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Semilocal convergence of an efficient fifth order method under weaker conditions

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Abstract The semilocal convergence of an efficient fifth order iterative method is established under weaker conditions for solving nonlinear equations. It is done by assuming omega continuity condition on second order Fréchet derivative. The novelty of our work lies in the fact that several examples are available where Lipschitz and Hölder condition fails but omega condition holds. Existence and uniqueness theorem is established along with R-order and error bounds. The R-order is found to be $4+q$, $q \in (0,1]$. Numerical experiments involving nonlinear integral equations are performed to show the applicability of the method. Finally the existence and uniqueness balls are found for all the examples.

Keywords Semilocal convergence · Lipschitz condition · Hölder condition · Hammerstein integral equation · Dynamical Systems

Mathematics Subject Classification (2000) 65G49 · 47H99

1 Introduction

Let X and Y are Banach spaces and consider solving

$$G(x) = 0$$

where $G : \Omega \subseteq X \to Y$ be a nonlinear operator in an open convex domain $\Omega_0 \subseteq \Omega$. Solution of various real life problems such as dynamical systems, boundary value problems etc. are obtained by solving these equations (see, [12,7,6,3]). The most well known quadratically convergent Newton's method to solve (1.1) is defined for $k \geq 0$, by

$$x_{k+1} = x_k - \Gamma_k G(x_k) \tag{1.1}$$

where, $\Gamma_k = G^{'}(x_k)^{-1}$ and x_0 is the initial point. Various modification of Newton's method are proposed to increase the order of convergence and efficiency. In literature [11,18,13,2,15,8,9,16], authors have established the semilocal convergence of higher order iterative methods under various continuity conditions.

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Recently, the semilocal convergence of an efficient fifth order method is established in [17] under Lipschitz condition on $F^{''}$. It is given for $k = 0, 1, 2 \dots$ by

$$\begin{aligned} y_k &= x_k - \Gamma_k G(x_k), \\ z_k &= y_k - \Gamma_k G(y_k), \\ x_{k+1} &= z_k - G^{'}(y_k)^{-1} G(z_k), \end{aligned} \tag{1.2}$$

In real life applications, various numerical examples are available which neither satisfies Lipschitz nor Hölder condition. Consider nonlinear Hammerstein type integral equation

$$x(r) + \sum_{i=1}^{m} \int_a^b K_i(r,s) S_i(x(s)) ds = f(r),\ r \in [a,b], \tag{1.3}$$

where functions f, K_i and S_i for $i = 1, 2, \dots m$ are known, the solution x is to be determined and $-\infty < a < b < +\infty$. In order to solve (1.3), we have to solve

$$G(x)(u) = x(u) + \sum_{i=1}^{m} \int_a^b K_i(u,v) S_i(x(v)) dv - f(u) \tag{1.4}$$

If $S_i^{'}(x(u))$ is (M_i, α_i)- Hölder continuous in Ω, then, under max-norm, we have

$$\|G^{''}(x) - G^{''}(y)\| \le \sum_{i=1}^{m} M_i \|x - y\|^{\alpha_i},\ M_i \ge 0,\ \alpha_i \in [0,1],\ \forall\ x, y \in \Omega. \tag{1.5}$$

For different α_i, $G^{''}$ neither satisfies Lipschitz nor Hölder condition but satisfies the weaker ω-condition. This motivate us to establish the semilocal convergence of an efficient fifth order method under weaker conditions.
The paper is organized as follows. In section 2, the semilocal convergence analysis of an efficient fifth order iterative method is established under weaker conditions for solving nonlinear equations. The existence and uniqueness theorem is established along with error bounds for the solution. The R-order is also derived. In section 3, numerical examples are solved to demonstrate the applicability of our approach. Finally, conclusions are included in section 4.

2 Semilocal convergence analysis

2.1 Preliminaries results

Let $\Gamma_0 = G^{'}(x_0)^{-1} \in BL(Y, X)$ exists at $x_0 \in \Omega$, where $BL(Y, X)$ denotes the set of bounded linear operators from Y to X and the following conditions hold.
(1) $\|\Gamma_0\| \le \beta_0$
(2) $\|\Gamma_0 G(x_0)\| \le \eta_0$
(3) $\|G^{''}(x)\| \le M$
(4) $\|G^{''}(x) - G^{''}(y)\| \le \omega(\|x - y\|), x, y \in \Omega$, for a continuous non-decreasing real function $\omega(x)$, $x > 0$, $\omega(0) \ge 0$ such that, $\omega(tx) \le t^q \omega(x)$ for $t \in [0,1]$, $x \in (0,\infty)$ and $q \in [0,1]$.
Let $r_0 = M\beta_0\eta_0$, $s_0 = \beta_0\eta_0\omega(\eta_0)$ and define sequences $\{r_k\}$, $\{s_k\}$ and $\{\eta_k\}$ for $k = 0, 1, 2 \dots$, by

$$r_{k+1} = r_k \phi(r_k)^2 \psi(r_k, s_k), \tag{2.1}$$

$$s_{k+1} = s_k \phi(r_k)^{2+q} \psi(r_k, s_k)^{1+q}, \tag{2.2}$$

$$\eta_{k+1} = \eta_k \phi(r_k) \psi(r_k, s_k), \tag{2.3}$$

where,

$$\phi(u) = \frac{1}{1-ug(u)}, \tag{2.4}$$

$$g(u) = \left(1+\frac{u}{2}+\frac{u^2}{2(1-u)}\left(1+\frac{u}{4}\right)\right), \tag{2.5}$$

and

$$\psi(u,v) = \frac{u^2}{2(1-u)}\left(1+\frac{u}{4}\right)\Bigg[\frac{v}{1+q}\left(\frac{u^{1+q}}{2^{1+q}}+\frac{1}{2+q}\left(\frac{u^2}{2(1-u)}\left(1+\frac{u}{4}\right)\right)^{1+q}\right) + \frac{u}{2}\left(u+\frac{u^2}{2(1-u)}\left(1+\frac{u}{4}\right)\right)\Bigg]. \tag{2.6}$$

Let $h(u) = g(u)\,u - 1$. Since, $h(0) = -1$ and $g(u)$ is increasing function, therefore, $h(u)$ has a real root ν. If $u \in (0,\nu)$, we get $g(u)\,u < 1$.

Lemma 1 *Let $\phi(u)$, $g(u)$ and $\psi(u,v)$ are given by* (2.4), (2.5) *and* (2.6) *respectively. If $0 < r_0 < \nu$ and $\phi(r_0)^2\psi(r_0,s_0) < 1$, then*
(i) $\phi(u)$ and $\psi(u)$ are increasing functions and $\phi(u) > 1$, $g(u) > 1$ for $u \in (0,\nu)$.
(ii) $\psi(u,v)$ is an increasing function of u, for $u \in (0,\nu)$.
(iii) $\{r_k\}, \{s_k\}$ and $\{\eta_k\}$ are decreasing sequences and $r_k g(r_k) < 1$ as well as $\phi(r_k)^2\psi(r_k,s_k) < 1$ for $k \geq 0$.

Proof The proof of (*i*) and (*ii*) are obvious. The proof of (*iii*) can be given in following manner. For $k = 0$, (2.1) gives $r_1 = r_0\phi(r_0)^2\psi(r_0,s_0) < r_0$. Using (2.2) and (2.3), we get $s_1 = s_0\phi(r_0)^{2+q}\psi(r_0,s_0)^{1+q} < s_0(\phi(r_0)^2\psi(r_0,s_0))^{1+q} < s_0$ and $\eta_1 = \phi(r_0)\psi(r_0,s_0)\eta_0 < \eta_0$. Thus, (*iii*) holds for $k = 0$. Since, $\phi(u)$ and $g(u)$ are the increasing functions, and therefore by using mathematical induction Lemma 1 holds $\forall\, k \geq 0$.

Lemma 2 *Let $\phi(u)$ and $\psi(u,v)$ are given by* (2.4) *and* (2.6)*, respectively. If $\gamma \in (0,1)$ then $\phi(\gamma t) < \gamma\phi(t)$ and $\psi(\gamma u, \gamma^{1+q}v) < \gamma^{3+q}\psi(u,v)$.*

Proof The proof is trivial.

Lemma 3 *Let $\gamma = \phi(r_0)^2\psi(r_0,s_0)$, $0 < r_0 < \nu$ and $\delta = \frac{1}{\phi(r_0)}$. Then,*
(i) $r_k \leq \gamma^{(4+q)^{k-1}} r_{k-1} \leq \gamma^{\frac{(4+q)^k-1}{3+q}} r_0$ and $s_k \leq \left(\gamma^{(4+q)^{k-1}}\right)^{1+q} s_{k-1} \leq \left(\gamma^{\frac{(4+q)^k-1}{3+q}}\right)^{1+q} s_0$.
(ii) $\phi(r_k)\psi(r_k,s_k) \leq \dfrac{\gamma^{(4+q)^k}}{\phi(r_0)}\ \forall\ k \in N$.
(iii) $\eta_k \leq \gamma^{\frac{(4+q)^k-1}{3+q}}\delta^k\eta_0$.

Proof Using $k = 0$ in (2.1) and (2.2), we get $r_1 = r_0\phi(r_0)^2\psi(r_0,s_0) = \gamma r_0$ and $s_1 = s_0\phi(r_0)^{2+q}$ $\psi(r_0,s_0)^{1+q} \leq \gamma^{1+q}s_0$. Thus Lemma, holds for $k = 0$. Suppose Lemma holds for $k = n$. Using induction, we will prove for $k = n+1$. Then, we have

$$\begin{aligned} r_{n+1} &= r_n\phi(r_n)^2\psi(r_n, s_n), \\ &\le \gamma^{(4+q)^{n-1}} r_{n-1}\phi\left(\gamma^{(4+q)^{n-1}} r_{n-1}\right)^2 \psi\left(\gamma^{(4+q)^{n-1}} r_{n-1}, (\gamma^{(4+q)^{n-1}})^{1+q} s_{n-1}\right), \\ &\le \gamma^{(4+q)^{n-1}} r_{n-1}\phi\left(r_{n-1}\right)^2 \left(\gamma^{(4+q)^{n-1}}\right)^{3+q} \psi(r_{n-1}, s_{n-1}), \\ &\le \left(\gamma^{(4+q)^{n-1}}\right)^{(4+q)} r_{n-1}\phi(r_{n-1})^2\psi(r_{n-1}, s_{n-1}), \\ &\le \gamma^{(4+q)^n} r_n. \end{aligned} \tag{2.7}$$

In a similar manner, we get

$$\begin{aligned} r_{n+1} &\le \gamma^{(4+q)^n} r_n \le \gamma^{(4+q)^n}\gamma^{(4+q)^{n-1}} r_{n-1} \\ &\le \cdots \le \gamma^{(4+q)^n}\gamma^{(4+q)^{n-1}} \ldots \gamma^{(4+q)^0} r_0 = \gamma^{\frac{(4+q)^{n+1}-1}{3+q}} r_0. \end{aligned} \tag{2.8}$$

Consider

$$\begin{aligned} s_{n+1} &= s_n\phi\left(r_n\right)^{(2+q)} \psi(r_n, s_n)^{1+q} \le s_n \left(\phi(r_n)^2\psi\left(r_n, s_n\right)\right)^{1+q} \\ &\le s_n\left(\frac{r_{n+1}}{r_n}\right)^{1+q} \le \left(\gamma^{(4+q)^n}\right)^{1+q} s_n \end{aligned}$$

Proceeding in this way, we get

$$\begin{aligned} s_{n+1} &\le \left(\gamma^{(4+q)^n}\right)^{1+q} s_n \le \left(\gamma^{(4+q)^n}\right)^{1+q}\left(\gamma^{(4+q)^{n-1}}\right)^{1+q} s_{n-1} \\ &\le \left(\gamma^{\frac{(4+q)^{n+1}-1}{3+q}}\right)^{1+q} s_0. \end{aligned} \tag{2.9}$$

Hence, (i) holds $\forall\ k \ge 0$ by using mathematical induction. Now, consider

$$\begin{aligned} \phi(r_k)\psi(r_k, s_k) &\le \phi(\gamma^{\frac{(4+q)^k-1}{3+q}} r_0)\psi\left(\gamma^{\frac{(4+q)^k-1}{3+q}} r_0, (\gamma^{\frac{(4+q)^k-1}{3+q}})^{1+q} s_0\right), \\ &\le \gamma^{(4+q)^k-1}\phi(r_0)\psi\left(r_0, s_0\right) = \gamma^{(4+q)^k}\delta. \end{aligned} \tag{2.10}$$

Thus (ii) is proved. From (2.3), we get

$$\begin{aligned} \eta_k &= \phi(r_{k-1})\psi(r_{k-1}, s_{k-1})\eta_{k-1} \le \prod_{n=0}^{k-1}\phi(r_n)\psi(r_n, s_n)\eta_0, \\ &\le \prod_{n=0}^{k-1}\frac{\gamma^{(4+q)^n}}{\phi(r_0)}\eta_0 \le \gamma^{\frac{(4+q)^k-1}{1+q}}\delta^k\eta_0. \end{aligned} \tag{2.11}$$

Thus, (iii) is proved.

2.2 **Recurrence relations**

In this section, we establish the recurrence relations for (1.2) under the assumption considered in the previous section. Consider

$$\|I - \Gamma_0 G^{'}(y_0)\| \leq \|\Gamma_0\| \ \|G^{'}(y_0) - G^{'}(x_0)\| \leq M\beta_0\eta_0 = r_0 \tag{2.12}$$

If $r_0 < 1$, then

$$\|G^{'}(y_0)^{-1} G^{'}(x_0)\| \leq \frac{1}{1 - r_0}. \tag{2.13}$$

Substitute $k = 0$ in (1.2), we get

$$z_0 - x_0 = -\Gamma_0 G(x_0) - \Gamma_0 G(y_0). \tag{2.14}$$

By using Taylor expansion of $G(y_0)$ about x_0, we get

$$\begin{aligned} G(y_0) &= G(x_0) + G^{'}(x_0)(y_0 - x_0) + \int_0^1 G^{''}(x_0 + \theta(y_0 - x_0))(y_0 - x_0)^2(1 - \theta)d\theta \\ &= \int_0^1 G^{''}(x_0 + \theta(y_0 - x_0))(y_0 - x_0)^2 \theta d\theta \end{aligned} \tag{2.15}$$

Using (2.15) in (2.14) and taking norm, we get

$$\begin{aligned} \|z_0 - x_0\| &= \|\Gamma_0 G(x_0)\| + \|\Gamma\| \frac{M}{2} \|y_0 - x_0\|^2 \\ &\leq \left(1 + \frac{r_0}{2}\right) \eta_0. \end{aligned} \tag{2.16}$$

Now,

$$\|z_0 - y_0\| = \|\Gamma_0 G(y_0)\| \leq \frac{r_0}{2} \eta_0.$$

Substitute $k = 0$ in (1.2) and taking norm, we get

$$\begin{aligned} \|x_1 - z_0\| &\leq \|G^{'}(y_0)^{-1} G(z_0)\| \leq \|G^{'}(y_0)^{-1} G^{'}(x_0)\| \|\Gamma_0 G(z_0)\|, \\ &\leq \frac{r_0^2}{2(1 - r_0)} \left(1 + \frac{r_0}{4}\right) \eta_0 \end{aligned} \tag{2.17}$$

Therefore,

$$\begin{aligned} \|x_1 - x_0\| &\leq \|x_1 - z_0\| + \|z_0 - x_0\|, \\ &\leq \left(1 + \frac{r_0}{2} + \frac{r_0^2}{2(1 - r_0)} \left(1 + \frac{r_0}{4}\right)\right) \eta_0 = g(r_0)\eta_0. \end{aligned} \tag{2.18}$$

Consider

$$\begin{aligned} \|I - \Gamma_0 G^{'}(x_1)\| &\leq \|\Gamma_0\| \|G^{'}(x_1) - G^{'}(x_0)\| \leq \beta_0 M \|x_1 - x_0\| \\ &\leq M\beta_0\eta_0 g(r_0) = r_0 g(r_0) < 1 \end{aligned}$$

Therefore, by Banach Lemma, we get

$$\|\Gamma_1\| \leq \frac{\|\Gamma_0\|}{1 - r_0 g(r_0)} = \|\Gamma_0\| \phi(r_0). \tag{2.19}$$

Also,

$$\|\Gamma_0\|\|y_0 - x_0\|\omega\left(\|y_0 - x_0\|\right) \leq \beta_0\eta_0\omega(\eta_0) = s_0$$

. Using Taylor expansion of $G(x_1)$ about z_0, we get

$$\begin{aligned} G(x_1) = &\int_0^1 \left(G^{''}(y_0 + t(z_0 - y_0)) - G^{''}(y_0)\right)(z_0 - y_0)(x_1 - z_0)dt + G^{''}(y_0)(z_0 - y_0)(x_1 - z_0) \\ &+ \int_0^1 G^{''}(z_0 + t(x_1 - z_0))(x_1 - z_0)^2(1 - t)dt + \frac{1}{2}G^{''}(z_0)\left(x_1 - z_0\right)^2 \end{aligned} \tag{2.20}$$

Therefore,

$$\begin{aligned} \|\Gamma_1 G(x_1)\| &\leq \phi(r_0)\|\Gamma_0\|\|G(x_1)\|, \\ &\leq \phi(r_0)\frac{{r_0}^2}{2(1 - r_0)}\left(1 + \frac{r_0}{4}\right)\left[\frac{s_0}{1+q}\left(\frac{{r_0}^{1+q}}{2^{1+q}} + \frac{1}{2+q}\left(\frac{{r_0}^2}{2(1-r_0)}\left(1 + \frac{r_0}{4}\right)\right)^{1+q}\right)\right. \\ &\left. + \frac{r_0}{2}\left(r_0 + \frac{{r_0}^2}{2(1-r_0)}\left(1 + \frac{r_0}{4}\right)\right)\right] \\ &= \phi(r_0)\psi(r_0, s_0)\eta_0 = \eta_1 \end{aligned} \tag{2.21}$$

Using (2.21), we get

$$\begin{aligned} M\|\Gamma_1\|\|\Gamma_1 G(x_1)\| &\leq M\phi(r_0)\|\Gamma_0\|\phi(r_0)\psi(r_0, s_0)\eta_0, \\ &\leq r_0\phi(r_0)^2\psi(r_0, s_0) = r_1 \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} \|\Gamma_1\|\|\Gamma_1 G(x_1)\|\omega\left(\|\Gamma_1 G(x_1)\|\right) &\leq \beta_0\eta_0\omega(\eta_0)\phi(r_0)^{1+q}\psi\left(r_0, s_0\right)^{1+q} \\ &\leq s_0\phi(r_0)^{2+q}\psi\left(r_0, s_0\right)^{1+q} = s_1. \end{aligned} \tag{2.23}$$

The following recurrence relations are proved for $k \geq 1$ using mathematical induction.
(I) $\|\Gamma_k\| \leq \phi(r_{k-1})\|\Gamma_{k-1}\|$,
(II) $\|\Gamma_k G(x_k)\| \leq \phi(r_{k-1})\psi(r_{k-1}, s_{k-1})\eta_{k-1}$,
(III) $M\|\Gamma_k\|\|\Gamma_k G(x_k)\| \leq r_k$,
(IV) $\|\Gamma_k\|\|\Gamma_k G(x_k)\|\omega\left(\|\Gamma_k G(x_k)\|\right) \leq s_k$,
(V) $\|x_k - x_{k-1}\| \leq g(r_{k-1})\eta_{k-1}$,
Hence, for $k = 1$, the recurrence relations (I)-(IV) follow from (2.19), (2.21), (2.22) and (2.23) respectively. The recurrence relation (V) is already established for $k = 1$ in (2.18). Using mathematical induction, these recurrence relations hold for all $k \geq 1$.

2.3 **Convergence Theorem**

Theorem 1 *Let* $r_0 = M\beta_0\eta_0 < \nu$, $s_0 = \beta_0\eta_0\omega(\eta_0)$ *and assumptions (1)-(4) hold. Then for* $\overline{B}(x_0, R\eta_0) \subseteq \Omega$, *where* $R = \dfrac{g(r_0)}{1 - \delta\gamma}$, *the sequence* $\{x_k\}$ *generated by* (1.2) *converges to the solution of* (1.1). *Moreover,* $y_k, z_k, x_{k+1}, x^* \in \overline{B}(x_0, R\eta_0)$ *and* x^* *is the unique solution in* $B\left(x_0, \frac{2}{L_1\beta_0} - R\eta_0\right) \cap \Omega$. *The error bound for iterates is given as follows.*

$$\|x_k - x^*\| \leq g(r_0)\delta^k \frac{\gamma^{\frac{(4+q)^k - 1}{3+q}}}{1 - \delta\gamma^{(4+q)^k}}\eta_0.$$

Proof To prove this theorem, we have to show that $\{x_k\}$ is a Cauchy sequence. Using (V), we get

$$\|x_{k+1}-x_k\| \le g(r_k)\eta_k \le g(r_0)\eta_k \le g(r_0)\prod_{j=0}^{k-1}\phi(r_j)\psi(r_j,s_j)\eta_0 \tag{2.24}$$

Consider

$$\begin{aligned}\|x_{k+m}-x_k\| &\le \|x_{k+m}-x_{k+m-1}\| + \|x_{k+m-1}-x_{k+m-2}\| + \ldots + \|x_{k+1}-x_k\| \\ &\le g(r_0)\prod_{j=0}^{k+m-2}\phi(r_j)\psi(r_j,s_j)\eta_0 + g(r_0)\prod_{j=0}^{k+m-1}\phi(r_j)\psi(r_j,s_j)\eta_0 + \ldots \\ &+ g(r_0)\prod_{j=0}^{k-1}\phi(r_j)\psi(r_j,s_j)\eta_0 \\ &\le g(r_0)\sum_{l=0}^{m-1}\left(\prod_{j=0}^{k+l-1}\phi(r_j)\psi(r_j,s_j)\eta_0\right)\end{aligned} \tag{2.25}$$

Using Lemma 3(iii), we get

$$\begin{aligned}\|x_{k+m}-x_k\| &\le g(r_0)\sum_{l=0}^{m-1}\delta^{k+l}\left(\gamma^{\frac{(4+q)^{k+l}-1}{3+q}}\right)\eta_0 \\ &\le g(r_0)\delta^k\left(\gamma^{\frac{(4+q)^k-1}{3+q}}\right)\sum_{l=0}^{m-1}\left(\delta\gamma^{(4+q)^k}\right)^l \\ &\le g(r_0)\delta^k\left(\gamma^{\frac{(4+q)^k-1}{3+q}}\right)\frac{1-(\delta\gamma^{(4+q)^k})^m}{1-\delta\gamma^{(4+q)^k}}\eta_0.\end{aligned} \tag{2.26}$$

Hence, $\{x_k\}$ is a Cauchy sequence and hence converges to x^* as $k \to \infty$. Taking $m \to \infty$ in (2.26), we get

$$\|x_k - x^*\| \le g(r_0)\delta^k\gamma^{\frac{(4+q)^k-1}{3+q}}\frac{1}{1-\delta\gamma^{(4+q)^k}}\eta_0. \tag{2.27}$$

Taking $k=0$ in (2.27), we get

$$\|x^*-x_0\| \le \frac{g(r_0)}{1-\delta\gamma}\eta_0 \le R\eta_0. \tag{2.28}$$

Hence, $x^* \in \overline{B}(x_0, R\eta_0)$. Now,

$$\|x_{k+1}-x_0\| \le \sum_{i=0}^{k}\|x_{i+1}-x_i\| \le \sum_{i=0}^{k}g(r_i)\eta_i \le g(r_0)\sum_{i=0}^{k}\eta_i \le R\eta_0,$$

and

$$\|y_k-x_0\| \le \|y_k-x_k\| + \|x_k-x_0\| \le \eta_k + \sum_{i=0}^{k-1}g(r_i)\eta_i \le g(r_0)\sum_{i=0}^{k}\eta_i \le R\eta_0.$$

Using (2.16), we get

$$\|z_k - x_0\| \leq \|z_k - x_k\| + \|x_k - x_0\| \leq \left(1 + \frac{r_0}{2}\right)\eta_k + \sum_{i=0}^{k-1} g(r_i)\eta_i \leq g(r_0)\sum_{i=0}^{k} \eta_i \leq R\eta_0.$$

Hence, $y_k, z_k, x_{k+1} \in \overline{B}(x_0, R\eta_0)$.

To prove the uniqueness part, let $z^* \in B\left(x_0, \frac{2}{M\beta} - R\eta_0\right) \cap \Omega$ be an another solution such that $G(z^*) = 0$, $z^* \neq x^*$. Then $0 = G(z^*) - G(x^*) = \int_0^1 G^{'}(x^* + t(z^* - x^*))dt(z^* - x^*) = P(z^* - x^*)$, where, $P = \displaystyle\int_0^1 G^{'}(x^* + t(z^* - x^*))dt$. Now,

$$\begin{aligned}
\|I - \Gamma_0 P\| &\leq \|\Gamma_0\| \int_0^1 \left\| \left(G^{'}(x^* + t(z^* - x^*)) - G^{'}(x_0) \right) \right\| dt \\
&\leq M\beta \int_0^1 \|(1-t)(x^* - x_0) + t(z^* - x_0)\| dt \\
&\leq \frac{M\beta}{2} \left(\|x^* - x_0\| + \|z^* - x_0\|\right) \\
&\leq \frac{M\beta}{2} \left(R\eta_0 + \frac{2}{M\beta} - R\eta_0 \right) \\
&= 1
\end{aligned}$$

Therefore, $\|I - \Gamma_0 P\| < 1$. Thus, by Banach Lemma P^{-1} exists and hence $z^* = x^*$.

3 Numerical examples

In this section, different numerical examples are worked out to demonstrate the efficiency of our approach.

Example 1 *Consider nonlinear integral equation*

$$G(x)(s) = x(s) - 1 + \int_0^1 H(s,t)\left(\frac{3}{5}x(t)^{7/3} + \frac{6}{15}x(t)^3\right)dt, \tag{3.29}$$

where, $s \in [0,1]$, $x \in \Omega = B(0,2) \subset X$.

Clearly,

$$\|G^{''}(x) - G^{''}(y)\| \leq \frac{7}{30}\|x - y\|^{1/3} + \frac{3}{10}\|x - y\|.$$

where, $\omega(\mu) = \frac{7}{30}\mu^{1/3} + \frac{3}{10}\mu$ and $q = \frac{1}{3}$. Therefore, neither Lipschitz nor Hölder condition holds but ω-condition holds. Taking $x_0(t) = 1$, all the assumptions are satisfied. Therefore the existence and uniqueness balls for integral equation is given by $\overline{B}(x_0, 0.21621)$ and $B(x_0, 1.2939)$ respectively. The values of the sequences $\{r_k\}$, $\{s_k\}$ and $\{\eta_k\}$ are given in Table 1.

Table 1: The values of r_k, s_k and η_k

k	r_k	s_k	η_k
0	0.24527	5.1729×10^{-2}	0.18519
1	7.7009×10^{-4}	2.126×10^{-5}	4.1532×10^{-4}
2	6.8009×10^{-17}	8.359×10^{-23}	3.665×10^{-17}
3	3.6373×10^{-82}	7.8181×10^{-110}	1.9601×10^{-82}
4	1.5916×10^{-408}	5.5956×10^{-545}	8.5771×10^{-409}
5	2.5535×10^{-2040}	1.0509×10^{-2720}	1.3761×10^{-2040}

The error bounds for (1.2) are given in Table 2.

Table 2: Error bounds for (1.2)

k	$\|x_k - x^*\|$
0	4.8381×10^{-4}
1	4.9181×10^{-15}
2	3.4921×10^{-62}
3	5.2372×10^{-266}
4	9.3095×10^{-1149}
5	1.6043×10^{-4973}

Example 2 *Consider nonlinear integral equation*

$$G(x)(s) = x(s) - f(s) - \lambda \int_0^1 \frac{s}{s+t} x(t)^{2+q} dt, \tag{3.30}$$

where, $s \in [0,1]$*,* $x, f \in C[0,1]$*,* $\lambda \in \mathbb{R}$*.*

Clearly,

$$\|G''(x) - G''(y)\| \leq |\lambda| \log 2(1+q)(2+q)\|x-y\|^q.$$

Here $\omega(\eta) = |\lambda| \log 2(1+q)(2+q)\eta^q$ Clearly, Lipschitz condition fails for $q \in (0,1)$ but Hölder condition holds. Taking $\lambda = \frac{1}{4}$, $f(s) = 1$, $x_0 = x_0(s) = 1$, and $q = \frac{1}{5}$, all the assumptions are satisfied. Therefore the existence and uniqueness balls for integral equation is given by $\overline{B}(x_0, 0.3174)$ and $B(x_0, 2.3879)$ respectively. The values of $\{r_k\}$, $\{s_k\}$ and $\{\eta_k\}$ are given in Table 3.

Table 3: The values of r_k, s_k and η_k

k	c_k	d_k	η_k
0	0.20704	0.16052	0.28005
1	3.4155×10^{-4}	6.9698×10^{-5}	3.5373×10^{-4}
2	1.1985×10^{-18}	3.1433×10^{-22}	1.2408×10^{-18}
3	6.1808×10^{-91}	5.6532×10^{-109}	6.399×10^{-91}
4	2.255×10^{-452}	1.0637×10^{-542}	2.3347×10^{-452}
5	1.4579×10^{-2259}	2.509×10^{-2711}	1.5094×10^{-2259}

The error bounds for (1.2) are given in Table 4.

Table 4: Error bounds for (1.2)

k	$\|x_k - x^*\|$
0	4.004×10^{-4}
1	6.3045×10^{-16}
2	3.9701×10^{-65}
3	2.1201×10^{-271}
4	2.2551×10^{-1137}
5	4.3337×10^{-4774}

4 Conclusions

Using recurrence relations, semilocal convergence of an efficient fifth order iterative method is presented under weaker conditions for solving nonlinear equations. The convergence theorem is established along with error bounds. Different examples involving nonlinear integral equations are solved to show the applicablity of the approach. Existence and uniqueness balls are obtained for the considered examples.

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