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Additional Information

The number of compatible totally bounded quasi-uniformities

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Abstract

We prove that a topological space that admits a nontransitive totally bounded quasi-uniformity, admits at least $2^{2^{\aleph_0}}$ nontransitive totally bounded quasi-uniformities. Furthermore we show that each infinite T_2-space admits at least $2^{2^{\aleph_0}}$ transitive totally bounded quasi-uniformities. In fact, each supersober space having a discrete subspace of infinite cardinality κ admits at least $2^{2^{\kappa}}$ transitive totally bounded quasi-uniformities.

1 Introduction

Losonczi [13] established that a topological space that admits more than one quasi-uniformity possesses at least $2^{2^{\aleph_0}}$ compatible (transitive) quasi-uniformities. He also proved that the reals equipped with their usual topology admit exactly $2^{2^{\aleph_0}}$ totally bounded quasi-uniformities. Subsequently Künzi [9] proved that each quasi-proximity class that contains a quasi-uniformity which is not totally bounded possesses at least $2^{2^{\aleph_0}}$ members. He also observed that there is a topological space that admits exactly two quasi-proximities (equivalently, totally bounded quasi-uniformities) [8].

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AMS (1991) Subject Classifications: 54E15, 54A25

In this note we prove that a topological space that admits a nontransitive totally bounded quasi-uniformity, admits at least $2^{2^{\aleph_0}}$ nontransitive totally bounded quasi-uniformities. Furthermore we show that each infinite T_2-space admits at least $2^{2^{\aleph_0}}$ transitive totally bounded quasi-uniformities. In fact, we establish that a supersober space having a discrete subspace of infinite cardinality κ admits at least $2^{2^{\kappa}}$ transitive totally bounded quasi-uniformities. Our results are illustrated by various examples.

We refer the reader to the monograph [1] written by Fletcher and Lindgren for the basic facts about quasi-uniform spaces.

2 Nontransitive totally bounded quasi-uniformities

We are going to prove first that if a topological space admits a nontransitive totally bounded quasi-uniformity, then it possesses at least $2^{2^{\aleph_0}}$ compatible nontransitive totally bounded quasi-uniformities. The concept of the handy relation was defined by Künzi in [7]. He also characterized it as the intersection of all compatible strong inclusions of a topological space [7, Lemma 2].

Lemma 1 *Let X be a topological space admitting a nontransitive totally bounded quasi-uniformity $\mathcal{V}$. Then X possesses at least $2^{\aleph_0}$ open sets that are not (relatively) handy in themselves.*

Proof. Let $<$ be the strong inclusion induced by $\mathcal{V}$ on X. Note first, that since $\mathcal{V}$ is totally bounded and nontransitive, there are $A, B \subseteq X$ such that $A < B$, but there is no open set G in X such that $G < G$ and $A \subseteq G \subseteq B$:

Otherwise we would have $\mathcal{V} = \text{fil}\{[(X \setminus A) \times X] \cup [X \times B] : A, B \subseteq X$ and $A < B\} \subseteq \text{fil}\{[(X \setminus G) \times X] \cup [X \times G] : G < G, G \text{ open in } X\} \subseteq \mathcal{V}$ by [1, Theorem 1.33] and thus $\mathcal{V}$ would be transitive —a contradiction.

We then find a sequence $(W_n)_{n \in \omega}$ of entourages of $\mathcal{V}$ such that $W_0(A) \subseteq B$ and $W_{n+1}^3 \subseteq W_n$ whenever $n \in \omega$. According to [1, Lemma 1.5] there is a quasi-pseudometric $d : X \times X \to \mathbf{R}^+$ such that $W_{n+1} \subseteq B_{2^{-n}} \subseteq W_n$ whenever $n \in \omega$. (Here, as usual, for each $\epsilon > 0$ we have put $B_\epsilon = \{(x, y) \in X \times X : d(x, y) < \epsilon\}$.)

Next, for each $\epsilon \in]0, 1[$, set $G_\epsilon = B_\epsilon(A)$. Observe that $A \subseteq G_\epsilon \subseteq B_1(A) \subseteq W_0(A) \subseteq B$ whenever $\epsilon \in]0, 1[$.

Fix $\epsilon \in]0,1[$. Note that G_ϵ is open in X, since the topology $\tau(d)$ is coarser than the topology of X. Suppose that there are $\epsilon_1, \epsilon_2 \in]0,1[$ such that $\epsilon_1 < \epsilon_2$ and $G_{\epsilon_1} = G_{\epsilon_2}$. Then $B_{\epsilon_2-\epsilon_1}(G_{\epsilon_2}) = B_{\epsilon_2-\epsilon_1}(G_{\epsilon_1}) \subseteq B_{\epsilon_2}(A) = G_{\epsilon_2}$, and hence $G_{\epsilon_2} < G_{\epsilon_2}$, since $B_\epsilon \in \mathcal{V}$ whenever $\epsilon > 0$ —a contradiction.

We conclude that $\{G_\epsilon : \epsilon \in]0,1[\}$ is a strictly increasing chain of open sets of X such that $G_\epsilon < G_{\epsilon'}$ whenever $\epsilon, \epsilon' \in]0,1[$ and $\epsilon < \epsilon'$; furthermore by our assumption there is no open set G in X such that $G < G$ and $G_\epsilon \subseteq G \subseteq G_{\epsilon'}$.

In particular, if some G_ϵ were handy in itself, then we would have $G_\epsilon < G_\epsilon$ according to [7, Lemma 2] —a contradiction. Thus no G_ϵ is handy in itself.

Proposition 1 *A topological space X that admits a nontransitive totally bounded quasi-uniformity $\mathcal{V}$ possesses at least $2^{2^{\aleph_0}}$ compatible nontransitive totally bounded quasi-uniformities finer than $\mathcal{V}$.*

Proof. We shall use the notation explained in the argument above. So let $\mathcal{V}$ be a compatible nontransitive totally bounded quasi-uniformity on X and let $\{G_\epsilon : \epsilon \in]0,1[\}$ be the strictly increasing chain of open sets of X constructed in the proof of the preceding lemma.

For each $\epsilon \in]0,1[\setminus[\frac{1}{3},\frac{2}{3}]$ set $V_\epsilon = [(X \setminus G_\epsilon) \times X] \cup [X \times G_\epsilon]$. For an arbitrary subset A of $]0,1[\setminus[\frac{1}{3},\frac{2}{3}]$ let $\mathcal{V}_A = \mathrm{fil}(\mathcal{V} \cup \{V_a : a \in A\})$. Then, clearly, each $\mathcal{V}_A$ is a compatible totally bounded quasi-uniformity on X finer than $\mathcal{V}$. Since $]0,1[\setminus[\frac{1}{3},\frac{2}{3}]$ has cardinality $2^{\aleph_0}$, we conclude that we have defined $2^{2^{\aleph_0}}$ quasi-uniformities provided that we can show that they are all distinct.

To this end, suppose that $A, B \subseteq]0,1[\setminus[\frac{1}{3},\frac{2}{3}]$ such that $A \neq B$; say, for instance, there exists some $a \in A \setminus B$. We shall show that $V_a \notin \mathcal{V}_B$: Assume the contrary. Then there are $V \in \mathcal{V}$ and some finite subset C of B such that $H := V \cap (\bigcap_{c\in C} V_c) \subseteq V_a$. Consequently $G_a \subseteq H(G_a) \subseteq V_a(G_a) = G_a$.

Set $C_2 = \{c \in C : c > a\}$ and $C_1 = \{c \in C : c < a\}$. Moreover, let $D_2 = X$ if $C_2 = \emptyset$, and $D_2 = G_{\min C_2}$ otherwise. Similarly, put $D_1 = \emptyset$ if $C_1 = \emptyset$, and $D_1 = G_{\max C_1}$ otherwise. Consider $x \in G_a \setminus D_1$. Then $H(x) = V(x) \cap D_2$. Thus $G_a \supseteq H(G_a) \supseteq H(G_a \setminus D_1) = V(G_a \setminus D_1) \cap D_2$. Furthermore $V(G_a \setminus D_1) \cap D_2 \supseteq V(G_a \setminus D_1) \cap B_{(\min C_2)-a}(G_a \setminus D_1)$ if $\min C_2$ is defined.

We deduce that $G_a \setminus D_1 < G_a$ in either case. Since $D_1 < G_a$, it follows that $G_a < G_a$ —a contradiction.

Consequently $\mathcal{V}_A$ and $\mathcal{V}_B$ are distinct. The assertion of Proposition 1 will follow provided that all the constructed quasi-uniformities are nontransitive.

In order to show this, fix $A^* \subseteq]0,1[\setminus[\frac{1}{3},\frac{2}{3}]$ and put $b_1 = \frac{1}{3}$, $b_2 = \frac{2}{3}$. To reach a contradiction, we assume that $\mathcal{V}_{A^*}$ is transitive. Since $G_{b_1} < G_{b_2}$

in $(X, \mathcal{V})$, by our assumption there is a transitive entourage $T \in \mathcal{V}_{A^*}$ such that $T \subseteq [(X \setminus G_{b_1}) \times X] \cup [X \times G_{b_2}]$. Then there are $W \in \mathcal{V}$ and a finite $F \subseteq A^*$ such that $(\bigcap_{a \in F} V_a) \cap W \subseteq T$. Set $C_1 = \{a \in F : a < b_1\}$ and $C_2 = \{a \in F : a > b_2\}$. Furthermore let $I_2 = G_{\min C_2}$ if $C_2 \neq \emptyset$, and $I_2 = X$ otherwise. Similarly, let $I_1 = G_{\max C_1}$ if $C_1 \neq \emptyset$, and $I_1 = \emptyset$ otherwise.

Find $V \in \mathcal{V}$ such that $V(I_1) \subseteq G_{b_1}$, $V(G_{b_2}) \subseteq I_2$ and $V \subseteq W$. Set $H = (\bigcap_{a \in F} V_a) \cap V$. Observe that $H \subseteq T$. Since there is no open set G in X such that $A \subseteq G \subseteq B$ and $G < G$, there is $k \in \omega \setminus \{0\}$ such that $V^k(G_{b_1}) \cap (X \setminus G_{b_2}) \neq \emptyset$. Find $z_0, z_1, \ldots, z_n \in X$, where $n \in \omega \setminus \{0\}$ is chosen minimal, such that $(z_0, z_1) \in V, \ldots, (z_{n-1}, z_n) \in V$, $z_0 \in G_{b_1}$ and $z_n \in X \setminus G_{b_2}$.

In particular by minimality of n, $z_1 \notin G_{b_1}$ and $z_{n-1} \in G_{b_2}$. Thus $z_0 \notin I_1$ and $z_n \in I_2$, since $V(I_1) \subseteq G_{b_1}$, $V(G_{b_2}) \subseteq I_2$ and $(z_0, z_1), (z_{n-1}, z_n) \in V$. Furthermore if $n > 1$, we have for i with $0 < i < n$ that $z_i \in G_{b_2} \setminus G_{b_1}$, since otherwise n would not be minimal. We conclude that $(z_0, z_1) \in H, \ldots, (z_{n-1}, z_n) \in H$, because $\{z_0, \ldots, z_n\} \subseteq I_2 \setminus I_1$. Thus $z_n \in H^n(z_0) \subseteq T(z_0) \subseteq G_{b_2}$, because $z_0 \in G_{b_1}$. We have reached a contradiction, since $z_n \notin G_{b_2}$. Therefore the quasi-uniformity $\mathcal{V}_{A^*}$ is nontransitive.

3 Transitive totally bounded quasi-uniformities

Next we wish to determine a lower bound for the number of transitive totally bounded quasi-uniformities that topological spaces satisfying some weak separation property admit. Recall that a base $\mathcal{B}$ of a topological space X is called an *l-base* [12] if it is closed under finite unions and finite intersections. In particular $\emptyset$ and X belong to $\mathcal{B}$. In [12] Losonczi observed that there is a one-to-one correspondence between the compatible transitive totally bounded quasi-uniformities and the l-bases of a topological space.

A filter $\mathcal{G}$ on the lattice of open sets of a topological space X is called *prime open* provided that for open sets G_1, G_2 of X, $G_1 \cup G_2 \in \mathcal{G}$ implies that $G_1 \in \mathcal{G}$ or $G_2 \in \mathcal{G}$. As usual, we shall say that a point x in X is a *limit point* of $\mathcal{G}$ if $\mathcal{G}$ contains all open neighborhoods of the point x. The set of limit points of $\mathcal{G}$ is called the *convergence set* of $\mathcal{G}$.

Lemma 2 *Let $\mathcal{B}$ be an l-base of a topological space (X, τ). Furthermore let $\mathcal{G}$ be a prime open filter on X. Finally, let $\bigvee \mathcal{G}$ be the supremum filter of the family $\{\eta(x) : x$ is a limit point of $\mathcal{G}\}$ on X. (Here $\eta(x)$ denotes the* *neighborhood filter of* $x \in X$; *in particular if the convergence set of* $\mathcal{G}$ *is empty, we set* $\bigvee \mathcal{G} = \{X\}$.*)*

Let $\mathcal{B}(\mathcal{G}) = \{G \in \mathcal{B} : G \in \bigvee \mathcal{G} \text{ or } G \notin \mathcal{G}\}$. *Then* $\mathcal{B}(\mathcal{G})$ *is an* l*-base of* X.

Proof. Let $x \in X$. If $(\eta(x) \cap \tau) \subseteq \mathcal{G}$, then $(\bigvee \mathcal{G}) \cap \mathcal{B}$ contains arbitrarily small neighborhoods of x. If $(\eta(x) \cap \tau) \not\subseteq \mathcal{G}$, then there are arbitrarily small neighborhoods of x in $\mathcal{B} \setminus \mathcal{G}$. Thus $\mathcal{B}(\mathcal{G})$ is a base of the topological space X.

Clearly $\emptyset \in \mathcal{B} \setminus \mathcal{G}$ and $X \in (\bigvee \mathcal{G}) \cap \mathcal{B}$. Thus $\emptyset, X \in \mathcal{B}(\mathcal{G})$. Suppose that $G_1, G_2 \in \mathcal{B}(\mathcal{G})$:

(1) If $G_1 \in \bigvee \mathcal{G}$ or $G_2 \in \bigvee \mathcal{G}$, then $G_1 \cup G_2 \in \bigvee \mathcal{G}$ and thus $G_1 \cup G_2 \in \mathcal{B}(\mathcal{G})$. If $G_1 \notin \bigvee \mathcal{G}$ and $G_2 \notin \bigvee \mathcal{G}$, then $G_1, G_2 \notin \mathcal{G}$, and thus $G_1 \cup G_2 \in \mathcal{B} \setminus \mathcal{G}$, because $\mathcal{G}$ is prime. Thus $G_1 \cup G_2 \in \mathcal{B}(\mathcal{G})$, too.

(2) If $G_1 \in \mathcal{B} \setminus \mathcal{G}$ or $G_2 \in \mathcal{B} \setminus \mathcal{G}$, then $G_1 \cap G_2 \in \mathcal{B} \setminus \mathcal{G}$. Thus $G_1 \cap G_2 \in \mathcal{B}(\mathcal{G})$.

If $G_1 \notin \mathcal{B} \setminus \mathcal{G}$ and $G_2 \notin \mathcal{B} \setminus \mathcal{G}$, then $G_1, G_2 \in \bigvee \mathcal{G}$ and consequently $G_1 \cap G_2 \in \bigvee \mathcal{G}$. Thus $G_1 \cap G_2 \in \mathcal{B}(\mathcal{G})$, too. Therefore $\mathcal{B}(\mathcal{G})$ is an l-base.

Remark 1 Let us note that, in Lemma 2, $\mathcal{B}(\mathcal{G}) = \mathcal{B}$ if and only if $\mathcal{B} \cap \mathcal{G} = \mathcal{B} \cap (\bigvee \mathcal{G})$. In the following we shall apply the preceding lemma to the l-base τ of a topological space (X, τ). In particular, in this case we have that $\tau(\mathcal{G}) = \tau$ if and only if $\mathcal{G} = (\bigvee \mathcal{G}) \cap \tau$.

Lemma 3 *Let* (X, τ) *be a topological space and let* $\mathcal{C}$ *be a collection of pairwise distinct prime open filters on* X *such that no convergent filter* $\mathcal{G}$ *in* $\mathcal{C}$ *is of the form* $\mathcal{G} = (\bigvee \mathcal{G}) \cap \tau$. *Then the* l*-bases* $\{\tau(\mathcal{G}) : \mathcal{G} \in \mathcal{C}\}$ *are pairwise distinct and, thus,* X *possesses at least* $|\mathcal{C}|$ *transitive totally bounded quasi-uniformities.*

Proof. Suppose that $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{C}$ and $\mathcal{G}_1 \neq \mathcal{G}_2$, say for instance there is $G \in \mathcal{G}_1 \setminus \mathcal{G}_2$.

Suppose that $(\bigvee \mathcal{G}_1) \cap \tau = \mathcal{G}_1$. Then $\mathcal{G}_1$ is not convergent by our assumption on $\mathcal{C}$. Thus $\bigvee \mathcal{G}_1 = \{X\}$ and therefore $\mathcal{G}_1 = \{X\}$ —contradicting the choice of G. Thus there is $G' \in \mathcal{G}_1 \setminus ((\bigvee \mathcal{G}_1) \cap \tau)$. Then $G'' := G \cap G'$ belongs to $(\mathcal{G}_1 \setminus \mathcal{G}_2) \cap (\mathcal{G}_1 \setminus \bigvee \mathcal{G}_1)$. Therefore $G'' \in \tau(\mathcal{G}_2)$, but $G'' \notin \tau(\mathcal{G}_1)$. Thus $\tau(\mathcal{G}_1) \neq \tau(\mathcal{G}_2)$.

The final assertion follows from the one-to-one correspondence between l-bases and compatible transitive totally bounded quasi-uniformities on topological spaces.

A topological space is called *supersober* [2, p. 310] provided that the convergence set of each convergent ultrafilter is the closure of some (unique) point. A compact supersober space is called *strongly sober* (see e.g. [10, p. 238]). Of course, a T_1-space is supersober if and only if it is a T_2-space.

Lemma 4 *Let (X, τ) be a supersober topological space and let A be a discrete subspace of infinite cardinality κ in X. Then there are at least 2^{2^κ} prime open filters $\mathcal{M}$ on X such that $\mathcal{M} \neq (\bigvee \mathcal{M}) \cap \tau$ whenever $\mathcal{M}$ is convergent.*

Proof. For each $x \in A$ let G_x be an open subset of X such that $G_x \cap A = \{x\}$. It is well known that there exist 2^{2^κ} (free) ultrafilters on the set A [14].

For each free ultrafilter $\mathcal{G}$ on A let $\mathcal{F}(\mathcal{G})$ be an ultrafilter on X that contains $\{H_B \setminus (\bigcup_{x \in A \setminus B} G_x) : B \in \mathcal{G}$ and H_B is an open neighborhood of the set B in $X\}$. Finally let $\mathcal{M}(\mathcal{G}) = \mathcal{F}(\mathcal{G}) \cap \tau$. Clearly $\mathcal{M}(\mathcal{G})$ is a prime open filter on X.

Note now that on the supersober space X each convergent prime open filter $\mathcal{H}$ satisfying $\mathcal{H} = (\bigvee \mathcal{H}) \cap \tau$ is of the form $\eta(x) \cap \tau$ for some $x \in X$:

Indeed, if for a prime open filter $\mathcal{H}$ with nonempty convergence set B we have $\mathcal{H} = (\bigvee \mathcal{H}) \cap \tau$, then an ultrafilter on X containing $\mathcal{D} = \mathcal{H} \cup \{X \setminus G : G \in \tau \setminus \mathcal{H}\}$ has B as its convergence set; note that the latter ultrafilter exists, because $\mathcal{D}$ has the finite intersection property, since $\mathcal{H}$ is prime. Thus by supersobriety of X, $B = \overline{\{x_0\}}$ for some $x_0 \in X$ and therefore $\mathcal{H} = (\bigvee \mathcal{H}) \cap \tau = \eta(x_0) \cap \tau$.

Next we show that each constructed filter $\mathcal{M}(\mathcal{G})$ is not an open neighborhood filter of a point in X: In order to reach a contradiction assume that $\mathcal{M}(\mathcal{G})$ is equal to $\eta(x) \cap \tau$ for some $x \in X$. If $A \cap \overline{\{x\}} \neq \emptyset$, then choose $y \in A \cap \overline{\{x\}}$. Since $\mathcal{G}$ is free, $A \setminus \{y\} \in \mathcal{G}$ and thus $X \setminus G_y \in \mathcal{F}(\mathcal{G})$. Note that $x \in G_y$. Thus by our assumption $G_y \in \mathcal{M}(\mathcal{G}) \subseteq \mathcal{F}(\mathcal{G})$ —a contradiction. Therefore $A \cap \overline{\{x\}} = \emptyset$. But then $X \setminus \overline{\{x\}} \in \mathcal{F}(\mathcal{G})$ and thus $X \setminus \overline{\{x\}} \in \mathcal{M}(\mathcal{G})$, although $X \setminus \overline{\{x\}}$ does not belong to $\eta(x) \cap \tau$. We conclude that $\mathcal{M}(\mathcal{G})$ is not an open neighborhood filter.

It remains to show that the constructed prime open filters are all distinct. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be free ultrafilters on A such that $\mathcal{G}_1 \neq \mathcal{G}_2$. So suppose that $D \in \mathcal{G}_1$, but $D \notin \mathcal{G}_2$. Then $A \setminus D \in \mathcal{G}_2$. It follows that $\bigcup_{a \in D} G_a \in \mathcal{M}(\mathcal{G}_1)$, but $X \setminus (\bigcup_{a \in D} G_a) \in \mathcal{F}(\mathcal{G}_2)$ and, thus, $\bigcup_{a \in D} G_a \notin \mathcal{M}(\mathcal{G}_2)$. Therefore $\mathcal{M}(\mathcal{G}_1) \neq \mathcal{M}(\mathcal{G}_2)$ and we conclude that the constructed prime open filters on X are all distinct.

Proposition 2 *A supersober space having a discrete subspace of infinite cardinality κ admits at least 2^{2^κ} transitive totally bounded quasi-uniformities. In particular, each infinite T_2-space admits at least $2^{2^{\aleph_0}}$ transitive totally bounded quasi-uniformities.*

Proof. The first result is an immediate consequence of the preceding two lemmas. The second result then follows from the well-known fact that each infinite T_2-space possesses a sequence $(G_n)_{n\in\omega}$ of pairwise disjoint nonempty open sets [3, proof of Theorem 0.13].

Recall that for a topological space X, $o(X) =$ (the number of open sets in X) $+ \omega$ (see [4]).

Corollary 1 *Each infinite metrizable space admits exactly $2^{o(X)}$ totally bounded transitive quasi-uniformities.*

Proof. Having a σ-discrete base, an infinite metrizable space X of network weight $nw(X)$ possesses a discrete subspace of cardinality $nw(X)$ (see [4, Theorem 8.1(d)]). By Proposition 2, X admits at least $2^{2^{nw(X)}}$ transitive totally bounded quasi-uniformities. The result follows, because according to an observation of Künzi [8, Corollary 2], X admits at most $2^{2^{nw(X)}}$ quasi-uniformities and because the number $o(X)$ of open sets in the metrizable space X is equal to $2^{nw(X)}$ (see e.g. [4, Theorem 8.1(e)]).

In connection with the preceding corollary let us mention the following inequality that the number of compatible quasi-proximities in any topological space satisfies.

Remark 2 *Let X be a topological space. Then there are at most $2^{o(X)}$ compatible quasi-proximities on X :*

Indeed, let δ be a compatible quasi-proximity on X. It is well known that for any $A, B \subseteq X$ we have $A\overline{\delta}B$ if and only if there is an open set G such that $A \subseteq G$ and $G\overline{\delta}\ \overline{B}$ [1, e.g., Proposition 1.28]. Thus each compatible quasi-proximity on X is uniquely determined by the pairs (G, F) where G is open in X, F is closed in X and $G\overline{\delta}F$. We conclude that the number of compatible quasi-proximities on X is at most $2^{o(X)}$.

Example 1 *Observe that some separation assumption is necessary in Proposition 2: In [11] a T_1-space is constructed that admits a unique quasi-proximity, but contains an infinite subset of isolated points.*

The following example extends Example 1 of Künzi [8] to higher cardinalities; he studied the special case that $\alpha = \omega + 1$.

Example 2 *Let X be an ordinal α equipped with the topology $\tau = \{[0,\beta] : \beta \in \alpha\} \cup \{\emptyset, \alpha\} \cup \{[0,\beta) : \beta \in L\}$ where L is the set of limit ordinals smaller than α. Then (X,τ) admits exactly $2^{|L|}$ totally bounded quasi-uniformities:* Indeed, it follows from the proof of Lemma 1 that a topological space that admits a nontransitive totally bounded quasi-uniformity necessarily has a strictly decreasing chain $(G_n)_{n\in\omega}$ of open sets. We conclude that all totally bounded quasi-uniformities that X admits are transitive. Since any set $[0,\beta]$, where $\beta \in \alpha$, is compact in X, $\mathcal{B}_0 := \{[0,\beta] : \beta \in \alpha\} \cup \{\emptyset, \alpha\}$ is a subset of each l-base of X. Consider now an arbitrary subset C of L. Obviously, $\mathcal{B}_0 \cup \{[0,\beta) : \beta \in C\}$ is an l-base for X. We conclude that X has exactly $2^{|L|}$ l-bases and, thus, $2^{|L|}$ totally bounded quasi-uniformities.

Note that the nonempty closed subsets of X are the closures of (unique) singletons. Hence X is supersober. We also observe that the prime open filters of X are exactly of the form $(\text{fil}\{G\}) \cap \tau$ where G is an arbitrary nonempty open subset of X, because each strictly decreasing sequence of open sets in X is finite and the open sets of X are linearly ordered under set-theoretic inclusion. Finally note that X is a locally compact strongly sober space if and only if α is a successor ordinal. It is known that for those spaces the coarsest compatible quasi-uniformity, which is associated with $\mathcal{B}_0$, is bicomplete (see [10, Proposition 2]).

Remark 3 Let $\mathcal{B}$ be an l-base of a topological space (X,τ) with infinite topology τ and let $\mathcal{U}_{\mathcal{B}} = \text{fil}\{[G \times G] \cup [(X \setminus G) \times X] : G \in \mathcal{B}\}$ be its associated transitive totally bounded quasi-uniformity on X (see [12]). Furthermore let $\mathcal{E}$ be the field of sets generated by $\mathcal{B}$ on X. From the theory of Boolean algebras [5, Theorem 5.31] it is known that $|\mathcal{E}| \leq |\text{Ult}(\mathcal{E})|$ where $\text{Ult}(\mathcal{E})$ denotes the set of ultrafilters on the Boolean algebra $(\mathcal{E}, \subseteq)$. Note that the minimal $(\mathcal{U}_{\mathcal{B}})^*$-Cauchy filters are exactly the filters on X generated by the ultrafilters of $(\mathcal{E}, \subseteq)$ (compare [1, Proposition 3.30]). Hence there are at least $|\mathcal{B}|$ minimal $(\mathcal{U}_{\mathcal{B}})^*$-Cauchy filters on X.

For the special case that $\mathcal{B} = \tau$, that is, that $\mathcal{U}_{\mathcal{B}}$ is the Pervin quasi-uniformity of X, we conclude that $o(X) \leq po(X)$ where $po(X)$ denotes the number of prime open filters on X, because, evidently, the minimal $(\mathcal{U}_\tau)^*$-Cauchy filters can be identified with the prime open filters on X : Indeed, if

$\mathcal{G}$ is a prime open filter on X, then the filter $\mathrm{fil}(\mathcal{G} \cup \{X \setminus G : G \in \tau \setminus \mathcal{G}\})$ on X is a minimal $(\mathcal{U}_\tau)^*$-Cauchy filter on X, and for any minimal $(\mathcal{U}_\tau)^*$-Cauchy filter $\mathcal{H}$ on X, $\mathcal{H} \cap \tau$ is a prime open filter on X.

Finally let us note that the condition about the prime open filters that appears in Remark 1 can be used to characterize topological spaces admitting a unique quasi-proximity.

Remark 4 *A topological space (X, τ) admits a unique totally bounded quasi-uniformity if and only if each prime open filter $\mathcal{G}$ on X satisfies $\mathcal{G} = (\bigvee \mathcal{G}) \cap \tau$. In particular, these spaces possess at most $o(X)$ prime open filters:*

Indeed, if a topological space X has a prime open filter $\mathcal{G}$ such that $\mathcal{G} \neq (\bigvee \mathcal{G}) \cap \tau$, then, by Remark 1, X possesses more than one l-base and thus more than one compatible transitive totally bounded quasi-uniformity. On the other hand, suppose that for each prime open filter $\mathcal{G}$ on X we have $\mathcal{G} = (\bigvee \mathcal{G}) \cap \tau$. Consider a proper open subset G of X and let $\mathcal{F}$ be an ultrafilter on X such that $G \in \mathcal{F}$. Then $G \in \mathcal{F} \cap \tau$ where $\mathcal{F} \cap \tau$ is a prime open filter on X. By our assumption, there is a finite collection $\mathcal{M}$ of open sets in X such that $\cap \mathcal{M} \subseteq G$ and each member of the collection $\mathcal{M}$ contains some limit point of $\mathcal{F} \cap \tau$. In the light of [6, Proposition and Theorem] we conclude that X admits a unique totally bounded quasi-uniformity.

The last assertion follows, since by the result just proved in a topological space admitting only one quasi-proximity each prime open filter is uniquely determined by its (closed) set of limit points.

Problem: *Characterize those cardinals κ for which there exists a topological space X such that X admits exactly κ totally bounded quasi-uniformities.*

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Remark: It is now known that for any cardinal κ there is a T_0-space admitting exactly κ totally bounded quasi-uniformities; see H.P.A. Künzi and A. Losonczi, *Semilattices of totally bounded quasi-uniformities*, in preparation.