Operators with the specification property*

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Abstract

We study a version of the specification property for linear dynamics. Operators having the specification property are investigated, and relationships with other well known dynamical notions such as mixing, Devaney chaos, and frequent hypercyclicity are obtained.

1 Introduction

A continuous map on a metric space is said to be chaotic in the sense of Devaney if it is topologically transitive and the set of periodic points is dense. Although there is no common agreement about what a chaotic map is, a notion of chaos stronger than Devaney's definition is the so called specification property. It was first introduced by Bowen [16] and since then, several kinds and degrees of this property have been stated [38]. We follow the definitions and terminology used in [5]. Some recent works on the specification property are [34, 35, 31, 21, 26, 30, 4, 3].

Definition 1 ([16])**.** A continuous map $f : X \to X$ on a compact metric space (X, d) has the *specification property* (SP) if for any $\delta > 0$ there is a positive integer N_δ such that for any integer $s \geq 2$, any set $\{y_1, \ldots, y_s\} \subset X$ and any integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ satisfying $j_{r+1} - k_r \geq N_\delta$ for $r = 1, \ldots, s-1$, there is a point $x \in X$ such that, for each positive integer $r \leq s$ and any integer i with $j_r \leq i \leq k_r$, the following conditions hold:

$$d(f^i(x), f^i(y_r)) < \delta,$$
$$f^n(x) = x, \text{ where } n = N_\delta + k_s.$$

Although there are weaker versions of this property, we will be using the above version, which is in fact, the strongest one. Compact dynamical systems with the SP are mixing and Devaney chaotic, among other basic dynamical properties (see, e.g., [20]).

Devaney chaos and mixing properties have been widely studied for linear operators on Banach and more general spaces [10, 12, 18, 22, 23, 24, 36]. The recent books [9] and [25] contain the basic theory, examples, and many results on chaotic linear dynamics.

Our aim is to study the SP in the context of continuous linear operators defined on separable F-spaces. In this situation, the first crucial problem is that these spaces are never compact, therefore, our first task should be the adjusting of this property to the new context. We recall that an F-space is a topological vector space whose topology is induced by a complete translation-invariant metric. In fact, if X is an F-space, there exists a complete translation-invariant metric d such that $\|x\| = d(x, 0)$ is an F-norm.

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Definition 2. Let X be a vector space. A map $\|\cdot\|$ from X to $\mathbb{R}^+$ is an *F-norm* provided for each $x, y \in X$ and $\lambda \in \mathbb{K}$ we have

1. $\|x+y\| \leq \|x\| + \|y\|$;
2. $\|\lambda x\| \leq \|x\|$ if $|\lambda| < 1$;
3. $\lim_{\lambda \to 0} \|\lambda x\| = 0$;
4. $\|x\| = 0$ implies $x = 0$.

A consequence from 1 and 2 above is that for any $x \in X$, and $\lambda \in \mathbb{K}$, we have

$$\|\lambda x\| \leq (|\lambda| + 1)\|x\|.$$

The class of F-spaces includes complete metrizable locally convex spaces (i.e., Fréchet spaces) and hence it also includes Banach spaces. For introductory texts on functional analysis that cover Fréchet spaces we refer to Rudin [37] and Meise and Vogt [32]. The notion of an F-norm can be found in Kalton, Peck and Roberts [29].

From now on the space X will a separable (infinite dimensional) F-space with F-norm $\|.\|$, and $T : X \to X$ will be a continuous linear operator (operator for short).

The following definition can be considered the natural extension of the SP in this setting.

Definition 3. An operator $T : X \to X$ on a separable F-space X has the *operator specification property* (OSP) if there exists an increasing sequence $(K_m)_m$ of T-invariant sets with $0 \in K_1$ and $\overline{\cup_{m \in \mathbb{N}} K_m} = X$ such that for each $m \in \mathbb{N}$ the map $T|_{K_m}$ has the SP, that is, for any $\delta > 0$ there is a positive integer $N_{\delta,m}$ such that for every $s \geq 2$, any set $\{y_1, \ldots, y_s\} \subset K_m$ and any integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ with $j_{r+1} - k_r \geq N_{\delta,m}$ for $1 \leq r \leq s-1$, there is a point $x \in K_m$ such that, for each positive integer $r \leq s$ and any integer i with $j_r \leq i \leq k_r$, the following conditions hold:

$$\|T^i(x) - T^i(y_r)\| < \delta,$$
$$T^n(x) = x, \text{ where } n = N_{\delta,m} + k_s.$$

Observation 4. We would like to point out that although we removed compactness of each K_m from our definition, it is hard to think of a map having the specification property outside of the compact setting; in other words, for all cases we know of operators having the OSP, the required sets K_m are always compact.

Its is natural to study any property in linear dynamics for the most typical operators in this context, namely the weighted shifts on sequence spaces. By a *sequence space* we mean a topological vector space X which is continuously included in ω, the countable product of the scalar field $\mathbb{K}$. A *sequence F-space* is a sequence space that is also an F-space. Given a sequence $w = (w_n)_n$ of positive weights, the associated *unilateral weighted backward shift* $B_w : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is defined by $B_w(x_1, x_2, \ldots) = (w_2 x_2, w_3 x_3, \ldots)$. If a sequence F-space X is invariant under certain weighted backward shift T, then T is also continuous on X by the closed graph theorem. In [4] we characterize when backward shift operators defined on certain Banach sequence spaces exhibit the OSP. We were able to extend these characterizations to the more general setting of sequence F-spaces in [3] where the next result is proved.

Theorem 5 ([3])**.** Let $B_w : X \to X$ be a unilateral weighted backward shift on a sequence F-space X in which $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis. Then the following conditions are equivalent:

(i) B_w is chaotic;

(ii) the series
$$\sum_{n=1}^{\infty}\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n$$
converges in X;

(iii) B_w has a nontrivial periodic point;

(iv) B_w has the OSP.

The paper is organised as follows: in Section 2 we study the basic properties for operators with the OSP. In Section 3 we show the connections of the OSP with other dynamical properties for linear operators like mixing, chaos in the sense of Devaney and frequent hypercyclicity. Section 4 provides several examples of operators with the OSP. In the final Section 5 we present the conclusions and a diagram containing the implications between the different dynamical properties discussed here.

2 Basic properties

We first show that the OSP behaves well by quasi-conjugation.

Proposition 6. Suppose $T_i : X_i \to X_i$ is an operator on a separable F-space X_i, $i = 1, 2$, and $\phi : X_1 \to X_2$ is a uniformly continuous map with dense range such that the diagram

$$\begin{tikzcd} X_1 \arrow[r, "T_1"] \arrow[d, "\phi"'] & X_1 \arrow[d, "\phi"'] \\ X_2 \arrow[r, "T_2"] & X_2 \end{tikzcd}$$

conmutes. If T_1 has the OSP then so does T_2.

Proof. Without loss of generality, we may assume that $\phi(0) = 0$, otherwise take $\tilde{\phi} := \phi - \phi(0)$. Since T_1 has the OSP, let $(K^1_m)_m$ the required sequence of T_1-invariant sets satisfying all the conditions given in Definition 3. Set $(K^2_m)_m := (\phi(K^1_m))_m$. Clearly $0 \in K^2_1$ and, since ϕ has dense range we have that $\overline{\cup_{m\in\mathbb{N}} K^2_m} = X_2$. The map ϕ is uniformly continuous on each K^1_m, therefore, fixed $\delta > 0$, there exists $\delta' > 0$ such that for each $x, y \in K^1_m$ with $\|x - y\| < \delta'$, we have $\|\phi(x) - \phi(y)\| < \delta$. Since $T_1|_{K^1_m}$ has the SP, there exits $N_{\delta',m}$. Taking now $N_{\delta,m} := N_{\delta',m}$, using the commutativity of the diagram, and the uniform continuity of ϕ, it is routine to see that T_2 has the OSP. □

Remark 7. Since uniform continuity and continuity are equivalent for linear transformations on F-spaces, we have that Proposition 6 is true when $\phi : X_1 \to X_2$ is a linear continuous transformation with dense range. Even more, if ϕ is a linear homeomorphism, then T_1 has the OSP if and only if T_2 does it.

Our next result shows that each iterate of an operator having the OSP inherits that property. This is a natural question in discrete dynamics, and the most important result in this direction in the linear setting was due to Ansari [1] who proved that T^n is hypercyclic whether T is. We recall that for continuous maps on separable complete metric spaces, topological transitivity is equivalent to the existence of a dense orbit, and this concept is known as hypercyclicity in our context (see [25]).

Proposition 8. If $T : X \to X$ has the OSP, then so does T^k for every $k \in \mathbb{N}$.

Proof. To show that T^k has the OSP, take the same sequence $(K_m)_m$ of T-invariant sets, which are obviously T^k-invariant. Since $T|_{K_m}$ has the SP, given $\delta > 0$, there exists $N_{\delta,m}$ satisfying all the requirements of Definition 1 and, taking a greater index if necessary, we may assume that $N_{\delta,m}$ is a multiple of k. Now, the positive integer $N_{\delta,m}/k$ would do the job to show that $T^k|_{K_m}$ has the SP. □

Next, we study how the OSP behaves by direct sums of operators. The motivation for this question in our linear setting comes from an old problem of Herrero [27]: He asked whether $T \oplus T$ is hypercyclic whenever T is. This problem turned out to be equivalent to the question whether every hypercyclic operator satisfies the so called Hypercyclicity Criterion. The negative answer was found by de la Rosa and Read [19]. In contrast, the OSP is inherited by taking direct sums.

Proposition 9. Suppose $T_i : X_i \to X_i$ is an operator on a separable F-space X_i, $1 \le i \le n$. If T_i has the OSP for $1 \le i \le n$, then $\oplus_{i=1}^n T_i : \oplus_{i=1}^n X_i \to \oplus_{i=1}^n X_i$ has the OSP.

Proof. It is enough to do the proof for the case of two operators. We recall that there are several equivalent F-norms on $X_1 \oplus X_2$, we use here the F-norm

$$\|(x_1, x_2)\|_{X_1 \otimes X_2} := \|x_1\|_{X_1} + \|x_2\|_{X_2}, \; (x_1, x_2) \in X_1 \oplus X_2,$$

where $\|.\|_{X_i}$ is the corresponding F-norm on X_i. To make notation simpler we will avoid to specify the underlying space.

Take $(K_m)_m := (K_m^1 \times K_m^2)_m$, where $(K_m^i)_m$ is the corresponding sequence of T_i-invariant sets required by Definition 3, $i = 1, 2$. To see that $(T_1 \oplus T_2)|_{K_m}$ has the SP, given $\delta > 0$, since $T_i|_{K_m}$ has the SP, $i = 1, 2$, there exists $N^i_{\delta/2,m}$ and we take $N_{\delta,m} := \max\{N^1_{\delta/2,m}, N^2_{\delta/2,m}\}$. Now, for every $s \ge 2$, any set $\{(y_1^1, y_1^2), \dots, (y_s^1, y_s^2)\} \subset K_m$ and any integers $0 = j_1 \le k_1 < j_2 \le k_2 < \cdots < j_s \le k_s$ with $j_{r+1} - k_r \ge N_{\delta,m}$ for $1 \le r \le s-1$, there are $x^1 \in K_m^1$ and $x^2 \in K_m^2$ such that, for each positive integer $r \le s$ and any integer i with $j_r \le i \le k_r$, the following conditions hold:

$$\begin{aligned}
&\|T_1^i(x^1) - T_1^i(y_r^1)\| < \delta/2, \\
&\|T_2^i(x^2) - T_2^i(y_r^2)\| < \delta/2, \\
&\left.\begin{aligned} T_1^n(x_1) = x_1 \\ T_2^n(x_2) = x_2 \end{aligned}\right\}, \; n = N_{\delta,m} + k_s.
\end{aligned}$$

Now, it is easy to check that

$$\begin{gathered}
\|(T_1 \oplus T_2)^i(x^1, x^2) - (T_1 \oplus T_2)^i(y_r^1, y_r^2)\| < \delta, \; 1 \le r \le s, \; j_r \le i \le k_r, \\
(T_1 \oplus T_2)^n(x^1, x^2) = (x^1, x^2), \; n = N_{\delta,m} + k_s,
\end{gathered}$$

which completes the proof. □

We finish this section with two additional properties. They may appear somehow artificial and technical but they do play a crucial role in the proof of one of the main results of the next section.

Proposition 10. Let $T : X \to X$ be an operator on a separable F-space X.

i) If $\lambda_i \in \mathbb{K}$ and $K_i \subset X$ is a T-invariant set such that $T|_{K_i}$ has the SP, $1 \le i \le k$, then $T|_{\sum_{i=1}^k \lambda_i K_i}$ has the SP.

ii) If X is locally convex and $K \subset X$ is a T-invariant set such that $T|_K$ has the SP and $\overline{\mathrm{co}(K)}$ is the closed convex envelope of K, then $T|_{\overline{\mathrm{co}(K)}}$ has the SP.

Proof. i) It suffices to prove the case of two sets. Given $\delta > 0$, take $\delta' := \delta/(|\lambda_1|+|\lambda_2|+2)$. There exists $N^i_{\delta'}$, $i = 1, 2$, and we take $N_\delta := \max\{N^1_{\delta'}, N^2_{\delta'}\}$. Now, for every $s \ge 2$, any set $\{(\lambda_1 y^1_1 + \lambda_2 y^2_1), \dots, (\lambda_1 y^1_s + \lambda_2 y^2_s)\} \subset \lambda_1 K_1 + \lambda_2 K_2$ and any integers $0 = j_1 \le k_1 < j_2 \le k_2 < \cdots < j_s \le k_s$ with $j_{r+1} - k_r \ge N_\delta$ for $1 \le r \le s-1$, there are $x^1 \in K_1$ and $x^2 \in K_2$ such that for each integer $r \in [1, s]$ and for any integer i with $j_r \le i \le k_r$, the following conditions hold:

$$\begin{aligned}
&\|T^i(x^1) - T^i(y^1_r)\| < \delta', \\
&\|T^i(x^2) - T^i(y^2_r)\| < \delta', \\
&\left.\begin{aligned} T^n(x^1) = x^1 \\ T^n(x^2) = x^2 \end{aligned}\right\}, \quad n = N_\delta + k_s.
\end{aligned}$$

Now, it is easy to check that

$$\begin{gathered}
\|T^i(\lambda_1 x^1 + \lambda_2 x^2) - T^i(\lambda_1 y^1_r + \lambda_2 y^2_r)\| < \delta, \ 1 \le r \le s, \ j_r \le i \le k_r, \\
T^n(\lambda_1 x^1 + \lambda_2 x^2) = \lambda_1 x^1 + \lambda_2 x^2, \ n = N_\delta + k_s.
\end{gathered}$$

ii) By the continuity of T and all its iterates, it is clear that if $T|_K$ has the SP, then $T|_{\overline{K}}$ has the SP. Therefore, it remains to show that $T|_{\mathrm{co}(K)}$ has the SP. Since X is a Fréchet space, we can fix an increasing sequence of seminorms $(\|\cdot\|_n)_n$ that generate the topology of X. The key point to prove this is to observe that if you have two (or more points) belonging to the convex hull of a set, you can always rewrite the convex combinations in such a way that their length and coefficients are the same. This fact may appear strange at first because one usually thinks about 'minimal' convex combinations but the fact is clear if we decompose terms of the convex combination in several terms 'as needed'. For example

$$\begin{aligned}
x &= 0.9x_1 + 0.1x_2 &&= 0.5x_1 + 0.3x_1 + 0.1x_1 + 0.1x_2 \\
y &= 0.5y_1 + 0.3y_2 + 0.2y_3 &&= 0.5y_1 + 0.3y_2 + 0.1y_3 + 0.1y_3
\end{aligned}$$

Using the above fact to express the points $\{y_1, \dots, y_s\} \subset \mathrm{co}(K)$ as convex combinations with the same length and coefficients, the fact that $T|_{\mathrm{co}(K)}$ has the SP can be obtained as in one i), by computing the corresponding inequalities for an arbitrary seminorm $\|\cdot\|_n$, $n \in \mathbb{N}$. $\square$

3 Connections with other dynamical properties

In this section we focus in the connections of the OSP with other well know dynamical properties. To be precise, we prove that operators with the OSP are mixing, chaotic in the sense of Devaney, and they have a strong version of hypercyclicity, introduced by Bayart and Grivaux [6, 7], called frequent hypercyclicity. For completeness we recall that an operator $T : X \to X$ is topologically transitive if for any non-empty open sets U and V, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Moreover, if the set $\{n \in \mathbb{N} : T^n(U) \cap V \neq \emptyset\}$ is cofinite, then T is mixing.

Proposition 11. If $T : X \to X$ has the OSP, then T is mixing.

Proof. Fix a non-empty open set U and a 0-neighbourhood W. We recall that the return set from U to W is defined as $N(U, W) := \{n \in \mathbb{N} : T^n(U) \cap W \neq \emptyset\}$. We claim that $N(U, W)$ and $N(W, U)$ are cofinite and this implies T is mixing (see [25, Proposition 2.37]).

Take $u \in U$ and $\delta > 0$ such that $B(u, 2\delta) \subset U$ and $B(0, 2\delta) \subset W$. Since T has the OSP, we may find a set K such that $T|_K$ has the SP and $K \cap B(u, \delta) \neq \emptyset$. There exists N_δ (depending on K and δ).

Take $y_1 \in K \cap B(u, \delta)$, $y_2 = 0$, $m \in \mathbb{N}$, $0 = j_1 = k_1 < j_2 = N_\delta < k_2 = N_\delta + m$. Since $j_2 - k_1 \geq N_\delta$, there exists $x \in K$ such that

$$\begin{aligned} \|T^i x - T^i y_1\| &< \delta, \ i = j_1, \ldots, k_1 \\ \|T^i x - T^i y_2\| &< \delta, \ i = j_2, \ldots, k_2. \end{aligned}$$

This implies $\|x - y_1\| < \delta$, so $\|x - u\| < 2\delta$ and hence $x \in U$. As $T^i y_2 = 0$, we have that $T^i x \in B(0, \delta)$ for $i = N_\delta, \ldots, N_\delta + m$, therefore $T^{N_\delta + m} x \in B(0, \delta) \subset W$. We have proved that $N_\delta + m \in N(U, W)$ for any $m \in \mathbb{N}$.

Take now $i = N_\delta$. Clearly $\|T^{N_\delta} x\| < \delta$, hence $T^{N_\delta} x \in B(0, \delta) \subset W$. Observing that x is periodic with period $N_\delta + k_2$, we have

$$T^{N_\delta + m}(T^{N_\delta} x) = T^{N_\delta + k_2} x = x \in U,$$

which means that $N_\delta + m \in N(W, U)$ for any $m \in \mathbb{N}$. This finishes the proof □

Proposition 12. If $T : X \to X$ has the OSP, then T is chaotic in the sense of Devaney, that is, T is topologically transitive and it admits a dense set of periodic points.

Proof. Clearly, T is transitive by the above proposition. By the mere definition of the specification property, it is also clear that any point in the space may be approximated by periodic points. □

Given $A \subset \mathbb{N}$, its lower density is defined by

$$\underline{\mathrm{dens}}(A) = \liminf_{n \to \infty} \frac{\mathrm{card}(A \cap [1, n])}{n},$$

An operator T is *frequently hypercyclic* if there exists $x \in X$ such that for every non-empty open subset U of X, the set $\{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density (see [7]). Frequent hypercyclicity is a way to measure the frequency of hitting times in an arbitrary non-empty open set for a dense orbit.

Theorem 13. If $T : X \to X$ has the OSP, then T is frequently hypercyclic.

Proof. Let $(K_m)_m$ be an increasing sequence of T-invariant sets associated with the OSP for T. Fix $v_m \in K_m$, $m \in \mathbb{N}$, such that $\overline{\{v_m; m \in \mathbb{N}\}} = X$. We set inductively $\widetilde{K}_1 = K_1$, $\widetilde{K}_m = K_m - \sum_{i=1}^{m-1} \widetilde{K}_i$, $m > 1$. We know that T satisfies the OSP with respect to $(\widetilde{K}_m)_m$. Let $(r_m)_m$ be an increasing sequence in $]0, 1[$ with

$$\lim_m \prod_{i=1}^{m} r_i > 0,$$

and fix $(p_m)_m \subset \mathbb{N}$ such that $(p_m - 1)/(p_m + 1) > r_m$, $m \in \mathbb{N}$. We apply the OSP with respect to $(\widetilde{K}_m)_m$. For $\delta_m := 2^{-m}$ we denote $N_m = N_{\delta_m,m}$, $m \in \mathbb{N}$. W.l.o.g., $(p_m + 1)N_m$ divides N_{m+1}, $m \in \mathbb{N}$. Given $m \in \mathbb{N}$ we set $j_1 = k_1 = 0$, $j_2 = N_m$, $k_2 = p_m N_m$. For $m = 1$ let $y_1 = v_1$ and, inductively, given $m > 1$ suppose we have $x_i \in \widetilde{K}_i$, $i = 1, \dots, m - 1$. Let

$$\begin{aligned} y_1 &= u_m := v_m - \sum_{i=1}^{m-1} T^{N_i} x_i \in \widetilde{K}_m, \\ y_2 &= 0. \end{aligned}$$

By assumption, there is $x_m \in \widetilde{K}_m$ such that

$$\|x_m - u_m\| < 2^{-m}, \quad \|T^i x_m\| < 2^{-m}, \quad i = N_m, \dots, p_m N_m, \quad \text{and}$$

$$T^n x_m = x_m \text{ for } n = (p_m + 1)N_m.$$

We will show that the vector $x := \sum_k T^{N_k} x_k$ is frequently hypercyclic for T.

Let $q_k := (p_k + 1)N_k$, $k \in \mathbb{N}$. Given $m > 1$, we have

$$\|T^{p_m N_m + j q_m} \left(\sum_{k=1}^{m} T^{N_k} x_k \right) - v_m\| = \| \left(\sum_{k=1}^{m-1} T^{N_k} x_k + u_m \right) - v_m\| < \frac{1}{2^m}, \tag{1}$$

for all $j \in \mathbb{N}_0$.

Fix $n > q_{m+1}$. There exists $m' > m$ such that $(p_{m'} - 1)N_{m'} < n \le (p_{m'+1} - 1)N_{m'+1}$. Since

$$\|T^j (T^{N_k} x_k)\| < \frac{1}{2^k}, \quad \forall k \ge m' + 1, \quad j = 0, \dots, (p_{m'+1} - 1)N_{m'+1},$$

we get,

$$\|T^j \left(\sum_{k \ge m'+1} T^{N_k} x_k \right)\| < \frac{1}{2^{m'}}, \quad j = 0, \dots, n. \tag{2}$$

It remains to show the inequalities $\|T^j (T^{N_k} x_k)\| < 2^{-k}$ for $m < k \le m'$, and for certain $j \le n$ of the form $j = p_m N_m + j' q_m$, $j' \in \mathbb{N}_0$. To do this, we have to count the number of elements of this form contained in suitable blocks of consecutive integers. Indeed, for each $i \in \mathbb{N}_0$, the block of integers $\{iN_{m+1} + j \ ; \ j = 0, \dots, N_{m+1} - 1\}$ contains N_{m+1}/q_m elements of the form $p_m N_m + j q_m$, $j \in \mathbb{N}_0$. Hence the block $\{i q_{m+1} + j \ ; \ j = 0, \dots, (p_{m+1} - 1)N_{m+1}\}$ contains $(p_{m+1} - 1)N_{m+1}/q_m$ elements of the form $p_m N_m + j q_m$, $j \in \mathbb{N}_0$.

Analogously, for each $i \in \mathbb{N}_0$, the block $\{iN_{k+1} + j \ ; \ j = 0, \dots, N_{k+1} - 1\}$ contains N_{k+1}/q_k blocks of the form $\{i' q_k + j \ ; \ j = 0, \dots, (p_k - 1)N_k\}$, $i' \in \mathbb{N}_0$. Hence the block $\{i q_{k+1} + j \ ; \ j = 0, \dots, (p_{k+1} - 1)N_{k+1}\}$ contains $(p_{k+1} - 1)N_{k+1}/q_k$ blocks of the form $\{i' q_k + j \ ; \ j = 0, \dots, (p_k - 1)N_k\}$, $i' \in \mathbb{N}_0$.

We assumed $n > (p_{m'}-1)N_{m'}$, then there is $k \geq 1$ such that $(p_{m'}-1)N_{m'}+(k-1)q_{m'} \leq n < (p_{m'}-1)N_{m'} + kq_{m'}$. This implies that the set of integers $A_n := \{i \in \mathbb{N}_0 \ ; \ i \leq n\}$ contains k blocks of the form $\{i'q_{m'} + j \ ; \ j = 0, \ldots, (p_{m'}-1)N_{m'}\}$, $i' \in \mathbb{N}_0$. Thus, the above considerations yield that A_n contains

$$k\left(\frac{(p_{m'}-1)N_{m'}}{q_{m'-1}}\right)\cdots\left(\frac{(p_{m+1}-1)N_{m+1}}{q_m}\right) > k(p_{m'}-1)N_{m'}\frac{\alpha_m}{q_m}$$

elements of the form $j = p_m N_m + j'q_m$, $j' \in \mathbb{N}_0$, such that $\|T^j(T^{N_l}x_l)\| < 2^{-l}$, for all $l \in \{m+1, \ldots, m'\}$, where

$$\alpha_m := \prod_{l>m} \frac{p_l - 1}{p_l + 1} > \prod_{l>m} r_l > 0.$$

Therefore,

$$\left|\{j \leq n \ ; \ \|T^j(\sum_{k>m} T^{N_k}x_k)\| < \frac{1}{2^m}\}\right| > \left(\frac{k}{k+1}\right)\left(\frac{p_{m'}-1}{p_{m'}+1}\right) n\beta_m \geq n\frac{\beta_m}{4}, \tag{3}$$

where $\beta_m := \alpha_m / q_m$.

From (1) and (3), we conclude

$$\underline{\mathrm{dens}}\{j \in \mathbb{N} \ ; \ \|T^j x - v_m\| < \frac{1}{2^{m-1}}\} \geq \frac{\beta_m}{4} > 0,$$

which finishes the proof. $\square$

The most usual way to prove that an operator is (frequently) hypercyclic is to use the so called (Frequent) Hypercyclicity Criterion (see [9, 25]). It should be noted that the proof of Theorem 13 does not use the Frequent Hypercyclicity Criterion, instead a frequent hypercyclicity vector is constructed. Next result shows that the Frequent Hypercyclicity Criterion is far stronger than any of the dynamical properties we have been working with in this paper. In particular, it implies the OSP. We will take the general version of the Frequent Hypercyclicity Criterion given in [14].

Theorem 14. Let $T : X \to X$ be an operator on a separable F-space X. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X$ such that, for each $x \in X_0$,

(i) $\sum_{n=0}^{\infty} T^n x$ converges unconditionally,

(ii) $\sum_{n=0}^{\infty} S_n x$ converges unconditionally, and

(iii) $T^n S_n x = x$ and $T^m S_n x = S_{n-m} x$ if $n > m$,

then the operator T has the OSP.

Proof. We suppose that $X_0 = \{x_n \ ; \ n \in \mathbb{N}\}$ with $x_1 = 0$ and $S_n 0 = 0$ for all $n \in \mathbb{N}$. Let $(U_n)_n$ be a basis of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$. By (i) and (ii), there exists an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\sum_{k>N_n} T^k x_{m_k} \in U_{n+1} \text{ and } \sum_{k>N_n} S_k x_{m_k} \in U_{n+1}, \text{ if } m_k \in \{1, \ldots, n\}, \text{ for each } n \in \mathbb{N}. \tag{4}$$

We let $B_m = \{1, \dots, m\}$ and define the map $\Phi : \bigcup_{m=1}^{\infty} B_m^{\mathbb{Z}} \to X$ given by

$$\Phi((n_k)_{k\in\mathbb{Z}}) = \sum_{k<0} S_{-k} x_{n_k} + x_{n_0} + \sum_{k>0} T^k x_{n_k}.$$

The map Φ is well-defined and $\Phi|_{B_m^{\mathbb{Z}}}$ is continuous for each $m \in \mathbb{N}$ by (4). We have that $K_m := \Phi(B_m^{\mathbb{Z}})$ is a compact subset of X, invariant under the operator T, and such that $T|_{K_m}$ is conjugated to $\sigma^{-1}|_{B_m^{\mathbb{Z}}}$ via Φ, for each $m \in \mathbb{N}$; where σ is the usual Bernoulli shift defined as $\sigma(\dots, n_{-1}, n_0, n_1, \dots) = (\dots, n_0, n_1, n_2, \dots)$. Since $\sigma^{-1}|_{B_m^{\mathbb{Z}}}$ has the SP (see for instance [38]), by conjugacy, we obtain that $T|_{K_m}$ satisfies the SP too, for every $m \in \mathbb{N}$ and, since $\bigcup_{m\in\mathbb{N}} K_m$ is dense in X because it contains X_0, we conclude that T has the OSP. □

4 Families of operators with the OSP

We already noticed in Theorem 5 that weighted backward shifts on sequence F-spaces having the OSP can be characterized in terms of the weight sequence, therefore examples of backward shifts with the OSP defined on the Banach spaces ℓ^p and c_0 are easy to find (see [4]). Also, any weighted shift so that every weight is non-zero on ω has the OSP. Theorem 14 is very useful to find more examples of operators having the OSP.

Example 15. We consider the Fréchet space $H(\mathbb{C})$ of entire functions endowed with the topology of uniform convergence on compact sets. Suppose that $T : H(\mathbb{C}) \to H(\mathbb{C})$, $T \neq \lambda I$, is an operator that commutes with the operator of differentiation D, that is, $TD = DT$. It is known that T satisfies the Frequent Hypercyclicity Criterion [13], so T has the OSP.

Example 16. Let φ be a nonconstant bounded holomorphic function on the unit disc $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, and let M_φ^* be the corresponding adjoint multiplication operator on the Hardy space H^2. Godefroy and Shapiro proved that M_φ^* is hypercyclic if and only if $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, where $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ (see [22]). They even proved that condition $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ is equivalent to M_φ^* being chaotic and also equivalent to M_φ^* being mixing. Later, Bayart and Grivaux improved this result showing that if $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, then M_φ^* satisfies the Frequent Hypercyclicity Criterion [7]. Taking into account Theorem 14 we have the following characterization.

Theorem 17. Let φ be a nonconstant bounded holomorphic function on $\mathbb{D}$ and let M_φ^* be the corresponding adjoint multiplier on H^2. Then the following assertions are equivalent: (i) M_φ^* is hypercyclic; (ii) M_φ^* is mixing; (iii) M_φ^* is chaotic; (iv) M_φ^* is frequently hypercyclic; (v) M_φ^* has the OSP; (vi) $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

Example 18. Let φ be an automorphism of the unit disk $\mathbb{D}$ and let $C_\varphi f = f \circ \varphi$ be the corresponding composition operator on the Hardy space H^2. Bourdon and Shapiro proved that C_φ is hypercyclic if and only if C_φ is mixing if and only if φ has no fixed point in $\mathbb{D}$ (see [15]). Hosokawa [28] proved that C_φ is chaotic whenever it is hypercyclic and his proof shows that in fact C_φ satisfies the Frequent Hypercyclicity Criterion whenever C_φ has no fixed point (see also [39]). Therefore we have the following characterization.

Theorem 19. Let $C_\varphi \in \mathrm{Aut}(\mathbb{D})$ and C_φ be the corresponding composition operator on H^2. Then the following assertions are equivalent: (i) C_φ is hypercyclic; (ii) C_φ is mixing; (iii) C_φ is chaotic; (iv) C_φ is frequently hypercyclic; (v) C_φ has the OSP; (vi) φ has no fixed point in $\mathbb{D}$.

Example 20. Let $\Omega \subset \mathbb{C}$ be a simply connected domain, and let $\varphi : \Omega \to \Omega$ be a holomorphic function. Bès (see Theorem 1 in [11]) characterized several dynamical properties for the composition operator C_φ on $H(\Omega)$, which included when C_ϕ satisfies the Frequent Hypercyclicity Criterion. As a consequence we obtain the following result.

Theorem 21. Then the following assertions are equivalent: (i) C_φ is hypercyclic; (ii) $P(C_\varphi)$ has the OSP for every non-constant polynomial P; (iii) φ is univalent and has no fixed point in Ω.

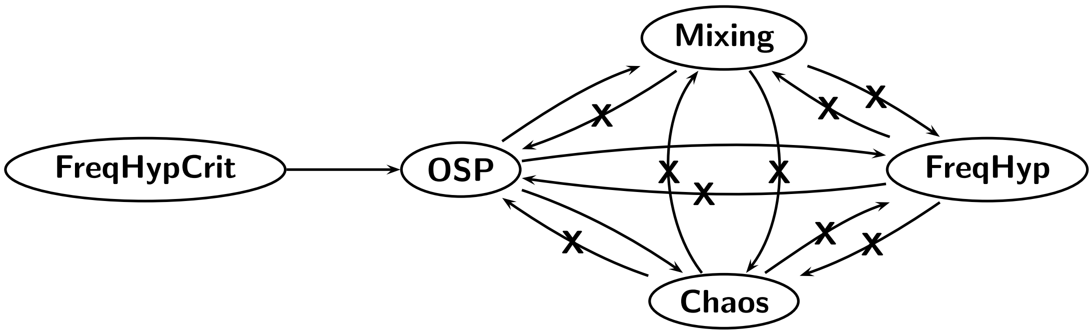
5 Concluding remarks

Theorems 11, 12, and 13 show that the OSP is in fact a strong dynamical property. Next we prove that neither converse of those theorems are true. Even more, we will show that there are operators defined on the Hilbert space ℓ^2 which are mixing, chaotic and frequently hypercyclic altogether but not having the OSP. To this aim we need a result from [17] concerning sets of periods of maps. We recall that $n \in \mathbb{N}$ is a period of T if there is $x \in X$ such that $T^n x = x$ but $T^i x \neq x$ for $0 < i < n$. The set of periods of T is defined as $\{n : n \text{ is a period of } T\}$.

Theorem 22 ([17])**.** A nonvoid subset $A \subset \mathbb{N}$ is the set of periods for certain bounded operator T on a separable complex Hilbert space H if and only if A contains $\mathrm{lcm}(a, b)$ for every $a, b \in A$. Moreover, if A is infinite, then it is possible to find a mixing, chaotic and frequently hypercyclic bounded operator T on H whose set of periods is exactly A.

Now, as set of periods take the powers of 2, that is, $A = \{2^i,\ i \in \mathbb{N}\}$. To fix ideas set the complex Hilbert space ℓ^2. Obviously A contains the least common multiple of any pair of elements of A so, by Theorem 22, there exists a mixing, chaotic and frequently hypercyclic operator $T : \ell^2 \to \ell^2$ whose periods are only the powers of 2. The operator T cannot have OSP because for any operator having this property there is a positive integer N such that any integer greater than N is a period.

At this point, relationships between OSP, mixing, chaos, frequent hypercyclicity and the Frequent Hypercyclicity Criterion are shown in the next figure.



To complete the figure, we would like to mention here the sources for the counter examples: Mixing operators which are not chaotic are easy to find; Bayart and Grivaux [8] constructed a weighted shift on c_0 that is frequently hypercyclic, but neither chaotic nor mixing; Badea and Grivaux [2] found operators on a Hilbert space that are frequently hypercyclic and chaotic but not mixing. Also, Bayart and Grivaux [7] provided easy examples of topologically mixing operators that are not frequently hypercyclic. Very recently, Menet constructed examples of chaotic operators which are not frequently hypercyclic in [33], which solved an important problem in linear dynamics.

References

[1] S. I. Ansari, *Hypercyclic and cyclic vectors*, J. Funct. Anal. **128** (1995), no. 2, 374–383.

[2] C. Badea and S. Grivaux, *Unimodular eigenvalues, uniformly distributed sequences and linear dynamics*, Adv. Math. **211** (2007), no. 2, 766–793.

[3] S. Bartoll, F. Martínez-Giménez, M. Murillo-Arcila, and A. Peris, *Cantor sets, Bernoulli shifts and linear dynamics*, Descriptive topology and functional analysis, Springer Proc. Math. Stat., vol. 80, Springer, Cham, 2014, pp. 195–207.

[4] S. Bartoll, F. Martínez-Giménez, and A. Peris, *The specification property for backward shifts*, J. Difference Equ. Appl. **18** (2012), no. 4, 599–605.

[5] W. Bauer and K. Sigmund, *Topological dynamics of transformations induced on the space of probability measures*, Monatsh. Math. **79** (1975), 81–92.

[6] F. Bayart and S. Grivaux, *Hypercyclicité: le rôle du spectre ponctuel unimodulaire*, C. R. Math. Acad. Sci. Paris **338** (2004), no. 9, 703–708.

[7] F. Bayart and S. Grivaux, *Frequently hypercyclic operators*, Trans. Amer. Math. Soc. **358** (2006), no. 11, 5083–5117 (electronic).

[8] F. Bayart and S. Grivaux, *Invariant Gaussian measures for operators on Banach spaces and linear dynamics*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 1, 181–210.

[9] F. Bayart and É. Matheron, *Dynamics of linear operators*, Cambridge Tracts in Mathematics, vol. 179, Cambridge University Press, Cambridge, 2009.

[10] T. Bermúdez, A. Bonilla, J. A. Conejero, and A. Peris, *Hypercyclic, topologically mixing and chaotic semigroups on Banach spaces*, Studia Math. **170** (2005), no. 1, 57–75.

[11] J. Bès, *Dynamics of composition operators with holomorphic symbol*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **107** (2013), 437–449.

[12] J. Bonet, F. Martínez-Giménez, and A. Peris, *A Banach space which admits no chaotic operator*, Bull. London Math. Soc. **33** (2001), no. 2, 196–198.

[13] A. Bonilla and K.-G. Grosse-Erdmann, *On a theorem of Godefroy and Shapiro*, Integral Equations Operator Theory **56** (2006), 151–162.

[14] A. Bonilla and K.-G. Grosse-Erdmann, *Frequently hypercyclic operators and vectors*, Ergodic Theory Dynam. Systems **27** (2007), no. 2, 383–404.

[15] P. S. Bourdon and J. H. Shapiro, *Cyclic phenomena for composition operators*, Mem. Amer. Math. Soc. **125** (1997), no. 596, x+105.

[16] R. Bowen, *Topological entropy and axiom* A, Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 23–41.

[17] J. Conejero, F. Martínez-Giménez, and A. Peris, *Sets of periods for chaotic linear operators*, preprint 2015.

[18] G. Costakis and M. Sambarino, *Topologically mixing hypercyclic operators*, Proc. Amer. Math. Soc. **132** (2004), no. 2, 385–389 (electronic).

[19] M. de la Rosa and C. Read, *A hypercyclic operator whose direct sum $T \oplus T$ is not hypercyclic*, J. Operator Theory **61** (2009), no. 2, 369–380.

[20] M. Denker, C. Grillenberger, and K. Sigmund, *Ergodic theory on compact spaces*, Lecture Notes in Mathematics, Vol. 527, Springer-Verlag, Berlin, 1976.

[21] M. Foryś, P. Oprocha, and P. Wilczyński, *Factor maps and invariant distributional chaos*, J. Differential Equations **256** (2014), no. 2, 475–502.

[22] G. Godefroy and J. H. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. **98** (1991), no. 2, 229–269.

[23] S. Grivaux, *Hypercyclic operators, mixing operators, and the bounded steps problem*, J. Operator Theory **54** (2005), no. 1, 147–168.

[24] K.-G. Grosse-Erdmann and A. Peris, *Weakly mixing operators on topological vector spaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **104** (2010), no. 2, 413–426.

[25] K.-G. Grosse-Erdmann and A. Peris Manguillot, *Linear chaos*, Universitext, Springer, London, 2011.

[26] G. Harańczyk, D. Kwietniak, and P. Oprocha, *Topological structure and entropy of mixing graph maps*, Ergodic Theory Dynam. Systems **34** (2014), no. 5, 1587–1614.

[27] D. A. Herrero, *Hypercyclic operators and chaos*, J. Operator Theory **28** (1992), no. 1, 93–103.

[28] T. Hosokawa, *Chaotic behavior of composition operators on the Hardy space*, Acta Sci. Math. (Szeged) **69** (2003), no. 3-4, 801–811.

[29] N. J. Kalton, N. T. Peck, and J. W. Roberts, *An F-space sampler*, London Mathematical Society Lecture Note Series, vol. 89, Cambridge University Press, Cambridge, 1984.

[30] D. Kwietniak and P. Oprocha, *A note on the average shadowing property for expansive maps*, Topology Appl. **159** (2012), no. 1, 19–27.

[31] M. Lampart and P. Oprocha, *Shift spaces, ω-chaos and specification property*, Topology Appl. **156** (2009), no. 18, 2979–2985.

[32] R. Meise and D. Vogt, *Introduction to functional analysis*, Oxford Graduate Texts in Mathematics, vol. 2, The Clarendon Press Oxford University Press, New York, 1997, Translated from the German by M. S. Ramanujan and revised by the authors.

[33] Q. Menet, *Linear chaos and frequent hypercyclicity*, to appear in Trans. Amer. Math. Soc.

[34] P. Oprocha, *Specification properties and dense distributional chaos*, Discrete Contin. Dyn. Syst. **17** (2007), no. 4, 821–833.

[35] P. Oprocha and M. Štefánková, *Specification property and distributional chaos almost everywhere*, Proc. Amer. Math. Soc. **136** (2008), no. 11, 3931–3940.

[36] A. Peris and L. Saldivia, *Syndetically hypercyclic operators*, Integral Equations Operator Theory **51** (2005), no. 2, 275–281.

[37] W. Rudin, *Functional analysis*, second ed., International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[38] K. Sigmund, *On dynamical systems with the specification property*, Trans. Amer. Math. Soc. **190** (1974), 285–299.

[39] M. Taniguchi, *Chaotic composition operators on the classical holomorphic spaces*, Complex Var. Theory Appl. **49** (2004), no. 7-9, 529–538.