A note on extreme points of C^∞-smooth balls in polyhedral spaces

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Abstract

Morris [Mo83] proved that every separable Banach space X that contains an isomorphic copy of c_0 has an equivalent strictly convex norm such that all points of its unit sphere S_X are unpreserved extreme, i.e., they are no longer extreme points of $B_{X^{**}}$. We use a result of Hájek [Ha95] to prove that any separable infinite-dimensional polyhedral Banach space has an equivalent C^∞-smooth and strictly convex norm with the same property as in Morris' result. We additionally show that no point on the sphere of a C^2-smooth equivalent norm on a polyhedral infinite-dimensional space can be strongly extreme, i.e., there is no point x on the sphere for which a sequence (h_n) in X with $\|h_n\| \not\to 0$ exists such that $\|x \pm h_n\| \to 1$.

1 Introduction

It is known that in non-superreflexive spaces, there exist no equivalent C^2-smooth norms that would be at the same time locally uniformly rotund (cf e.g. [FHHMZ, Exercise 9.16]). We show in this note that yet, in separable polyhedral spaces —all of which non-superreflexive—, there exist C^∞-smooth norms with various degrees of rotundity weaker than local uniform rotundity.

If $(X, \|\cdot\|)$ is a normed space, its closed unit ball (its unit sphere) will be denoted alternatively by B_X, $B_{\|\cdot\|}$, or even $B_{(X,\|\cdot\|)}$ (respectively S_X, $S_{\|\cdot\|}$, or $S_{(X,\|\cdot\|)}$), according to the circumstances. If $x \in X$ and $\delta > 0$, we put $B_X(x;\delta)$, $B_{\|\cdot\|}(x;\delta)$, or even $B_{(X,\|\cdot\|)}(x;\delta)$, for $x + \delta B_X$. The norm on X, its dual norm on X^*, and its bidual norm on X^{**}, are denoted by the same notation. For standard notation, results, and undefined terms we refer, e.g., to [FHHMZ].

Extreme points of B_X that are not extreme of $B_{X^{**}}$ are called *unpreserved*. On the other side, points in S_X that are extreme points of $B_{X^{**}}$ are called *preserved extreme points* (see Figure 1). Obviously, every preserved extreme point of B_X is itself an extreme point of B_X.
The preserved extreme points coincide with the w-strongly extreme points of B_X (see [GLT92] and references therein). A point $x \in S_X$ is called *(w-) strongly extreme* of B_X if given two sequences $\{y_n\}$ and $\{z_n\}$ in B_X such that $(y_n + z_n) \to 2x$, then $y_n \to x$ (respectively, $y_n \xrightarrow{w} x$). A norm $\|\cdot\|$ such that all points in $S_{\|\cdot\|}$ are strongly extreme is said to be *midpoint locally uniformly rotund* (for this notion, see, e.g., [LPT09] and references therein).
Solving a question by Phelps, Katznelson (see the reference in [Mo83]) proved that the closed unit ball of the disk algebra contains unpreserved extreme points.

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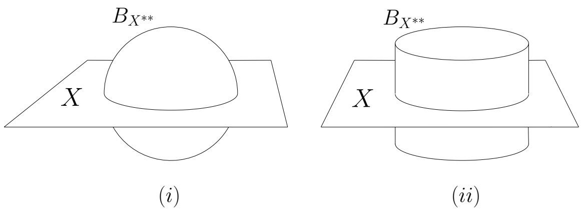

Figure 1: In (i), all points in S_X are preserved extreme, none in (ii)

Let $x \in S_X$. The point x is said to be *strongly exposed* (by a functional $f \in S_{X^*}$) if $f(x) = 1$ and $\operatorname{diam} S(f, \delta) \to 0$ as $\delta \downarrow 0$, where $S(f, \delta) := \{x \in B_X : f(x) > 1 - \delta\}$ is a *section* of B_X determined by f. The point x is said to be *denting* if for every $\varepsilon > 0$ it is contained in a section of B_X having diameter less than ε. It is easy to show that strongly exposed $\Rightarrow$ denting $\Rightarrow$ strongly extreme $\Rightarrow$ w-strongly extreme (= preserved extreme) $\Rightarrow$ extreme, and that if X is locally uniformly rotund, then every point in S_X is strongly exposed. For an example showing how big the gap between being strongly or w-strongly extreme is, see Theorem 4. It is simple to show that a denting point of $S_{X^{**}}$ must belong to X, hence the example in Remark 5.2 hints also at the difference between being strongly extreme and denting.

Morris proved in [Mo83] the following result.

(M1) *Any separable Banach space X containing an isomorphic copy of c_0 can be renormed in such a way that all points of S_X are unpreserved extreme points.* (Observe that the new norm is then strictly convex.)

The space c_0 has the property that the set $\operatorname{Ext}(B_{X^*})$ of extreme points of the closed dual unit ball is countable. The set $\operatorname{Ext}(B_{X^*})$ is an example of a *James boundary*, i.e., a subset of B_{X^*} where each element $x \in X$ attains its supremum on B_{X^*}. A Banach space with a countable James boundary has a separable dual space (this follows, e.g., from the fact that a countable James boundary is strong, i.e., its closed convex hull is the closed dual unit ball ([Ro81], see also [Go87]).

A Banach space X is called *polyhedral* if the ball of every finite-dimensional subspace (equivalently every two-dimensional subspace, see [K59]) of X has only a finite number of extreme points. Every polyhedral separable space has a countable James boundary ([Fo80], see also [Ve00]).

An example of polyhedral space is c_0 in its canonical norm ([K60], see also [GM72] and [Go01]). The argument in [Go01] is so nice that we cannot help but to reproduce it here. It relies on the fact that the $\|\cdot\|_\infty$-norm on c_0 *depends locally on a finite number of coordinates* (se the precise definition of this term below). Let E be a finite-dimensional subspace of c_0. For each $x \in S_E$ there exists $\varepsilon(x) > 0$ and a finite subset $F(x)$ of X^* such that $\|y\|_\infty = \sup\{|\langle y, x^*\rangle| : x^* \in F(x)\}$ for all $y \in B_E(x; \varepsilon(x))$. Since S_E is compact, there are $x_1, \ldots, x_n$ in S_E such that

$$S_E \subset \bigcup_{i=1}^{n} B_E(x_i, \varepsilon(x_i)).$$

Put $F := \bigcup_{i=1}^{n} F(x_i)$. Then F is a finite subset of X^* such that

$$\|x\|_\infty = \sup\{|\langle x, x^*\rangle| : x^* \in F\}$$

for all $x \in E$, hence E is isometric to a subspace of $(\mathbb{R}^{|F|}, \|\cdot\|_\infty)$, a polyhedral space.
On the other side, the space c in its canonical norm is not polyhedral. The following argument was kindly provided by L. Veselý (personal communication): Consider the points $P_n := \exp\{\mathrm{i}(1 - 1/\mathrm{n})\pi/4\}$ in the plane, for all $n \in \mathbb{N}$ (see Figure 2). Let $a_n x + b_n y = 1$ be the equation of the line through P_n and P_{n+1} for all $n \in \mathbb{N}$, and $a_0 x + b_0 y = 1$ the equation of the line through $P_\infty := \exp(\pi/4)$ and $P_0 := (-1, 0)$. Then $a := (a_n)_{n\geq 0}$ and $b := (b_n)_{n\geq 0}$ are elements in c, and

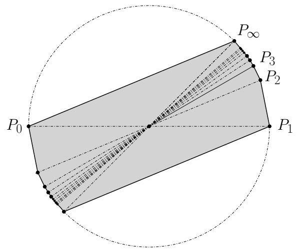

Figure 2: The construction to prove that c is not polyhedral

their linear span L is isometric to a plane equipped with the norm whose closed unit ball is the set $\overline{\mathrm{conv}}\,\{\pm P_1, \pm P_2, \ldots, \pm P_\infty\}$.
There is no infinite-dimensional reflexive polyhedral space ([L64]). Actually, no infinite-dimensional $C(K)$ space in its canonical norm is polyhedral—although such space has, if K is a countable compact topological space, obviously, a countable James boundary—. As seen below (see (H)), every $C(K)$ space with K a countable and compact topological space is isomorphic to a polyhedral space.

We will need the following result:

(Z) *Banach spaces with a countable James boundary are c_0-saturated*, i.e., each closed subspace contains an isomorphic copy of c_0 ([Fo77], [PWZ81], see also [FHHMZ, Theorem 10.9]).

In this note we slightly modify Morris technique by means of a result of P. Hájek ([Ha95], see also [FHHMZ, Theorem 10.12]) on normed spaces with a countable James boundary —a characterization quoted below as (H)— to add, under these circumstances, smoothness —in fact, C^∞-smoothness— to the kind of renorming shown by Morris.

The norm $\|\cdot\|$ of a Banach space is said *to depend locally on a finite number of coordinates* if given any $x_0 \in S_X$ there exists $\delta > 0$, continuous linear functionals $\{\psi_1, \psi_2, \ldots, \psi_n\} \subset X^*$, and a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that, for every $x \in B(x_0;\delta)$ we have $\|x\| = f(\psi_1(x), \psi_1(x), \ldots, \psi_1(x))$. The result of Hájek [Ha95] (see also [FHHMZ, Theorem 10.12]) mentioned above, an improvement of results in [Fo77] and [PWZ81], is the equivalence (i) to (iv) in the following. For the property (v) see [FLP01, Proposition 6.19] and, e.g., [Ve00].

(H) *For a Banach space X, the following are equivalent:* (i) *X has a countable James boundary.* (ii) *X has a James boundary that can be covered by a countable number of $\|\cdot\|$-compact subsets of X^*.* (iii) *X is separable and has an equivalent norm that depends locally on a finite number of coordinates.* (iv) *X is separable and has an equivalent norm that is C^∞-smooth away from the origin and depends locally on a finite number of coordinates.* (v) *X is separable and isomorphic to a polyhedral Banach space.*

The following result appears in [Mo83], with a different argument, as an ingredient of the proof of (M1) above; it will also be used in the proof of our main result.

(M2) *There exists an infinite-dimensional w^*-closed subspace M_0 of ℓ_∞ such that $M_0 \cap c_0 = \{0\}$.*

To see this, first note that every separable Banach space is isometric to a subspace of ℓ_∞, thus in particular ℓ_∞ contains an isometric copy Z of a given infinite-dimensional separable reflexive space. By a result of Rosenthal (see, e.g., [FHHMZ, Lemma 4.62]), Z is w^*-closed. Observe that $Z \cap c_0$ must be finite-dimensional, as any infinite-dimensional subspace of c_0 contains a copy of c_0. Then, a finite-codimensional subspace M_0 of Z is what we need to finish the proof.

2 The results

Theorem 1 *Let $(X, \|\cdot\|_0)$ be a Banach space having a countable James boundary. Then there exists an equivalent (strictly convex) norm $|||\cdot|||$ on X that is C^∞-smooth away from the origin and* *such that every point in $S_{|||\cdot|||}$ is an unpreserved extreme point of $B_{|||\cdot|||}$.*

Proof. By (H) above, the space X has an equivalent C^∞-smooth norm $\|\cdot\|$ that depends locally on a finite number of coordinates. Moreover, it contains an isomorphic copy Z of c_0 (see (Z) above). The space Z^{**} can be canonically identified to a closed subspace of X^{**}. Let M be a w^*-closed infinite-dimensional subspace of Z^{**} such that $M \cap Z = \{0\}$; it exists thanks to (M2) above. It is clear, too, that $M \cap X = \{0\}$.
Let $N := M_\perp \subset X^*$ (the orthogonal is taken with respect to the duality $\langle X^{**}, X^*\rangle$). Find a sequence $\{\phi_n\}$ in N such that $\overline{\text{span}}\,\{\phi_n : n \in \mathbb{N}\} = N$ and $\sum_{n=1}^\infty \|\phi_n\|^2 < +\infty$. Define a linear operator $T : X \to \ell_2$ by $Tx := (\langle x, \phi_n\rangle)_{n=1}^\infty$ for $x \in X$; then T is clearly bounded and one-to-one, and the mapping $x \to \|Tx\|_2$ from X into $\mathbb{R}$ is certainly C^∞-smooth away from the origin.
Define a norm $|||\cdot|||$ on X by

$$|||x||| := \|x\| + \|Tx\|_2 \text{ for all } x \in X. \tag{1}$$

Clearly $|||\cdot|||$ is strictly convex (see e.g. [DGZ, Chapter II]) and C^∞-smooth away from the origin. Let us show that every point x_0 in $S_{|||\cdot|||}$ is unpreserved extreme. Find $\delta > 0$ such that $\|\cdot\|$ depends on $B_{\|\cdot\|}(x_0; \delta)$ on finitely many coordinates $\{\psi_1, \psi_2, \dots, \psi_n\}$, i.e., $\|x\| = f(\psi_1(x), \psi_2(x), \dots, \psi_n(x))$ for $x \in B_{\|\cdot\|}(x_0; \delta)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Due to the fact that M is infinite-dimensional, we can find $h^{**} \in M \cap \bigcap_{k=1}^n \ker \psi_k$ with $0 < \|h^{**}\| \le \delta$.
Find a net $\{h_i : i \in I, \le\}$ in $B_{\|\cdot\|}(0; \delta)$ that w^*-converges to h^{**}. Observe that $x_0 + h_i \in B_{\|\cdot\|}(x_0; \delta)$, hence

$$\|x_0 + h_i\| = f(\psi_1(x_0 + h_i), \psi_2(x_0 + h_i), \dots, \psi_n(x_0 + h_i)), \text{ for all } i \in I. \tag{2}$$

Note that $\psi_k(x_0 + h_i) \to \psi_k(x_0 + h^{**})$ for all $k = 1, 2, \dots, n$, and so, by (2),

$$\begin{aligned} \|x_0 + h_i\| &= f(\psi_1(x_0 + h_i), \psi_2(x_0 + h_i), \dots, \psi_n(x_0 + h_i)) \\ &\to f(\psi_1(x_0 + h^{**}), \psi_2(x_0 + h^{**}), \dots, \psi_n(x_0 + h^{**})) \\ &= f(\psi_1(x_0), \psi_2(x_0), \dots, \psi_n(x_0)) = \|x_0\|. \end{aligned} \tag{3}$$

Since

$$x_0 + h_i \xrightarrow{w^*} x_0 + h^{**}, \tag{4}$$

we get from (3) and (4) that $\|x_0 + h^{**}\| \le \|x_0\|$. In the same way we get $\|x_0 - h^{**}\| \le \|x_0\|$, so finally by a standard convexity argument, $\|x_0\| = \|x_0 + h^{**}\| = \|x_0 - h^{**}\|$. Regarding the norm $|||\cdot|||$, we have then

$$|||x_0 + h^{**}||| = \|x_0 + h^{**}\| + \|T(x_0 + h^{**})\|,$$

as it is easy to show, hence, since $T(h^{**}) = 0$,

$$|||x_0 + h^{**}||| = \|x_0\| + \|Tx_0\| = |||x_0||| = 1. \tag{5}$$

Analogously,

$$|||x_0 - h^{**}||| = |||x_0||| = 1. \tag{6}$$

Equations (5) and (6) together show that x_0 is an unpreserved extreme point of $B_{|||\cdot|||}$. □

The following result extends what formerly was known for C^2-smooth LUR norms (see, e.g., [FHHMZ, Exercise 9.16]) and later for C^2-smooth norms with a strongly exposed point on its unit sphere [FWZ83, Theorem 3.3].

Theorem 2 *Let $(X, \|\cdot\|)$ be an infinite-dimensional C^2-smooth Banach space. If there exists a strongly extreme point of $B_{\|\cdot\|}$, then X is superreflexive.*

Proof. Assume that x is a strongly extreme point of B_X. The C^2-differentiability of $\|\cdot\|$ implies that there exists $\delta > 0$ such that the first derivative of $\|\cdot\|$ is uniformly continuous on a 2δ-ball around x. Let g be the supporting functional to the ball at x. For $h \in g^{-1}(0)$, let $f(h) = \|x+h\|+\|x-h\|-2$.

Then $f(h) \geq 0$, $f(0) = 0$ and $\inf_{\|h\|=\delta} f > 0$. Indeed, otherwise there exists a sequence $\{h_n\}_{n=1}^{\infty}$ in $g^{-1}(0)$ such that $\|h_n\| = \delta$ for all $n \in \mathbb{N}$, and $f(h_n) \to 0$, meaning that $\|x + h_n\| \to 1$ and $\|x - h_n\| \to 1$, as $\|x \pm h_n\| \geq g(x \pm h_n) = g(x) = 1$. Thus, by the definition of the strong extremality of x, $\|h_n\| \to 0$, a contradiction. Hence, by standard methods we can construct a bump function (i.e. a function with bounded nonempty support) on $g^{-1}(0)$ with uniformly continuous derivative, meaning that X is superreflexive (see, e.g., [FHHMZ, Theorem 9.19]). $\square$

Corollary 3 *Let $(X, \|\cdot\|)$ be an infinite-dimensional C^2-smooth Banach space. Assume that X does contain an isomorphic copy of c_0 (in particular, assume that X is isomorphic to a polyhedral space). Then no point of $S_{\|\cdot\|}$ is a strongly extreme point of $B_{\|\cdot\|}$.*

Proof. Otherwise, according to Theorem 2, the space X would be superreflexive. This is impossible since X contains an isomorphic copy of c_0. In case that X is isomorphic to a polyhedral space, so it is every separable subspace of X, thus the containment of c_0 follows from (Z) and (H) above. $\square$

Theorem 4 *Let X be a separable infinite-dimensional polyhedral Banach space. Then there exists an equivalent norm $|||\cdot|||$ on X such that every point in $S_{|||\cdot|||}$ is preserved extreme non-strongly extreme of $B_{|||\cdot|||}$.*

Proof. Let $\|\cdot\|$ be an equivalent C^2-smooth norm on X (such a norm always exists, see (H) above). Let $\{f_i :\ i \in \mathbb{N}\}$ be a countable norm-dense subset of $B_{(X^*,\|\cdot\|)}$ (recall that X is Asplund). Then the equivalent norm $|||\cdot|||$ on X defined by $|||x||| := \left(\|x\|^2 + \sum \frac{1}{2^i} f_i^2(x)\right)^{\frac{1}{2}}$ for all $x \in X$, is *weakly uniformly rotund*, i.e., whenever x_n, y_n are in $S_{(X,|||\cdot|||)}$ and $|||x_n + y_n||| \to 2$, then $x_n - y_n \to 0$ in the weak topology of X. This means that, in particular, the bidual norm of $|||\cdot|||$ is rotund (indeed, assume that $2x^{**} = y^{**} + z^{**}$ for some $x^{**} \in S_{(X^{**},|||\cdot|||)}$, where y^{**} and z^{**} are both in $B_{(X^{**},|||\cdot|||)}$ and $y^{**} \neq z^{**}$. Since X^* is separable, there exist sequences $\{y_n\}$ and $\{z_n\}$ in $B_{(X,|||\cdot|||)}$ such that $y_n \to y^{**}$ and $z_n \to z^{**}$ in the w^*-topology. This leads immediately to a contradiction). Moreover, the norm $|||\cdot|||$ on X is clearly C^2-smooth. Thus all points in $S_{(X,|||\cdot|||)}$ are preserved extreme points and yet, no point there is strongly extreme point of $B_{(X,|||\cdot|||)}$ by Corollary 3 (indeed, X is not superreflexive, as it contains an isomorphic copy of c_0). $\square$

Remark 5

1. Note that, in the setting of Theorem 4, no point in $S_{(X,|||\cdot|||)}$ is a point where the norm and weak topologies coincide, as otherwise, by a result in [LLT88], such a point would be a strongly extreme point of $B_{(X,|||\cdot|||)}$.
2. The James space J can be renormed by a norm the second bidual norm of which has the property that all its point on its sphere are strongly extreme points ([MOTV01], see also [LPT09]). None of the points in $S_{X^{**}} \setminus X$ can be denting. Recall that a space is reflexive if its dual space admits an equivalent Fréchet differentiable dual norm ([FHHMZ, Corollary 7.26]).
3. The space ℓ_∞ cannot be renormed so that all points on the sphere would be preserved extreme points ([HMS]).
4. Hájek ([Ha98]) showed that, if Γ is uncountable, then there exists no C^2-smooth and strictly convex norm on $c_0(\Gamma)$.
5. We refer to, e.g., [HMZ12], for a survey on related topics.

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