\|\|F'(x_0) - F'(x_n)\| \leq \beta K R\eta < 1,$$

so, by Banach's lemma [8], the result is satisfied.

□

Notice that, if we denote $\tilde{\beta}_R = \dfrac{\beta}{1 - \beta K R\eta}$, obviously $\tilde{\beta}_R > \beta$.

4.1 Recurrence relations

We will first analyze the case $n = 0$ for different values of j. From now, we denote $\eta_0 = \eta$.

In first place, for $j = 1$, we have

$$\|x_0^{(1)} - x_0\| \leq \|\Gamma_0 F(x_0)\| \leq \eta_0 < R\eta, \tag{7}$$

if $1 < R$. So, $x_0^{(1)} \in B(x_0, R\eta)$.

Now by considering the following Taylor expansion

$$F(x_0^{(1)}) = F(x_0) + F'(x_0)(x_0^{(1)} - x_0) + \int_{x_0}^{x_0^{(1)}} [F'(z) - F'(x_0)]\mathrm{dz}$$

we get

$$\Gamma_0[F(x_0^{(1)}) - F(x_0)] = (I - \int_0^1 \Gamma_0[F'(x_0 + \tau(x_0^{(1)} - x_0)) - F'(x_0)]\mathrm{d}\tau(x_0^{(1)} - x_0).$$

Then, for $j = 2$, by taking norms in the previous expression, using (C_2) and denoting $a_0 = \tilde{\beta}_R K\eta_0$ one has:

$$\begin{aligned}\|x_0^{(2)} - x_0\| = \|\Gamma_0[F(x_0^{(1)}) - F(x_0)]\| &\leq (1 + \|\Gamma_0\|\frac{1}{2}K\|x_0^{(1)} - x_0\|)\|x_0^{(1)} - x_0\| \\ &\leq (1 + \frac{a_0}{2})\eta_0 < R\eta,\end{aligned}$$

if $(1 + \frac{a_0}{2}) < R$. So, $x_0^{(2)} \in B(x_0, R\eta)$. Moreover, it follows

$$\|x_0^{(2)} - x_0^{(1)}\| = \|x_0^{(2)} - x_0\| + \|x_0 - x_0^{(1)}\| \leq (2 + \frac{a_0}{2})\eta_0.$$

Now, for bounding $F(x_0^{(2)})$ we consider the following Taylor's development:

$$\begin{aligned}F(x_0^{(2)}) &= F(x_0^{(1)}) + F'(x_0^{(1)})(x_0^{(2)} - x_0^{(1)}) + \int_{x_0^{(1)}}^{x_0^{(2)}} [F'(z) - F'(x_0^{(1)})]\mathrm{dz} \\ &= F(x_0^{(1)}) + [F'(x_0^{(1)} - F'(x_0)](x_0^{(2)} - x_0^{(1)}) + F'(x_0)(x_0^{(2)} - x_0^{(1)}) \\ &+ \int_0^1 [F'(x_0^{(1)} + \tau(x_0^{(2)} - x_0^{(1)})) - F'(x_0^1)]\mathrm{d}\tau(x_0^{(2)} - x_0^{(1)}), \\ &= [F'(x_0^{(1)}) - F'(x_0)](x_0^{(2)} - x_0^{(1)}) + \int_0^1 [F'(x_0^{(1)} + \tau(x_0^{(2)} - x_0^{(1)})) - F'(x_0^{(1)})]\mathrm{d}\tau(x_0^{(2)} - x_0^{(1)}),\end{aligned}$$

where we have used that $F(x_0^{(1)}) + F'(x_0)(x_0^{(2)} - x_0^{(1)}) = 0$. Now, we have the corresponding bounds:

$$\begin{aligned}\|F(x_0^{(2)})\| &\leq K\|x_0^{(1)} - x_0\|\|x_0^{(2)} - x_0^{(1)}\| + \frac{K}{2}\|x_0^{(2)} - x_0^{(1)}\|^2 \\ &= K(\|x_0^{(1)} - x_0\| + \frac{1}{2}\|x_0^{(2)} - x_0^{(1)}\|)\|x_0^{(2)} - x_0^{(1)}\| \\ &\leq K(R\eta + (1 + \frac{a_0}{4})\eta_0)\|x_0^{(2)} - x_0^{(1)}\| = (R + 1 + \frac{a_0}{4})K\eta_0\|x_0^{(2)} - x_0^{(1)}\|,\end{aligned} \tag{8}$$

then, for $j = 3$, by denoting $M_0 = a_0(R + 1 + \frac{a_0}{4})$, we obtain

$$\|x_0^{(3)} - x_0^{(2)}\| \leq \|\Gamma_0\|\|F(x_0^{(2)})\| \leq (R + 1 + \frac{a_0}{4})a_0\|x_0^{(2)} - x_0^{(1)}\| = M_0\|x_0^{(2)} - x_0^{(1)}\|$$

$$\begin{aligned}\|x_0^{(3)} - x_0\| &\leq \|x_0^{(3)} - x_0^{(2)}\| + \|x_0^{(2)} - x_0\| \leq M_0\|x_0^{(2)} - x_0^{(1)}\| + (1 + \frac{a_0}{2})\eta_0 \\ &\leq (M_0(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2})\eta_0 < R\eta,\end{aligned}$$

if $M_0(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2} < R$. In this case, $x_0^{(3)} \in B(x_0, R\eta)$.

Now, with a similar reasoning that in (8) and if $M_0 < 1$, we have

$$\begin{aligned}\|F(x_0^{(3)})\| &\leq K(\|x_0^{(2)} - x_0\| + \frac{1}{2}\|x_0^{(3)} - x_0^{(2)}\|)\|x_0^{(3)} - x_0^{(2)}\| \\ &\leq K(R\eta + \frac{1}{2}M_0(2 + \frac{a_0}{2})\eta_0)\|x_0^{(3)} - x_0^{(2)}\| < (R + 1 + \frac{a_0}{4})K\eta_0\|x_0^{(3)} - x_0^{(2)}\|, \quad (9)\end{aligned}$$

then for $j = 4$, it holds

$$\begin{aligned}\|x_0^{(4)} - x_0^{(3)}\| &\leq \|\Gamma_0\|\|F(x_0^{(3)})\| \leq (R + 1 + \frac{a_0}{4})a_0\|x_0^{(3)} - x_0^{(2)}\| = M_0\|x_0^{(3)} - x_0^{(2)}\| \\ &\leq M_0^2\|x_0^{(2)} - x_0^{(1)}\| \leq \|x_0^{(2)} - x_0^{(1)}\|\end{aligned}$$

and

$$\begin{aligned}\|x_0^{(4)} - x_0\| &\leq \|x_0^{(4)} - x_0^{(3)}\| + \|x_0^{(3)} - x_0\| \leq M_0^2\|x_0^{(2)} - x_0^{(1)}\| + \|x_0^{(3)} - x_0\| \\ &\leq M_0^2(2 + \frac{a_0}{2})\eta + (M_0(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2}))\eta \\ &\leq [(M_0 + M_0^2)(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2}]\eta < R\eta,\end{aligned}$$

if $(M_0 + M_0^2)(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2} < R$. So, $x_0^{(4)} \in B(x_0, R\eta)$.

This previous study leads us to state an inductive procedure and then we can obtain the following result.

Lemma 4 *The following conditions are verified for* $3 \leq j \leq k$

i) $\|x_0^{(j)} - x_0^{(j-1)}\| \leq M_0^{j-2}\|x_0^{(2)} - x_0^{(1)}\| \leq \|x_0^2 - x_0^{(1)}\|$

ii) $\|x_0^{(j)} - x_0\| \leq [(M_0 + M_0^2 + \ldots + M_0^{j-2})(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2}]\eta_0 < R\eta,$

iii) $\|F(x_0^{(j)})\| \leq K(R + 1 + \frac{a_0}{4})\|x_0^{(j)} - x_0^{(j-1)}\|,$

if $(M_0 + M_0^2 + \ldots + M_0^{j-2})(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2} < R.$

$\square$

Now, we analyze the method for $n = 1$. By remembering the notation we have that $x_1 = x_1^{(0)} = x_0^{(k)}$ and then, by the precedent results, we have already established that:

$$\|x_1^{(0)} - x_0^{(k-1)}\| \leq M_0 \|x_0^{(k-1)} - x_0^{(k-2)}\| \tag{10}$$
$$\|x_1^{(0)} - x_0\| \leq [(M_0 + M_0^2 + \ldots + M^{k-2})(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2}]\eta_0 < R\eta.$$

Then, if $(M_0 + M_0^2 + \ldots + M^{k-2})(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2} < R$, it follows that $x_1 \in B(x_0, R\eta)$.

To continue, for $j = 1$, we consider

$$\begin{aligned}
F(x_1^{(0)}) &= F(x_0^{(k-1)}) + F'(x_0^{(k-1)})(x_1^{(0)} - x_0^{(k-1)}) + \int_{x_0^{(k-1)}}^{x_1^{(0)}} [F'(z) - F'(x_0^{(k-1)})]\mathrm{dz} \\
&= F(x_0^{(k-1)}) + [F'(x_0^{(k-1)}) - F'(x_0)](x_1^{(0)} - x_0^{(k-1)}) + F'(x_0)(x_1^{(0)} - x_0^{(k-1)}) \\
&+ \int_0^1 [F'(x_0^{(k-1)} + \tau(x_1^{(0)} - x_0^{(k-1)})) - F'(x_0^{(k-1)})]\mathrm{d}\tau(x_1^{(0)} - x_0^{(k-1)}) \\
&= [F'(x_0^{(k-1)}) - F'(x_0)]((x_1^{(0)} - x_0^{(k-1)}) \\
&+ \int_0^1 [F'(x_0^{(k-1)} + \tau(x_1^{(0)} - x_0^{(1)})) - F'(x_0^{(k-1)})]\mathrm{d}\tau(x_1^{(0)} - x_0^{(k-1)}),
\end{aligned}$$

then, by taking norms and using (C_2), we have

$$\begin{aligned}
\|F(x_1^{(0)})\| &\leq K\|x_0^{(k-1)} - x_0\|\|x_1^{(0)} - x_0^{(k-1)}\| + \frac{k}{2}\|x_1^{(0)} - x_0^{(k-1)}\|^2 \\
&\leq K(\|x_0^{(k-1)} - x_0\| + \frac{1}{2}\|x_1^{(0)} - x_0^{(k-1)}\|)\|x_1^{(0)} - x_0^{(k-1)}\| \\
&\leq (R + 1 + \frac{a_0}{4})K\eta_0\|x_1^{(0)} - x_0^{(k-1)}\|, \qquad (11)
\end{aligned}$$

and consequently by denoting $\eta_1 = M_0^{k-1}(2 + \frac{a_0}{2})\eta_0$ one has:

$$\begin{aligned}
\|x_1^{(1)} - x_1^{(0)}\| &\leq \|\Gamma_1\|\|F(x_1^{(0)})\| \leq \beta_R K\eta_0(R + 1 + \frac{a_0}{4})\|x_1^{(0)} - x_0^{(k-1)}\| \\
&= M_0\|x_1^{(0)} - x_0^{(k-1)}\| \leq M_0^{k-1}\|x_0^{(2)} - x_0^{(1)}\| \\
&\leq M_0^{k-1}(2 + \frac{a_0}{2})\eta_0 = \eta_1 \leq R\eta_1, \qquad (12)
\end{aligned}$$

and also can be written:

$$\begin{aligned}\|x_1^{(1)}-x_0\| &\leq \|x_1^{(1)}-x_1^{(0)}\|+\|x_1^{(0)}-x_0\| \\ &\leq M_0^{k-1}(2+\frac{a_0}{2})\eta_0+\left[\left(\sum_{j=1}^{k-2}M_0^i\right)(2+\frac{a_0}{2})+1+\frac{a_0}{2}\right]\eta_0 \\ &= \left[\left(\sum_{j=1}^{k-1}M_0^j\right)(2+\frac{a_0}{2})+1+\frac{a_0}{2}\right]\eta_0 < R\eta,\end{aligned}$$

if $\left(\sum_{j=1}^{k-1}M_0^j\right)(2+\frac{a_0}{2})+1+\frac{a_0}{2}<R$. So, we obtain: $x_1^{(1)}\in B(x_0,R\eta)$.

Following the previous process, we have

$$F(x_1^{(1)})=F(x_1)+F'(x_1)(x_1^{(1)}-x_1))+\int_{x_1}^{x_1^{(1)}}[F'(z)-F'(x_0^{(k-1)})]\mathrm{dz},$$

then, we get

$$\Gamma_1[F(x_1^{(1)})-F(x_1)]=(I-\int_0^1\Gamma_1[F'(x_1+\tau(x_1^{(1)}-x_1))-F'(x_1)]\mathrm{d}\tau)(x_1^{(1)}-x_1).$$

So, for $j=2$, by noting that $x_1^{(0)}=x_1$, applying (C_2) and taking norms in the previous expression, we obtain

$$\begin{aligned}\|x_1^{(2)}-x_1^{(0)}\| = \|\Gamma_1[F(x_1^{(1)})-F(x_1)]\| &\leq (1+\|\Gamma_1\|\frac{1}{2}K\|x_1^{(1)}-x_1\|)\|x_1^{(1)}-x_1\| \\ &\leq (1+\tilde{\beta}_R\frac{1}{2}K\eta_1)\eta_1=(1+\frac{a_1}{2})\eta_1,\end{aligned}$$

denoting $a_1=\tilde{\beta}_Rk\eta_1$, and from (12) we also get

$$\|x_1^{(2)}-x_1^{(1)}\|\leq\|x_1^{(2)}-x_1^{(0)}\|+\|x_1^{(1)}-x_1^{(0)}\|\leq(1+\frac{a_1}{2})\eta_1+\eta_1\leq(2+\frac{a_1}{2})\eta_1.$$

On the other hand, from (10) we have

$$\begin{aligned}\|x_1^{(2)}-x_0\| &= \|x_1^{(2)}-x_1^{(0)}\|+\|x_1^{(0)}-x_0\| \\ &\leq (1+\frac{a_1}{2})\eta_1+\left[\left(\sum_{j=1}^{k-2}M_0^j\right)(2+\frac{a_0}{2})+1+\frac{a_0}{2}\right]\eta_0<R\eta,\end{aligned} \tag{13}$$

if $(1+\frac{a_1}{2})\eta_1 + \left[\left(\sum_{j=1}^{k-2} M_0^j\right)(2+\frac{a_0}{2}) + 1 + \frac{a_0}{2}\right]\eta_0 < R$. Then, $x_1^{(2)} \in B(x_0, R\eta)$.

Now, bounding $F(x_0^{(2)})$, we have

$$\begin{aligned}
\|F(x_1^{(2)})\| &\leq K\|x_1^{(1)} - x_1\|\|x_1^{(2)} - x_1^{(1)}\| + \frac{k}{2}\|x_1^{(2)} - x_1^{(1)}\|^2 \\
&\leq K(\|x_1^{(1)} - x_1\| + \frac{1}{2}\|x_1^{(2)} - x_1^{(1)}\|)\|x_1^{(2)} - x_1^{(1)}\| \\
&\leq K\eta_1(R+1+\frac{a_1}{4})\|x_1^{(2)} - x_1^{(1)}\|, \qquad (14)
\end{aligned}$$

then, for $j=3$, by writing $M_1 = a_1(R+1+\frac{a_1}{4})$ it follows:

$$\begin{aligned}
\|x_1^{(3)} - x_1^{(2)}\| &= \|\Gamma_1[F(x_1^{(2)})\| \leq a_1(R+1+\frac{a_1}{4})\|x_1^{(2)} - x_1^{(1)}\| \\
&\leq M_1\|x_1^{(2)} - x_1^{(1)}\| \leq M_1(2+\frac{a_1}{2})\eta_1 \leq (2+\frac{a_1}{2})\eta_1
\end{aligned}$$

so, we have

$$\begin{aligned}
\|x_1^{(3)} - x_0\| &= \|x_1^{(3)} - x_1^{(2)}\| + \|x_1^{(2)} - x_0\| \\
&\leq M_1(2+\frac{a_1}{2})\eta_1 + (1+\frac{a_1}{2})\eta_1 + \left[\left(\sum_{j=1}^{k-2} M_0^j\right)(2+\frac{a_0}{2}) + 1 + \frac{a_0}{2}\right]\eta_0 < R\eta,
\end{aligned}$$

if $M_1(2+\frac{a_1}{2})\eta_1 + (1+\frac{a_1}{2})\eta_1 + \left[\left(\sum_{j=1}^{k-2} M_0^j\right)(2+\frac{a_0}{2}) + 1 + \frac{a_0}{2}\right]\eta_0 < R$. Then, $x_1^{(3)} \in B(x_0, R\eta)$.

Then, we can establish the following result by means an inductive procedure.

Lemma 5 *The following conditions are verified for $3 \leq j \leq k$:*

i) $\|x_1^{(j)} - x_1^{(j-1)}\| \leq M_1^{j-2}\|x_1^{(2)} - x_1^{(1)}\| \leq \|x_1^{(2)} - x_1^{(1)}\|$

ii) $\|x_1^{(j)} - x_0\| \leq \left[\left(\sum_{i=1}^{j-2} M_1^i\right)(2+\frac{a_1}{2}) + 1 + \frac{a_1}{2}\right]\eta_1 + \left[\left(\sum_{l=1}^{k-2} M_0^l\right)(2+\frac{a_0}{2}) + 1 + \frac{a_0}{2}\right]\eta_0$

iii) $\|F(x_1^{(j)})\| \leq K\eta_1(R+1+\frac{a_1}{4})\|x_1^{(j)} - x_1^{(j-1)}\|$

$\square$

As a consequence of these previous lemmas, we define the following scalar sequences:

$$\begin{aligned}
a_n &= \beta_R K\eta_n, \quad n \geq 0, \\
M_n &= a_n(R+1+\frac{a_n}{4}), \quad n \geq 0, \qquad (15) \\
\eta_n &= M_{n-1}^{k-1}(2+\frac{a_{n-1}}{2})\eta_{n-1}, \quad n \geq 1,
\end{aligned}$$

in order to set the recurrence relations for the sequences $\{x_n^{(j)}\}$, defined from the iterative process (6).

Lemma 6 *Then following conditions are verified for $n \in \mathbb{N}$:*

(i) $\|x_n^{(1)} - x_n^{(0)}\| \leq \eta_n$

(ii) $\|x_n^{(2)} - x_n^{(1)}\| \leq (2 + \frac{a_n}{2})\eta_n$

and for $3 \leq j \leq k$

(iii) $\|x_n^{(j)} - x_n^{(j-1)}\| \leq M_n^{j-2}\|x_n^{(2)} - x_n^{(1)}\| \leq \|x_n^{(2)} - x_n^{(1)}\|$

(iv) $\|x_n^{(j)} - x_0\| \leq \left[\left(\sum_{i=1}^{j-2} M_n^i\right)(2 + \frac{a_n}{2}) + 1 + \frac{a_n}{2}\right]\eta_n + \sum_{i=0}^{n-1}\left[\left(\sum_{l=1}^{k-1} M_i^l\right)(2 + \frac{a_i}{2}) + 1 + \frac{a_i}{2}\right]\eta_i$

(v) $\|F(x_n^{(j)})\| \leq K\eta_n(R + 1 + \frac{a_n}{4})\|x_n^{(j)} - x_n^{(j-1)}\|$

Moreover, $x_n^{(j)} \in B(x_0, R\eta)$, for $1 \leq j \leq k$ and $n \in \mathbb{N}$, if the following conditions are verified:

a) $\{a_n\}$, $\{M_n\}$*and* $\{\eta_n\}$ *are decreasing scalar sequences,*

b) $M_0^{k-1}(2 + \frac{a_0}{2}) < 1$,

c) $\left[\frac{M_0 - M_0^k}{1 - M_0}(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2}\right]\frac{1}{1 - M_0^{k-2}(2 + \frac{a_0}{2})} \leq R.$

Proof: For obtaining these conditions we follow an induction process, so from previous reasonings we assume that the results are true for $n-1$.

In order to prove (i), let us remember that $x_n^{(0)} = x_{n-1}^{(k)}$, then by the construction of the method (6) we have:

$$\begin{aligned}\|x_n^{(1)} - x_n^{(0)}\| &\leq \|\Gamma_n\|\|F(x_{n-1}^{(k)})\| \leq \beta_R K\eta_{n-1}(R + 1 + \frac{a_{n-1}}{4})\|x_{n-1}^{(k)} - x_{n-1}^{(k-1)}\| \\ &= M_{n-1}\|x_{n-1}^{(k)} - x_{n-1}^{(k-1)}\| \leq M_{n-1}^{k-1}\|x_{n-1}^{(2)} - x_{n-1}^{(1)}\| \\ &\leq M_n^{k-1}(2 + \frac{a_{n-1}}{2})\eta_{n-1} = \eta_n\end{aligned}$$

so the result holds for n. Now from second step and following the induction we have;

$$\begin{aligned}\|x_n^{(2)} - x_n^{(0)}\| &= \|\Gamma_n[F(x_n^{(1)}) - F(x_n)]\| \leq (1 + \|\Gamma_n\|\frac{1}{2}K\|x_n^{(1)} - x_n\|)\|x_n^{(1)} - x_n\| \\ &\leq (1 + \tilde{\beta}_R\frac{1}{2}K\eta_n)\eta_n = (1 + \frac{a_n}{2})\eta_n,\end{aligned}$$

and then,

$$\|x_n^{(2)} - x_n^{(1)}\| = \|x_n^{(2)} - x_n^{(0)}\| + \|x_n^{(1)} - x_n^{(0)}\| \leq (1 + \frac{a_n}{2})\eta_n + \eta_n \leq (2 + \frac{a_n}{2})\eta_n,$$

so (ii) holds. Items (iii)-(v) can easily be deduced, again, from previous reasonings by the induction process.

Obviously $x_n^{(1)} = x_{n-1}^{(k)}$ so, from (iv) and following the induction $x_n^{(1)} \in B(x_0, R\eta)$. For $x_n^{(2)}$, using (iv), we have:

$$\begin{aligned}
\|x_n^{(2)} - x_0\| &\leq \|x_n^{(2)} - x_n^{(0)}\| + \|x_n^{(0)} - x_0\| \leq (1 + \frac{a_n}{2})\eta_n + \|x_{n-1}^{(k)} - x_0\| \leq (1 + \frac{a_n}{2})\eta_n \\
&\quad + \left[\left(\sum_{i=1}^{k-2} M_{n-1}^i\right)(2 + \frac{a_{n-1}}{2}) + 1 + \frac{a_{n-1}}{2}\right]\eta_{n-1} \\
&\quad + \sum_{i=0}^{n-2}\left[\left(\sum_{l=1}^{k-1} M_i^l\right)(2 + \frac{a_i}{2}) + 1 + \frac{a_i}{2}\right]\eta_i
\end{aligned}$$

Notice that we can bound this inequality in the same way that the corresponding for $3 \leq j \leq k$, that is:

$$\begin{aligned}
\|x_n^{(j)} - x_0\| &\leq \left[\left(\sum_{i=1}^{j-2} M_n^i\right)(2 + \frac{a_n}{2}) + 1 + \frac{a_n}{2}\right]\eta_n + \sum_{i=0}^{n-1}\left[\left(\sum_{l=1}^{k-1} M_i^l\right)(2 + \frac{a_i}{2}) + 1 + \frac{a_i}{2}\right]\eta_i \\
&< \left[\left(\sum_{l=1}^{k-1} M_0^l\right)(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2}\right]\sum_{i=0}^{n}\eta_i \\
&\leq \left[\frac{M_0 - M_0^k}{1 - M_0}(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2}\right]\sum_{i=0}^{n}\left(M_0^{k-1}(2 + \frac{a_0}{2})\right)^i \eta_0 \\
&\leq \left[\frac{M_0 - M_0^k}{1 - M_0}(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2}\right]\frac{1}{1 - M_0^{k-1}(2 + \frac{a_0}{2})}\eta_0 \leq R\eta.
\end{aligned}$$

where we have taken into account that$M_i \leq M_0$, $a_i \leq a_0$, and $\eta_i = M_{i-1}^{k-1}(2 + \frac{a_{i-1}}{2})\eta_{i-1}$, $\forall i \in \mathbb{N}$, so it holds that $x_n^{(j)} \in B(x_0, R\eta), \forall j \in \mathbb{N}$.

□

4.2 Main result

Now we are obtaining a semilocal convergence result for iterative processes given in (6). In principle, we must demand the indicated condition in Lemma 1 and conditions a), b) and c) in Lemma 6, without forgetting the existence of R. However, to continue, we are going to see that we can change these conditions and obtain new simpler ones. First, we analyze the scalar sequences given in (15).

Lemma 7 *If condition b) in Lemma 6 is verified then, the scalar sequences $\{a_n\}, \{M_n\}$ and $\{\eta_n\}$, are decreasing.*

Proof:
First of all by condition of b) in Lemma 6, we have $\eta_1 = M_0^{k-1}(2+\frac{a_0}{2})\eta_0 \leq \eta_0$, then $a_1 \leq a_0$ and $M_1 \leq M_0$. Now, by an inductive procedure hypothesis is obtained. $\square$

In relation to the existence of R, if we observe the condition c) in Lemma 6, we can consider a value of R given by

$$R = \left[\frac{M_0 - M_0^k}{1 - M_0}(2 + \frac{a_0}{2}) + 1 + \frac{a_0}{2}\right] \frac{1}{1 - M_0^{k-1}(2 + \dfrac{a_0}{2})}, \tag{16}$$

note that both M_0 and a_0 also depend on the value R.

Then already we are able to obtain the result of semilocal convergence for given iterative processes in (6).

Theorem 8 *Let $F : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a nonlinear function defined on a non-empty open convex domain D. Suppose that conditions (C_1)–(C_2) are satisfied. If there exists a positive real value R given by (16), with $M_0^{k-1}(2+\frac{a_0}{2}) < 1$, $\beta K R\eta < 1$ and $B(x_0, R\eta) \subset D$, then the iterative process given by (6), starting at x_0, is well-defined and converges to a solution x^* de $F(x) = 0$. Moreover, the solution x^* and the iterates x_n belong to $\overline{B(x_0, R)}$ and x^* is unique in $\overline{B(x_0, R)}$.*

Proof:
The iterative process is well defined as we have proved in the precedent results.

Now for obtaining that $\{x_n\}$ is a Cauchy sequence, first, remembering that $x_{n+1} = x_n^{(k)}$, we establish the following inequality and then we use the previous lemmas :

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \sum_{i=0}^{k-1} \|x_n^{(i+1)} - x_n^{(i)}\| = \sum_{i=1}^{k-1} \|x_n^{(i+1)} - x_n^{(i)}\| + \|x_n^{(1)} - x_n^{(0)}\| \\
&\leq \sum_{i=1}^{k-1} M_n^{i-1} \|x_n^{(2)} - x_n^{(1)}\| + \|x_n^{(1)} - x_n^{(0)}\| \\
&\leq (1 + M_n + M_n^2 + \ldots + M_n^{k-2}) \|x_n^{(2)} - x_n^{(1)}\| + \|x_n^{(1)} - x_n^{(0)}\| \\
&\leq \frac{1 - M_0^{k-1}}{1 - M_0} \|x_n^{(2)} - x_n^{(1)}\| + \|x_n^{(1)} - x_n^{(0)}\| \\
&\leq \frac{1 - M_0^{k-1}}{1 - M_0}(2 + \frac{a_n}{2})\eta_n + \eta_n \leq \left[\frac{1 - M_0^{k-1}}{1 - M_0}(2 + \frac{a_0}{2}) + 1\right] \eta_n
\end{aligned}$$

and then we have, for $m \geq 1$:

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \sum_{i=1}^{m} \|x_{n+i} - x_{n+i-1}\| \leq \sum_{i=1}^{m} \left[\frac{1-M_0^k}{1-M_0}(2+\frac{a_0}{2})+1\right]\eta_{n+i} \\
&\leq \left[\frac{1-M_0^k}{1-M_0}(2+\frac{a_0}{2})+1\right]\sum_{i=1}^{m}\eta_{n+i} \\
&\leq \left[\frac{1-M_0^k}{1-M_0}(2+\frac{a_0}{2})+1\right]\sum_{i=1}^{m}\left[M_0^{k-1}(2+\frac{a_0}{2})\right]^{n+i}\eta_0
\end{aligned}$$

So, by using that $M_0^{k-1}(2+\frac{a_0}{2}) < 1$ we have that $\{x_n\}$ is a Cauchy sequence and converge to x^* in the Banach space $\mathbb{R}^m$. By setting $n = 0$ we get:

$$\|x_m - x_0\| \leq \left[\frac{1-M_0^k}{1-M_0}(2+\frac{a_0}{2})+1\right]\frac{M_0^{k-1}(2+\frac{a_0}{2}) - (M_0^{k-1}(2+\frac{a_0}{2}))^{m+1}}{1-M_0^{k-1}(2+\frac{a_0}{2})}$$

Then, if $m \to +\infty$ we have $x^* \in \overline{B(x_0, R)}$. Moreover x^* is a solution of $F(x) = 0$ since

$$\begin{aligned}
\|F'(x_n)\| &\leq \|F'(x_0)\| + \|F'(x_n) - F'(x_0)\| \\
&\leq \|F'(x_0)\| + KR\eta_0.
\end{aligned}$$

Then, $F'(x_n)$ is bounded and, using that

$$\|\Gamma_n F(x_n)\| = \|\Gamma_n\|\|F(x_{n-1}^{(k)})\| \leq M_n^{k-1}(2+\frac{a_{n-1}}{2})\eta_{n-1} = \eta_n$$

$\|\Gamma_n F(x_n)\| \to 0$ by taking $n \to +\infty$ and $\|F(x_n)\| \leq \|F^{'}(x_n)\|\|\Gamma_n F(x_n)\|$, by the continuity of F we get $F(x^*) = 0$. □

4.3 Application.

In this paper we shall consider the special case of a nonlinear conservative system described by the equation

$$\frac{d^2x(t)}{dt^2} + \phi(x(t)) = 0 \tag{17}$$

with the boundary conditions

$$x(0) = x(1) = 0, \tag{18}$$

in which the damping force is zero and there is consequently no dissipation of energy. Extensive discussions with applications to a variety of physical problems, can be found in the classical references [1] and [13].

We study the existence of a unique solution for a special case of a nonlinear conservative system described by the equation (17). In order to study the application of iterative methods (6) for the numerical solution of differential equation problems, we illustrate the theory for the case of particular second-order ordinary differential equation (17) subject to the boundary conditions (18).

Initially, we transform the problem (17)–(18) into a finite dimensional problem. For this, we approximate the second derivative by a standard numerical formula.

For the direct numerical solution of problem (17)–(18), we introduce the points $t_j = jh$, $j = 0, 1, \ldots, m+1$, where $h = \frac{1}{m+1}$ and m is an appropriate integer. A scheme is then designed for the determination of numbers x_j, it is hoped, approximate the values $x(t_j)$ of the true solution at the points t_j. A standard approximation for the second derivative at these points is

$$x_j'' \approx \frac{x_{j-1} - 2x_j + x_{j+1}}{h^2}, \quad j = 1, 2, \ldots, m.$$

A natural way to obtain such a scheme is to demand that the x_j satisfy at each interior mesh point t_j the difference equation

$$x_{j-1} - 2x_j + x_{j+1} + h^2\phi(x_j) = 0. \tag{19}$$

Since x_0 and x_{m+1} are determined by the boundary conditions, the unknowns are $x_1, x_2, \ldots, x_m$.

A further discussion is simplified by the use of matrix and vector notation. Introducing the vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \qquad v_{\mathbf{x}} = \begin{pmatrix} \phi(x_1) \\ \phi(x_2) \\ \vdots \\ \phi(x_m) \end{pmatrix}$$

and the matrix

$$A = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{pmatrix},$$

the system of equations, arising from demanding that (19) holds for $j = 1, 2, \ldots, m$, can be written compactly in the form

$$F(\mathbf{x}) \equiv A\mathbf{x} + h^2 v_{\mathbf{x}} = 0, \tag{20}$$

which is a function from $\mathbb{R}^m$ into $\mathbb{R}^m$.

If the function $\phi(x)$ is not linear in x, we cannot hope to solve system (20) by algebraic methods. Some iterative procedures must be resorted to. Then, we analyze iterative methods (6) for this purpose.

Now, we consider a particular case of (17), for example, we choose the following polynomial

$$\phi(x(s)) = 3 + x(s) + 2x(s)^2 + x(s)^3. \tag{21}$$

According to the above-mentioned, $v_{\mathbf{x}} = (3 + x_1 + 2x_1^2 + x_1^3, 3 + x_2 + 2x_2^2 + x_2^3, \ldots, 3 + x_m + 2x_m^2 + x_m^3)^T$, where $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T$. In addition, the first derivative of the function F defined in (20) is given by

$$F'(\mathbf{x}) = A + h^2 D(d_{\mathbf{x}}),$$

where $d_{\mathbf{x}} = (1 + 4x_1 + 3x_1^2, 1 + 4x_2 + 3x_2^2, \ldots, 1 + 4x_m + 3x_m^2)^T$ and $D(d_{\mathbf{x}}) = \operatorname{diag}\{1 + 4x_1 + 3x_1^2, 1 + 4x_2 + 3x_2^2, \ldots, 1 + 4x_m + 3x_m^2\}$.

Usually these systems are solved by Newton's method, however we show that the application of the proposed iterative methods (6) is more favorable to apply the classical Newton method. To do this, we consider a combination of indexes considered previously, the efficiency index and the computational efficiency index. Note that, if we consider a particular problem, we can calculate the operational cost required to evaluate F and $F^{'}$. So, we consider another measure of the efficiency of an iterative process which takes into account both the operational cost of the functional evaluations that are required and the operational cost of doing an step of the algorithm. Notice that when the operator F is known both operational costs can be computed.

Thus, we define the measure of the efficiency of an iterative process applied to an operator F given as follows

$$E(method(6), F) = (k+1)^{1/(\mu+\sigma)},$$

where the operational cost of the functional evaluations and the operational cost of doing an step of the algorithm are denoted by μ and σ, respectively. In this case the number of operations related to evaluate $F(x_n)$ and $F'(x_n)$ are $3m+1$ and $7m-1$, respectively. As each iteration of the iterative methods (6) require $(m^3 + 3km^2 + m)/3$ operations, then, we obtain:

$$E(method(6), F) = (k+1)^{\frac{3}{3(3m+1+k(7m-1))+m^3+3km^2+m}}.$$

To continue, we are going to apply the more efficient iterative process of (6) to approximate a solution of the nonlinear system (20). In order to ensure the convergence of iterative process of (6), we will apply the result of semilocal convergence studied in the previous section. So, it is necessary firstly that the operator $F^{'}$ is Lipschitz. To prove this, we will apply the Mean Value Theorem. Then, we consider

$$F''(\mathbf{x})\mathbf{y}\,\mathbf{z} = (y_1, y_2, \ldots, y_m)F''(\mathbf{x})(z_1, z_2, \ldots, z_m),$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_m)^T$ and $\mathbf{z} = (z_1, z_2, \ldots, z_m)^T$, so that

$$F''(\mathbf{x})\mathbf{y}\,\mathbf{z} = h^2\left((4+6x_1)y_1z_1, (4+6x_2)y_2z_2, \ldots, (4+6x_m)y_mz_m\right)^T.$$

As

$$\|F''(\mathbf{x})\| = \sup_{\substack{\|\mathbf{y}\|=1\\ \|\mathbf{z}\|=1}} \|F''(\mathbf{x})\mathbf{y}\,\mathbf{z}\|, \tag{22}$$

and

$$\|F''(\mathbf{x})\mathbf{y}\,\mathbf{z}\| \leq h^2 \left\| \begin{pmatrix} (4+6x_1)y_1z_1 \\ (4+6x_2)y_2z_2 \\ \vdots \\ (4+6x_m)y_mz_m \end{pmatrix} \right\| \leq h^2(4+6\|\mathbf{x}\|)\|\mathbf{y}\|\|\mathbf{z}\|,$$

we observe that $\|F''(\mathbf{x})\|$ is not bounded in general, since the function $\chi(t) = 4 + 6t$ is increasing.

To solve the last difficulty a common alternative is to locate a solution of equation (20) in a domain and look for a bound for $\|F''(\mathbf{x})\|$ there (see [4]). For this, taking into account that the solution of (17)–(18) with $\phi(x(s))$ defined in (21) is a solution of the following Fredholm integral equation (see [12])

$$x(s) = -\int_0^1 G(s,t)\phi(x(t))\,dt, \tag{23}$$

where the kernel G is the Green function in $[0,1]\times[0,1]$.

Then, we can locate a solution $x^*(s)$ in some domain since

$$\|x^*(s)\| - \frac{1}{8}\left(3 + \|x^*(s)\| + 2\|x^*(s)\|^2 + \|x^*(s)\|^3\right) \leq 0,$$

where $\frac{1}{8} = \max_{[0,1]} \int_0^1 |G(s,t)|\,dt$, so that $\|x^*(s)\| \in [0,\rho_1]\cup[\rho_2,+\infty]$, where $\rho_1 = 0.5301\ldots$ and $\rho_2 = 1.4291\ldots$ are the two positive real roots of the scalar equation $t - \frac{1}{8}(3 + t + 2t^2 + t^3) = 0$.

Now, we are going to consider the convergence of iterative process (6) to a solution $x^*(s)$ such that $\|x^*(s)\| \in [0,\rho_1]$. For this, we can consider the domain

$$\Omega = \left\{ x(s) \in \mathcal{C}^{(2)}([0,1]);\ \|x(s)\| < \frac{2}{3},\ s\in[0,1] \right\},$$

since $\rho_1 < \frac{2}{3} < \rho_2$.

In view of what the domain Ω is for equation (17), we then consider (20) with $F : \Lambda \subset \mathbb{R}^m \to \mathbb{R}^m$ and $\Lambda = \left\{\mathbf{x} \in \mathbb{R}^m;\ \|\mathbf{x}\| < \frac{2}{3}\right\}$.

If we choose $m = 24$ and the starting point $\mathbf{x_0} = (0,0,\ldots,0)^T$, we see that conditions of the semilocal convergence theorem 8 are satisfied for the optimal iterative process of (6). In first place, we obtain:

$$\|F''(\mathbf{x})\| \leq \frac{8}{25^2} = K, \quad \beta = 11.1694\ldots, \quad \eta = 0.4136\ldots \quad \text{and} \quad K\beta\eta = 0.0591\ldots.$$

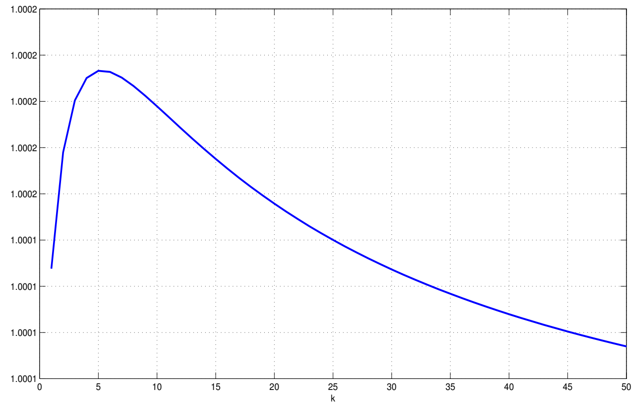

Figure 5: Efficiency of the iterative processes (6) depending on the number of steps k.

Once set m = 24, we find the most efficient iterative process (6). The Figure 5 represents the efficiency of the iterative processes (6) depending on the number of steps k. As you can be easily seen, the optimal situation appears for $k = 5$.

Then, in this situation we obtain that there exists $R = 1.5533\ldots.$ such that the conditions of Theorem 8 are verified and iterative process (6) converges to the solution $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_8^*)^T$ shown in Table 1 after 4 iterations with a tolerance 10^{-25}. Observe that $\|\mathbf{x}^*\| = 0.472604\ldots \leq \frac{2}{3}$.

In Table 2 we show the errors $\|\mathbf{x}^* - \mathbf{x_n}\|$, using the stopping criterion $\|\mathbf{x_n} - \mathbf{x_{n-1}}\| < 10^{-25}$, and the sequence $\{\|F(\mathbf{x_n})\|\}$. Notice that the vector shown in Table 1 is a good approximation of the solution of system (20)–(21) with $m = 24$, since $\|F(\mathbf{x}^*)\| \leq 2.98 \times 10^{-110}$.

i	x_i^*	i	x_i^*	i	x_i^*
1	$0.069481\ldots$	9	$0.434146\ldots$	17	$0.408742\ldots$
2	$0.134035\ldots$	10	$0.453321\ldots$	18	$0.37724\ldots$
3	$0.193514\ldots$	11	$0.466164\ldots$	19	$0.339794\ldots$
4	$0.247751\ldots$	12	$0.472604\ldots$	20	$0.296571\ldots$
5	$0.296571\ldots$	13	$0.472604\ldots$	21	$0.247751\ldots$
6	$0.339794\ldots$	14	$0.466164\ldots$	22	$0.193514\ldots$
7	$0.377240\ldots$	15	$0.453321\ldots$	23	$0.134035\ldots$
8	$0.408742\ldots$	16	$0.434146\ldots$	24	$0.0694813\ldots$

Table 1: Numerical solution $\mathbf{x}^*$ of (20) with $\phi(x)$ defined in (21).

n	$\|\mathbf{x}^* - \mathbf{x_n}\|$	$\|F(\mathbf{x_n})\|$
0	6.95×10^{-2}	2.35×10^{-2}
1	4.13×10^{-5}	1.38×10^{-5}
2	8.02×10^{-23}	2.74×10^{-23}
3	5.34×10^{-81}	1.77×10^{-80}
4	0	2.98×10^{-110}

Table 2: Absolute errors and $\|F(\mathbf{x_n})\|$.

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