SPECTRUM AND COMPACTNESS OF THE CESÀRO OPERATOR ON WEIGHTED ℓ_P SPACES

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(May 4, 2015)

Abstract

An investigation is made of the continuity, the compactness and the spectrum of the Cesàro operator C when acting on the weighted Banach sequence spaces $\ell_p(w)$, $1 < p < \infty$, for a positive, decreasing weight w, thereby extending known results for C when acting on the classical spaces ℓ_p. New features arise in the weighted setting (e.g., existence of eigenvalues, compactness) which are not present in ℓ_p.

Keywords and phrases: Cesàro operator, weighted ℓ_p space, spectrum, compact operator.

1. Introduction

The discrete Cesàro operator C is defined on the linear space $\mathbb{C}^{\mathbb{N}}$ (consisting of all scalar sequences) by

$$\mathsf{C}x := \left(x_1, \frac{x_1 + x_2}{2}, \ldots, \frac{x_1 + \ldots + x_n}{n}, \ldots\right), \quad x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}. \quad (1)$$

The operator C is said to act in a vector subspace $X \subseteq \mathbb{C}^{\mathbb{N}}$ if it maps X into itself. Of particular interest is the situation when X is a Banach space. The fundamental questions in this case are: Is $\mathsf{C}\colon X \to X$ continuous and, if so, what is the spectrum of $\mathsf{C}\colon X \to X$? Amongst the classical Banach spaces $X \subseteq \mathbb{C}^{\mathbb{N}}$ where precise answers are known we mention ℓ_p $(1 < p < \infty)$, [6], [15], and c_0, [15], [19], and both c, ℓ_∞, [1], [15], as well as ces_p, $p \in \{0\} \cup (1, \infty)$, [8], the Bachelis spaces N^p, $2 \le p < \infty$, [9], and the spaces of bounded variation bv_0, [18], and bounded p-variation bv_p, $1 \le p < \infty$, [2]. In all of these cases, the operator norm of $\mathsf{C}\colon X \to X$ equals

Mathematics Subject Classification 2010: Primary 47A10, 47B37; Secondary 46B45, 47A16, 47A30.

its spectral radius and C has at most one eigenvalue, namely 1. There is no claim that this list of spaces (and references) is complete.

The aim of this paper is to investigate the two questions mentioned above for C acting on the weighted Banach spaces $\ell_p(w)$. To be precise, let $w = (w(n))_{n=1}^{\infty}$ be a bounded sequence, always assumed to be *strictly* positive. Define the space

$$\ell_p(w) := \left\{ x = (x_n)_{n\in N} \in \mathbb{C}^{\mathbb{N}} \colon \|x\|_{p,w} := \left(\sum_{n=1}^{\infty} |x_n|^p w(n) \right)^{1/p} < \infty \right\},$$

for each $1 < p < \infty$, equipped with the norm $\|\cdot\|_{p,w}$. Observe that $\ell_p(w)$ is isometrically isomorphic to ℓ_p via the linear multiplication operator

$$\Phi_w \colon \ell_p(w) \to \ell_p, \quad x = (x_n)_{n\in\mathbb{N}} \to \Phi_w(x) := (w(n)^{1/p} x_n)_{n\in\mathbb{N}}.$$

Therefore, each $\ell_p(w)$ is a Banach space. The dual space $(\ell_p(w))'$ of $\ell_p(w)$ is the Banach space $\ell_{p'}(v)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ (i.e., p' is the conjugate exponent of p) and $v(n) = w(n)^{-p'/p}$ for $n \in \mathbb{N}$. In particular, $\ell_p(w)$ is reflexive and separable for $1 < p < \infty$. Moreover, the canonical unit vectors $e_k := (\delta_{kn})_{n\in\mathbb{N}}$, for $k \in \mathbb{N}$, form an unconditional basis in $\ell_p(w)$ for $1 < p < \infty$. If $\inf_{n\in\mathbb{N}} w(n) > 0$, then $\ell_p(w) = \ell_p$ with equivalent norms and we are in the standard situation. Accordingly, we are mainly interested in the case when $\inf_{n\in\mathbb{N}} w(n) = 0$.

By Hardy's inequality, [14, Theorem 326, p.239], for every $1 < p < \infty$ the restriction of the Cesàro operator $\mathsf{C} \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ as given in (1) defines a bounded linear operator from ℓ_p into itself with operator norm equal to p'. Denote these operators by $\mathsf{C}^{(p)}$ so that $\|\mathsf{C}^{(p)}\| = p'$. In Section 2, where the papers [5], [11], [12] are relevant, we discuss various aspects of the continuity of C when it is restricted to $\ell_p(w)$, $1 < p < \infty$; denote this operator by $\mathsf{C}^{(p,w)}$ whenever it is continuous.

For any Banach space X, let I denote the identity operator on X and $\mathcal{L}(X)$ denote the space of all continuous linear operators from X into itself. The *spectrum* and the *resolvent set* of $T \in \mathcal{L}(X)$ are denoted by $\sigma(T)$ and $\rho(T)$, respectively; see [10, Ch. VII], for example. The set of all *eigenvalues* of T, called the *point spectrum* of T, is denoted by $\sigma_{pt}(T)$. The *spectral radius* $r(T) := \sup\{|\lambda| \colon \lambda \in \sigma(T)\}$ of T always satisfies $r(T) \le \|T\|$, [10, p.567].

Section 3 is devoted to an analysis of the spectrum of C when acting in $\ell_p(w)$. The main result is Theorem 3.3; it is complemented by Examples 3 which clarify the scope of this theorem. Unlike for $\mathsf{C}^{(p)}$, it can happen that $\sigma_{pt}(\mathsf{C}^{(p,w)}) \neq \emptyset$. Actually, $\mathsf{C}^{(p,w)}$ can even have infinitely many eigenvalues; see Proposition 3.4. The final section deals with the *compactness* of $\mathsf{C}^{(p,w)}$. Relevant is how fast w decreases to 0; see Proposition 4.1, Theorem 4.2, Corollary 4.3 and Proposition 4.4. Unlike for C acting in the classical Banach spaces mentioned in the opening paragraph, it may happen in $\ell_p(w)$ that $r(\mathsf{C}^{(p,w)}) < \|\mathsf{C}^{(p,w)}\|$; see Proposition 4.1.

2. Continuity of C in weighted ℓ_p spaces

Some of the concepts and results from [12] that are quoted in this section actually have their origins in the paper [11]. We begin with the following fact.

LEMMA 2.1. *Let $w = (w(n))_{n=1}^{\infty}$ be a positive sequence and $1 < p < \infty$. Then the Cesàro operator C maps $\ell_p(w)$ continuously into itself if, and only if,*

$$\sup_{m\in\mathbb{N}} \left(\sum_{k=1}^{m} w(k)^{-p'/p}\right)^{-1} \left(\sum_{n=1}^{m} \frac{w(n)}{n^p} \left(\sum_{k=1}^{n} w(k)^{-p'/p}\right)^{p}\right) < \infty,$$

i.e., if, and only if, there exists $K > 0$ such that

$$\sum_{n=1}^{m} \frac{w(n)}{n^p} \left(\sum_{k=1}^{n} w(k)^{-p'/p}\right)^{p} \leq K \left(\sum_{k=1}^{m} w(k)^{-p'/p}\right), \quad m \in \mathbb{N}. \tag{2}$$

Moreover, if the constant K satisfying (2) is chosen as small as possible, then the operator norm of C is at most $p'K^{1/p}$.

PROOF. Let $T_w\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ denote the linear operator defined by

$$T_w x := \left(\frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} w(k)^{-1/p} x_k\right)_{n\in\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}. \tag{3}$$

Then $\Phi_w C = T_w \Phi_w$. Since Φ_w is isometric from $\ell_p(w)$ onto ℓ_p, it follows that C maps $\ell_p(w)$ continuously into itself if, and only if, T_w maps ℓ_p continuously into itself. But, the matrix of T_w is factorable (cf. [5, §4] with $a_n = w(n)^{1/p}/n$ and $b_k = w(k)^{-1/p}$ for $1 \leq k \leq n$) and so it follows from [5, Theorem 2] that $T_w \in \mathcal{L}(\ell^p)$ if, and only if, (2) holds.

The proof of Theorem 2 in [5] yields that the operator norm of C is at most $p'K^{1/p}$. □

PROPOSITION 2.2. *Let $w = (w(n))_{n=1}^{\infty}$ be a decreasing, positive sequence and $1 < p < \infty$. Then the Cesàro operator $\mathsf{C}^{(p,w)} \in \mathcal{L}(\ell_p(w))$ and satisfies*

$$1 < \left(\frac{1}{w(1)} \sum_{n=1}^{\infty} \frac{w(n)}{n^p}\right)^{1/p} \leq \|\mathsf{C}^{(p,w)}\| \leq p'. \tag{4}$$

PROOF. Fix $m \in \mathbb{N}$. Because w is decreasing, we have

$$\begin{aligned}\sum_{n=1}^{m} \frac{w(n)}{n^p} \left(\sum_{k=1}^{n} w(k)^{-p'/p} \right)^p &= \sum_{n=1}^{m} \left(\frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} w(k)^{-p'/p} \right)^p \\ \leq \sum_{n=1}^{m} \left(\frac{w(n)^{1/p}}{n} \cdot \frac{n}{w(n)^{p'/p}} \right)^p &= \sum_{n=1}^{m} w(n)^{-p'/p},\end{aligned}$$

which is precisely (2) with $K = 1$. So, Lemma 2.1 implies that C is continuous on $\ell_p(w)$ with $\|\mathsf{C}^{(p,w)}\| \leq p'$.

For an alternate proof of the continuity of $\mathsf{C}^{(p,w)}$, based directly on Hardy's inequality in ℓ_p, see [12, Proposition 5.1].

Since $T_w = \Phi_w \mathsf{C}^{(p,w)} \Phi_w^{-1}$, with Φ_w mapping the closed unit ball of $\ell_p(w)$ onto that of ℓ_p and Φ_w^{-1} mapping the closed unit ball of ℓ_p onto that of $\ell_p(w)$, it follows that $\|T_w\| = \|\mathsf{C}^{(p,w)}\|$. Of course,

$$\Phi_w^{-1} x = (w(n)^{-1/p} x_n)_{n \in \mathbb{N}}, \quad x \in \ell_p.$$

Substituting $x = e_1$ into (3) it follows that

$$\|\mathsf{C}^{(p,w)}\| = \|T_w\| \geq \|T_w e_1\|_p = \left(\frac{1}{w(1)} \sum_{n=1}^{\infty} \frac{w(n)}{n^p} \right)^{1/p} \geq \left(1 + \frac{w(2)}{w(1) 2^p} \right)^{1/p} > 1.$$

See also [12, Proposition 5.5]. □

Some comments regarding Proposition 2.2 are in order. As noted above, for each $1 < p < \infty$ we have $\|\mathsf{C}^{(p)}\| = p'$ and, for a positive, decreasing weight w, that (4) holds. These estimates are not the best possible in general. Denote by $\delta_p(w)$ the set of all decreasing, non-negative sequences in $\ell_p(w)$ and define

$$\Delta_{p,w}(\mathsf{C}^{(p,w)}) := \sup\{\|\mathsf{C}^{(p,w)} x\|_{p,w} \colon x \in \delta_p(w),\ \|x\|_{p,w} = 1\} \leq \|\mathsf{C}^{(p,w)}\|.$$

The following result follows from Propositions 6.3, 6.5 and 6.6 of [12].

PROPOSITION 2.3. *Let $1 < p < \infty$ and $w(n) = 1/n^\alpha$, $n \in \mathbb{N}$, for a fixed $\alpha > 0$. Then*

$$\max\{m_1, m_2\} \leq \Delta_{p,w}(\mathsf{C}^{(p,w)}) \leq \|\mathsf{C}^{(p,w)}\| \leq M_2(r) := [r\zeta(r+\alpha)]^{r/p}, \quad (5)$$

for $1 \leq r \leq p$, where $m_1 := p/(p+\alpha-1)$ and $m_2 := \zeta(p+\alpha)^{1/p}$, with ζ denoting the Riemann zeta function. Moreover, for $\alpha \leq r < (p+\alpha)$, it is also the case that

$$\|\mathsf{C}^{(p,w)}\| \leq M_3(r) := \left(\frac{p}{p+\alpha-r} \right)^{1/p'} \zeta \left(1 + \frac{r}{p'} + \frac{\alpha}{p} \right)^{1/p}. \quad (6)$$

We provide some relevant examples.

EXAMPLE 1. (i) For $w(n) = 1/n^\alpha$, if $\alpha = 0.9$ and $p = 1.1$, then $\max\{m_1, m_2\} \simeq 1.572$ and $M_2(1) = M_3(0.9) \simeq 1.663$ (see pp.15-16 of [12]) and so Proposition 2.3 shows that

$$1.572 \leq \|\mathsf{C}^{(p,w)}\| \leq 1.663.$$

On the other hand, $p' = 11$ and so Proposition 2.2 only yields $\|\mathsf{C}^{(p,w)}\| \leq 11$.

(ii) Still for $w(n) = 1/n^\alpha$, but now with $\alpha = 0.5$ and $p = 2$, we have $m_1 = 4/3$ and $M_3(3/4) \simeq 1.593$ (see p.16 of [12]) so that Proposition 2.3 reveals that

$$\frac{4}{3} \leq \|\mathsf{C}^{(p,w)}\| \leq 1.593.$$

In this case, $p' = 2$ and so Proposition 2.2 only yields $\|\mathsf{C}^{(p,w)}\| \leq 2$.

(iii) Again for $w(n) = 1/n^\alpha$, with $\alpha > 0$, it follows (in the notation of Proposition 2.3) that

$$\left(\frac{1}{w(1)} \sum_{n=1}^{\infty} \frac{w(n)}{n^p}\right)^{1/p} = \left(\sum_{n=1}^{\infty} \frac{1}{n^{p+\alpha}}\right)^{1/p} = \zeta(p+\alpha)^{1/p} = m_2.$$

Hence, the lower bound in (4) reduces to $m_2 \leq \|\mathsf{C}^{(p,w)}\|$ whereas (5) yields $\max\{m_1, m_2\} \leq \|\mathsf{C}^{(p,w)}\|$. Of course, (4) applies to more general weights w.

The following example is not a consequence of Proposition 2.3.

EXAMPLE 2. Let $p = 2$ and set $w(n) = 2^{-n}$ for $n \in \mathbb{N}$. The proof of Proposition 2.2 yields that $\|\mathsf{C}^{(2,w)}\| = \|T_w\|$. Recall, via (3), that

$$T_w x = \left(\frac{1}{n 2^{n/2}} \sum_{k=1}^{n} 2^{k/2} x_k\right)_{n\in\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}} \in \ell_2.$$

For every $x \in \ell_2$, it follows via the Cauchy-Schwarz inequality and the identity $\sum_{k=1}^{n} r^k = (r - r^{n+1})/(1-r)$, for $r \neq 1$, that

$$\begin{aligned}\|T_w x\|_2^2 &= \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} \left|\sum_{k=1}^{n} 2^{k/2} x_k\right|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} (\sum_{k=1}^{n} 2^k)(\sum_{k=1}^{n} |x_k|^2) \\ &\leq \|x\|_2^2 \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} (2^{n+1} - 2) = \|x\|_2^2 \sum_{n=1}^{\infty} \frac{2(1-2^{-n})}{n^2}.\end{aligned}$$

Accordingly, $\|T_w\| \leq \left(\sum_{n=1}^{\infty} \frac{2(1-2^{-n})}{n^2}\right)^{1/2}$. Observe that

$$\sum_{n=1}^{\infty} \frac{(1-2^{-n})}{n^2} = \frac{\pi^2}{6} - \int_0^{1/2} \frac{-\log(1-t)}{t}\, dt,$$

because of the fact that $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ and the identity

$$\int_0^{1/2} \frac{-\log(1-t)}{t}\, dt = \int_0^{1/2} \sum_{n=0}^{\infty} \frac{t^n}{(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2 2^n}.$$

The function $f(t) = \frac{-\log(1-t)}{t}$ for $t \in (0,1]$, with $f(0) := 1$, is positive, continuous and increasing on $[0,1)$ and so

$$1 = f(0) \leq f(t) \leq f\left(\frac{1}{2}\right) = 2\log 2, \quad t \in [0, 1/2],$$

which implies that $-\log 2 \leq -\int_0^{1/2} \frac{-\log(1-t)}{t}\, dt \leq -\frac{1}{2}$. Consequently,

$$\sum_{n=1}^{\infty} \frac{2(1-2^{-n})}{n^2} \leq 2(\frac{\pi^2}{6} - \frac{1}{2}) \simeq 2.2898$$

and so

$$\|\mathsf{C}^{(2,w)}\| = \|T_w\| \leq \sqrt{\left(\frac{\pi^2}{3} - 1\right)} \simeq 1.513 < p' = 2.$$

Direct calculation yields

$$\|T_w e_1\|_2 = \left(2\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}\right)^{1/2} \geq \left(2\sum_{n=1}^{3} \frac{1}{n^2 2^n}\right)^{1/2} \simeq 1.073$$

and so we have

$$1.073 \leq \|\mathsf{C}^{(2,w)}\| \leq \sqrt{\left(\frac{\pi^2}{3} - 1\right)} \simeq 1.513;$$

see also Proposition 2.2.

3. Spectrum of $\mathsf{C}^{(p,w)}$

The aim of this section is to provide some detailed knowledge of the spectrum of $\mathsf{C}^{(p,w)}$. Unlike for the classical Cesàro operators $\mathsf{C}^{(p)} \in \mathcal{L}(\ell_p)$, for $1 < p < \infty$, it can now happen that eigenvalues appear.

Given a (strictly) positive, bounded sequence $w = (w(n))_{n\in\mathbb{N}}$ and $1 < p < \infty$, let $S_w(p) := \{s \in \mathbb{R} \colon \sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} < \infty\}$. In case $S_w(p) \neq \emptyset$ we define $s_p := \inf S_w(p)$. Note that $\frac{p'}{p} = \frac{1}{p-1}$, for every $1 < p < \infty$. Moreover, let $R_w := \{t \in \mathbb{R} \colon \sum_{n=1}^{\infty} n^t w(n) < \infty\}$. In case $R_w \neq \mathbb{R}$ we define $t_0 := \sup R_w$.

Fix $1 < p < \infty$ and let $w(n) = 2^{-np/p'}$ for $n \in \mathbb{N}$. Then $S_w(p) = \emptyset$, i.e., it can happen that $S_w(p)$ is empty. However, in the event that $S_w(p) \neq \emptyset$, then $s_p \geq 1$. Indeed, for any fixed $s \in \mathbb{R}$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} \geq \|w\|_{\infty}^{-p'/p} \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{7}$$

So, whenever $s \in S_w(p)$ it follows that $\sum_{n=1}^{\infty} \frac{1}{n^s} < \infty$, that is, $s > 1$. Hence, $S_w(p) \subseteq (1, \infty)$ which implies that $s_p \geq 1$. Moreover, for any $r > s \in S_w(p)$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^r w(n)^{p'/p}} < \sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}}$$

and so also $r \in S_w(p)$. Accordingly, whenever $S_w(p) \neq \emptyset$, then it is an infinite interval, i.e., $S_w(p) = [s_p, \infty)$ or $S_w(p) = (s_p, \infty)$ with $s_p \geq 1$. It is a consequence of (7) that $1 \notin S_w(p)$, for all $1 < p < \infty$ and all positive, bounded sequences w.

In the event that $a_w := \inf_{n \in \mathbb{N}} w(n) > 0$ it follows that necessarily $s_p = 1$. Indeed, in this case $w(n)^{-p'/p} \leq a_w^{-p'/p}$, $n \in \mathbb{N}$, which implies that $\frac{1}{n^s w(n)^{p'/p}} \leq \frac{a_w^{-p'/p}}{n^s}$, for all $n \in \mathbb{N}$ and $s \in \mathbb{R}$. Hence, $(1, \infty) \subseteq S_w(p)$ and so $s_p \leq 1$. Since we are assuming that $S_w(p) \neq \emptyset$, we already know that $s_p \geq 1$. Accordingly, $s_p = 1$.

Let $1 < p < \infty$ and fix $\alpha > 0$. For $w(n) = 1/n^{\alpha p/p'}$ and any $s \in \mathbb{R}$ it follows that $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} = \sum_{n=1}^{\infty} \frac{1}{n^{s-\alpha}} < \infty$ precisely when $s > (1 + \alpha)$ and so $s_p = 1 + \alpha$. Hence, given any $\beta > 1$ and $1 < p < \infty$, there exists a positive, decreasing weight $w \downarrow 0$ such that $S_w(p) = (\beta, \infty)$, i.e., $s_p = \beta$.

Concerning the set R_w, a similar discussion applies. For $w(n) = 2^{-n}$ it turns out that $R_w = \mathbb{R}$ with $t_0 = \infty$. However, if $R_w \neq \mathbb{R}$, then t_0 is finite with $t_0 \geq -1$ and $R_w = (-\infty, t_0)$ or $R_w = (-\infty, t_0]$. Moreover, $R_w = \emptyset$ is not possible as $\sum_{n=1}^{\infty} n^t w(n) \leq \|w\|_{\infty} \sum_{n=1}^{\infty} n^t < \infty$ whenever $t < -1$. If $a_w > 0$, then necessarily $t_0 = -1$ but, $-1 \notin R_w$ as $\sum_{n=1}^{\infty} n^t w(n) \geq a_w \sum_{n=1}^{\infty} n^t$ for all $t \in \mathbb{R}$.

The following result clarifies the connection between s_p and t_0.

Proposition 3.1. *Let $w = (w(n))_{n \in \mathbb{N}}$ be a bounded, strictly positive sequence.*

(i) *For each $1 < p < \infty$ such that $S_w(p) \neq \emptyset$ we have*

$$t_0 \leq \frac{s_p p}{p'} = (p-1)s_p.$$

In particular, $R_w \neq \mathbb{R}$ whenever there exists $p \in (1, \infty)$ with $S_w(p) \neq \emptyset$.

(ii) *If $R_w \neq \mathbb{R}$, then $S_w(p) \subseteq [1 + \frac{t_0}{(p-1)}, \infty)$, for every $1 < p < \infty$.*

(iii) *Suppose that* $1 < p < \infty$ *satisfies* $S_w(p) \neq \emptyset$. *Then*

$$S_w(p) \subseteq S_w(q), \quad q \in [p, \infty).$$

In particular, $S_w(q) \neq \emptyset$ *and* $s_q \leq s_p$ *whenever* $q \geq p$.

(iv) *If* $S_w(p) = \emptyset$ *for some* $1 < p < \infty$, *then* $S_w(q) = \emptyset$ *for all* $1 < q \leq p$.

PROOF. (i) Suppose that $S_w(p) \neq \emptyset$. Fix $s > s_p$. Since $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} < \infty$, there exists $N \in \mathbb{N}$ such that $\frac{1}{n^s w(n)^{p'/p}} \leq 1$ for $n \geq N$ and hence, $n^{sp/p'} w(n) \geq 1$ for $n \geq N$. So, the series $\sum_{n=1}^{\infty} n^{sp/p'} w(n)$ diverges which yields that $t_0 \leq \frac{sp}{p'}$. Accordingly, $t_0 \leq \frac{s_p p}{p'}$. In particular, $R_w \neq \mathbb{R}$.

(ii) Fix $p \in (1, \infty)$ and any $t < t_0$, in which case $\sum_{n=1}^{\infty} n^t w(n) < \infty$. Hence, there exists $K \in \mathbb{N}$ such that $n^t \leq \frac{1}{w(n)}$ for $n \geq K$, that is, $n^{tp'/p} \leq \frac{1}{w(n)^{p'/p}}$ for $n \geq K$. So, for any $s \in \mathbb{R}$ we have (as $\frac{1}{n^s} > 0$ for $n \in \mathbb{N}$) that

$$\frac{1}{n^{s-(tp'/p)}} = \frac{n^{tp'/p}}{n^s} \leq \frac{1}{n^s w(n)^{p'/p}}, \quad n \geq K.$$

Choose now $s \leq 1 + \frac{tp'}{p}$. It follows from the previous inequality that $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}}$ diverges. Hence, $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}}$ diverges whenever $s \leq 1 + \frac{tp'}{p}$, for some $t < t_0$, that is, whenever $s \in (-\infty, 1 + \frac{t_0 p'}{p})$. So, $S_w(p) \subseteq [1 + \frac{t_0 p'}{p}, \infty) = [1 + \frac{t_0}{(p-1)}, \infty)$.

(iii) Fix $s \in S_w(p)$, i.e., $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} < \infty$. For every $1 < q < \infty$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{q'/q}} = \sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} \cdot w(n)^{\frac{p'}{p} - \frac{q'}{q}} \leq \|w\|_{\infty}^{\frac{p'}{p} - \frac{q'}{q}} \sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}},$$

which is finite provided that $\frac{p'}{p} \geq \frac{q'}{q}$. This is equivalent to $(p'-1) \geq (q'-1)$, that is, to $q \geq p$. Hence, whenever $q \geq p$ we have $S_w(p) \subseteq S_w(q)$ which clearly implies $S_w(q) \neq \emptyset$ and $s_q \leq s_p$.

(iv) Follows immediately from part (iii). □

Define $\Sigma := \{\frac{1}{n} \colon n \in \mathbb{N}\}$ and let $\Sigma_0 := \{0\} \cup \{\frac{1}{n} \colon n \in \mathbb{N}\}$ be its closure. The following inequalities will be needed later.

LEMMA 3.2. (i) *Let* $\lambda \in \mathbb{C} \setminus \Sigma_0$ *and set* $\alpha := \operatorname{Re}\left(\frac{1}{\lambda}\right)$. *Then there exist constants* $d > 0$ *and* $D > 0$ *(depending on* α*) such that*

$$\frac{d}{n^{\alpha}} \leq \prod_{k=1}^{n} \left|1 - \frac{1}{k\lambda}\right| \leq \frac{D}{n^{\alpha}}, \quad n \in \mathbb{N}. \tag{8}$$

(ii) *For each $m \in \mathbb{N}$ we have that*

$$\frac{(n-1)!}{(n-m)!} \simeq n^{m-1}, \quad \textit{for all large } n \in \mathbb{N}. \tag{9}$$

(iii) Let $1 < p < \infty$ and $w = (w(n))_{n\in\mathbb{N}}$ be a positive, decreasing sequence. Then

$$(n^m w(n))_{n\in\mathbb{N}} \in \ell_p, \quad \forall m \in \mathbb{N}, \tag{10}$$

if and only if

$$(n^m w(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p, \quad \forall m \in \mathbb{N}, \tag{11}$$

Proof. (i) The inequalities in (8) follow as in the proof of Lemma 7 in [19], where the restriction $\alpha < 1$ is assumed. Indeed, with $\frac{1}{\lambda} = \alpha + i\beta$ (for $\alpha,\ \beta \in \mathbb{R}$) and using $1 + x \le e^x$ for $x > 0$, we have

$$\begin{aligned}\prod_{k=1}^{n}\left|1-\frac{1}{k\lambda}\right| &= \prod_{k=1}^{n}\left(1-\frac{2\alpha}{k}+\frac{\alpha^2+\beta^2}{k^2}\right)^{1/2} \\ &\le \exp\sum_{k=1}^{n}\left(-\frac{\alpha}{k}+\frac{C}{k^2}\right) \le \exp\left(-\alpha\log(n)+v\right) \le \frac{D}{n^{\alpha}}.\end{aligned}$$

An application of Taylor's formula to $x \mapsto (1+x)^{-1/2}$, for $x > -1$, yields

$$\begin{aligned}\prod_{k=1}^{n}\left|1-\frac{1}{k\lambda}\right|^{-1} &= \prod_{k=1}^{n}\left(1-\frac{2\alpha}{k}+\frac{\alpha^2+\beta^2}{k^2}\right)^{-1/2} \le \prod_{k=1}^{n}\left(1+\frac{\alpha}{k}+\frac{C'}{k^2}\right) \\ &\le \exp\sum_{k=1}^{n}\left(\frac{\alpha}{k}+\frac{C'}{k^2}\right) \le \exp\left(\alpha\log(n)+v'\right) = d^{-1}n^{\alpha}.\end{aligned}$$

(ii) Fix $m \in \mathbb{N}$. Then, for all large $n > m$, we have

$$\frac{(n-1)!}{(n-m)!} = (n-1)\cdot\ldots\cdot(n-m+1) = n^{m-1}\cdot\left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{m-1}{n}\right) \simeq n^{m-1}.$$

(iii) Suppose that (10) holds. Fix $m \in \mathbb{N}$. Let $k \in \mathbb{N}$ satisfy $k \ge (2+mp)$. Since $(n^k w(n))_{n\in\mathbb{N}} \in \ell_p$, there exists $N \in \mathbb{N}$ such that

$$w(n) \le \frac{1}{n^k} \le \frac{1}{n^{2+mp}}, \quad n > N.$$

It follows that

$$\sum_{n=1}^{\infty} n^{mp} w(n) \le \sum_{n=1}^{N} n^{mp} w(n) + \sum_{n=N+1}^{\infty} n^{mp}\left(\frac{1}{n^{2+mp}}\right) < \infty,$$

that is, $(n^m w(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p$. Accordingly, (11) is satisfied.

Conversely, suppose that (11) holds. Since $(nw(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p$, there exists $K \in \mathbb{N}$ such that $w(n) \leq 1$ for $n \geq K$ and hence, $w(n) \leq w(n)^{1/p}$ for $n \geq K$. Fix $m \in \mathbb{N}$. Then $n^m w(n) \leq n^m w(n)^{1/p}$ for $n \geq K$. Since $(n^m w(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p$, we can conclude that also $(n^m w(n))_{n\in\mathbb{N}} \in \ell_p$. Hence, (10) is satisfied. □

If $S_w(p) \neq \emptyset$, then $s_p \geq 1$ and so $\frac{p'}{2s_p} \leq \frac{p'}{2}$, which is relevant for the following results. Also relevant is that $\|\mathsf{C}^{(p,w}\| < p'$ is possible; see Section 2.

We now come to the main result of this section.

THEOREM 3.3. *Let $w = (w(n))_{n\in\mathbb{N}}$ be a positive, decreasing sequence.*

(i) *Suppose that $S_w(p) \neq \emptyset$ for some $1 < p < \infty$. Then for the dual operator $(\mathsf{C}^{(p,w)})' \in \mathcal{L}((\ell_p(w))')$ of $\mathsf{C}^{(p,w)}$ we have*

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| < \frac{p'}{2s_p}\right\} \cup \Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(p,w)})') \tag{12}$$

and

$$\sigma_{pt}((\mathsf{C}^{(p,w)})') \setminus \Sigma \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\}. \tag{13}$$

For the Cesàro operator $\mathsf{C}^{(p,w)}$ itself we have

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\} \cup \Sigma \subseteq \sigma(\mathsf{C}^{(p,w)}) \tag{14}$$

and

$$\sigma(\mathsf{C}^{(p,w)}) \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\right\} \cap \left\{\lambda \in \mathbb{C}\colon |\lambda| \leq \|\mathsf{C}^{(p,w)}\|\right\}. \tag{15}$$

(ii) *Suppose that $R_w \neq \mathbb{R}$, i.e., $t_0 < \infty$. Then, for every $1 < p < \infty$, we have*

$$\left\{\frac{1}{m}\colon m \in \mathbb{N},\ 1 \leq m < \frac{t_0}{p} + 1\right\} \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \left\{\frac{1}{m}\colon m \in \mathbb{N},\ 1 \leq m \leq \frac{t_0}{p} + 1\right\}. \tag{16}$$

If $R_w = \mathbb{R}$, then

$$\sigma_{pt}(\mathsf{C}^{(p,w)}) = \Sigma, \quad \forall\ 1 < p < \infty. \tag{17}$$

PROOF. The proof is via a series of steps.

(i) By Proposition 2.2 we have $\mathsf{C}^{(p,w)} \in \mathcal{L}(\ell_p(w))$ with $\|\mathsf{C}^{(p,w)}\| \leq p'$. The dual operator $A := (\mathsf{C}^{p,w})' \in \mathcal{L}(\ell_{p'}(w^{-p'/p}))$ also satisfies $\|A\| \leq p'$ and is given by

$$Ay = \left(\sum_{k=n}^{\infty} \frac{y_k}{k}\right)_{n\in\mathbb{N}}, \quad y = (y_n)_{n\in\mathbb{N}} \in \ell_{p'}(w^{-p'/p}). \tag{18}$$

Step 1. $0 \notin \sigma_{pt}(A)$.

Observe that $Ay = 0$, for some $y \in \ell_{p'}(w^{-p'/p})$, implies that $z_n := \sum_{k=n}^{\infty} \frac{y_k}{k} = 0$ for all $n \in \mathbb{N}$. Hence, $y_n = n(z_n - z_{n+1}) = 0$, for $n \in \mathbb{N}$, and so A is injective.

Step 2. $\Sigma \subseteq \sigma_{pt}(A)$.

Let $\lambda \in \Sigma$, i.e., $\lambda = \frac{1}{m}$ for some $m \in \mathbb{N}$. Via (19) below, the non-zero vector $y = (y_n)_{n\in\mathbb{N}}$ defined via $y_1 \in \mathbb{C}\backslash\{0\}$ arbitrary, $y_n := y_1 \prod_{k=1}^{n-1} \left(1 - \frac{1}{\lambda k}\right)$ for $1 < n \leq m$ and $y_n := 0$ for $n > m$, which belongs to $\ell_{p'}(w^{-p'/p})$, satisfies $Ay = \lambda y$.

Step 3. $\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| < \frac{p'}{2s_p}\right\} \subseteq \sigma_{pt}(A)$.

Let $\lambda \in \mathbb{C} \setminus \{0\}$. Then $Ay = \lambda y$ for some non-zero $y \in \ell_{p'}(w^{-p'/p})$ if, and only if, $\lambda y_n = \sum_{k=n}^{\infty} \frac{y_k}{k}$ for all $n \in \mathbb{N}$. This yields, for every $n \in \mathbb{N}$, that $\lambda(y_n - y_{n+1}) = \frac{y_n}{n}$ and so $y_{n+1} = \left(1 - \frac{1}{\lambda n}\right) y_n$. It follows that

$$y_{n+1} = y_1 \prod_{k=1}^{n} \left(1 - \frac{1}{\lambda k}\right), \quad n \in \mathbb{N}, \tag{19}$$

with $y_1 \neq 0$. In particular, each eigenvalue of A is simple.

Let now $\lambda \in \mathbb{C}\backslash\Sigma$ satisfy $\left|\lambda - \frac{p'}{2s_p}\right| < \frac{p'}{2s_p}$ (equivalently, $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right) > \frac{s_p}{p'}$, i.e., $\alpha p' = \mathrm{Re}\left(\frac{p'}{\lambda}\right) > s_p$). For such a λ the vector $y = (y_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ defined by (19) actually belongs to $\ell_{p'}(w^{-p'/p})$. Indeed, via Lemma 3.2(i) there exists $c = c(\lambda) > 0$ such that

$$\prod_{k=1}^{n} \left|1 - \frac{1}{\lambda k}\right|^{p'} \leq c n^{-\mathrm{Re}(p'/\lambda)}, \quad n \in \mathbb{N}.$$

It then follows from (19) that

$$\begin{aligned}\sum_{n=1}^{\infty} |y_n|^{p'} w(n)^{-p'/p} &= |y_1|^{p'} w(1)^{-p'/p} + |y_1|^{p'} \sum_{n=2}^{\infty} \prod_{k=1}^{n} \left|1 - \frac{1}{\lambda k}\right|^{p'} w(n)^{-p'/p} \\ &\leq |y_1|^{p'} w(1)^{-p'/p} + c|y_1|^{p'} \sum_{n=2}^{\infty} n^{-\mathrm{Re}(p'/\lambda)} w(n)^{-p'/p},\end{aligned}$$

where the series $\sum_{n=2}^{\infty} n^{-\mathrm{Re}(p'/\lambda)} w(n)^{-p'/p}$ converges because $\mathrm{Re}(p'/\lambda) \in S_w(p)$, that is, $y \in \ell_{p'}(w^{-p'/p})$. Hence, $\lambda \in \sigma_{pt}(A)$.

Step 4. $\sigma_{pt}(A) \setminus \Sigma_0 \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\}$.

Fix $\lambda \in \sigma_{pt}(A) \setminus \Sigma_0$. According to (8) there exists $\beta = \beta(\lambda) > 0$ such that

$$\prod_{k=1}^{n} \left|1 - \frac{1}{\lambda k}\right|^{p'} \geq \beta \cdot n^{-\mathrm{Re}(p'/\lambda)}, \quad n \in \mathbb{N}. \tag{20}$$

But, as argued in Step 2 (for any $y_1 \in \mathbb{C} \setminus \{0\}$) the eigenvector $y = (y_n)_{n\in\mathbb{N}}$ corresponding to the eigenvalue λ of A, which necessarily belongs to $\ell_{p'}(w^{-p'/p})$, i.e., $\sum_{n=1}^{\infty} |y_n|^{p'} w(n)^{-p'/p} < \infty$, is given by (19). Then (20) implies that also $\sum_{n=1}^{\infty} \frac{1}{n^{\mathrm{Re}(p'/\lambda)} w(n)^{p'/p}} < \infty$, i.e., $\mathrm{Re}\left(\frac{p'}{\lambda}\right) \in S_w(p)$ and so $\mathrm{Re}\left(\frac{p'}{\lambda}\right) \geq s_p$. Equivalently, $\mathrm{Re}\left(\frac{1}{\lambda}\right) \geq \frac{s_p}{p'}$, i.e., $\lambda \in \left\{\mu \in \mathbb{C} \colon \left|\mu - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\}$.

It is clear that Steps 1-4 establish the two containments in (12) and (13).

For every $T \in \mathcal{L}(X)$ with X a Banach space, it is known that $\sigma_{pt}(T') \subseteq \sigma(T)$, [10, p.581], with $\sigma(T)$ a closed subset of $\mathbb{C}$. Accordingly, (14) follows from (12).

Step 5. $\sigma(\mathsf{C}^{(p,w)}) \subseteq \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\right\}$.

It suffices to show that every $\lambda \in \mathbb{C}$ with $\left|\lambda - \frac{p'}{2}\right| > \frac{p'}{2}$ belongs to $\rho(\mathsf{C}^{(p,w)})$. To do this we argue as in [7]. We recall the formula for $(\mathsf{C} - \lambda I)^{-1} \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ whenever $\lambda \notin \Sigma_0$, [19, p.266]. For $n \in \mathbb{N}$ the n–th row of the matrix for $(\mathsf{C} - \lambda I)^{-1}$ has the entries

$$\frac{-1}{n\lambda^2 \prod_{k=m}^{n} \left(1 - \frac{1}{\lambda k}\right)}, \quad 1 \leq m < n,$$

$$\frac{n}{1 - n\lambda} = \frac{1}{\frac{1}{n} - \lambda}, \quad m = n,$$

and all the other entries in row n are equal to 0. So, we can write

$$(\mathsf{C} - \lambda I)^{-1} = D_\lambda - \frac{1}{\lambda^2} E_\lambda, \tag{21}$$

where the diagonal operator $D_\lambda = (d_{nm})_{n,m\in\mathbb{N}}$ is given by $d_{nn} := \frac{1}{\frac{1}{n} - \lambda}$ and $d_{nm} := 0$ if $n \neq m$. The operator $E_\lambda = (e_{nm})_{n,m\in\mathbb{N}}$ is then the lower triangular matrix with $e_{1m} = 0$ for all $m \in \mathbb{N}$, and for every $n \geq 2$ with $e_{nm} := \frac{1}{n \prod_{k=m}^{n} (1 - \frac{1}{\lambda k})}$ if $1 \leq m < n$ and $e_{nm} := 0$ if $m \geq n$.

If $\lambda \notin \Sigma_0$, then $d(\lambda) := \mathrm{dist}(\lambda, \Sigma_0) > 0$ and $|d_{nn}| \leq \frac{1}{d(\lambda)}$ for $n \in \mathbb{N}$. Hence, for every $x \in \ell_p(w)$, we have

$$\begin{aligned} \|D_\lambda(x)\|_{p,w} &= \left(\sum_{n=1}^{\infty} |d_{nn} x_n|^p w(n)\right)^{1/p} \\ &\leq \frac{1}{d(\lambda)} \left(\sum_{n=1}^{\infty} |x_n|^p w(n)\right)^{1/p} = \frac{1}{d(\lambda)} \|x\|_{p,w}. \end{aligned}$$

This means that $D_\lambda \in \mathcal{L}(\ell_p(w))$. So, by (21) it remains to show that $E_\lambda \in \mathcal{L}(\ell_p(w))$ whenever $\lambda \in \mathbb{C}$ satisfies $\left|\lambda - \frac{p'}{2}\right| > \frac{p'}{2}$. To this end, we note that if $\lambda \in \mathbb{C} \setminus \Sigma_0$ then, with $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$, it follows from (8) that

$$|e_{n1}| \leq \frac{d^{-1}}{n^{1-\alpha}}, \quad n \geq 2,$$
$$|e_{nm}| \leq \frac{d^{-1}D'}{n^{1-\alpha}m^{\alpha}}, \quad 2 \leq m < n,$$

for some constants $d > 0$ and $D' > 0$ depending on λ. So, for every $\lambda \in \mathbb{C}\setminus\Sigma_0$ there exists $c = c(\lambda) > 0$ such that

$$|(E_\lambda(x))_n| \leq c(G_\lambda(|x|))_n, \quad x \in \mathbb{C}^{\mathbb{N}},\ n \in \mathbb{N}, \tag{22}$$

where $(G_\lambda(x))_n := \sum_{k=1}^{n} \frac{x_k}{n^{1-\alpha}k^{\alpha}}$ with $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$ and for all $x \in \mathbb{C}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Clearly (22) implies that $E_\lambda \in \mathcal{L}(\ell_p(w))$ whenever $G_\lambda \in \mathcal{L}(\ell_p(w))$.

Claim: $G_\lambda \in \mathcal{L}(\ell_p(w))$ *whenever* $\lambda \in \mathbb{C}$ *satisfies* $\left|\lambda - \frac{p'}{2}\right| > \frac{p'}{2}$.

To establish this claim fix $\lambda \in \mathbb{C}$ with $\left|\lambda - \frac{p'}{2}\right| > \frac{p'}{2}$. Then necessarily $\lambda \notin \Sigma_0$ with $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right) < \frac{1}{p'}$ and so $(1-\alpha)p > 1$. This implies that $\alpha < 1$. Observe that $G_\lambda \in \mathcal{L}(\ell_p(w))$ if, and only if, the operator $\tilde{G}_\lambda \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ given by

$$(\tilde{G}_\lambda(x))_n = w(n)^{1/p} \sum_{k=1}^{n} \frac{w(k)^{-1/p}}{n^{1-\alpha}k^{\alpha}} x_k, \quad x \in \mathbb{C}^{\mathbb{N}},\ n \in \mathbb{N},$$

defines a continuous linear operator on ℓ_p (the proof of this is along the lines of that of Lemma 2.1). To prove that indeed $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$ we need to distinguish the three cases; a) $\alpha = 0$; b) $\alpha < 0$ and c) $0 < \alpha < 1$ and establish relevant inequalities in each case.

Case a). Since w is decreasing, we have, for every $n \in \mathbb{N}$, that

$$\sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)}k^{\alpha p/(p-1)}} = \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)}} \leq \frac{n}{w(n)^{1/(p-1)}}$$

and hence, for every $m \in \mathbb{N}$, that

$$\sum_{n=1}^{m} \left(\frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)}} \right)^p \leq \sum_{n=1}^{m} \frac{1}{w(n)^{1/(p-1)}}. \tag{23}$$

Case b). Observe, for every $n \in \mathbb{N}$, that

$$\sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)}k^{\alpha p/(p-1)}} \leq \frac{1}{w(n)^{1/(p-1)}} \int_1^{n+1} x^{-\alpha p/(p-1)}\, dx$$
$$= \frac{1}{w(n)^{1/(p-1)}} \frac{((n+1)^{-\frac{\alpha p}{p-1}+1} - 1)}{-\frac{\alpha p}{p-1} + 1} \leq \frac{(p-1)}{(p(1-\alpha)-1)} \frac{(n+1)^{\frac{p(1-\alpha)-1}{p-1}}}{w(n)^{1/(p-1)}}.$$

Setting $c := \frac{p-1}{p(1-\alpha)-1} > 0$ it follows, for every $m \in \mathbb{N}$, that

$$\sum_{n=1}^{m}\left(\frac{w(n)^{1/p}}{n^{1-\alpha}}\sum_{k=1}^{n}\frac{1}{w(k)^{1/(p-1)}k^{\alpha p/(p-1)}}\right)^p \le c^p \sum_{n=1}^{m}\frac{(n+1)^{\frac{p[p(1-\alpha)-1]}{p-1}}}{w(n)^{1/(p-1)}n^{(1-\alpha)p}}$$
$$\le 2^{\frac{p(p(1-\alpha)-1)}{p-1}} c^p \sum_{n=1}^{m}\frac{1}{w(n)^{1/(p-1)}n^{\alpha p/(p-1)}}. \tag{24}$$

Case c). We have, for every $n \in \mathbb{N}$, still with $c = \frac{p-1}{p(1-\alpha)-1}$, that

$$\sum_{k=2}^{n}\frac{1}{w(k)^{1/(p-1)}k^{\alpha p/(p-1)}} \le \frac{1}{w(n)^{1/(p-1)}}\int_1^n \frac{1}{x^{\alpha p/(p-1)}}\,dx$$
$$= \frac{c}{w(n)^{1/(p-1)}}(n^{\frac{p(1-\alpha)-1}{p-1}} - 1).$$

Since $(1-\alpha)p > 1$ (i.e., $(1-\alpha)p - 1 > 0$) and $\alpha p > 0$ with $\frac{1}{w(1)} \le \frac{1}{w(n)}$, this implies, for every $n \in \mathbb{N}$, that

$$\left(\frac{w(n)^{1/p}}{n^{1-\alpha}}\sum_{k=1}^{n}\frac{1}{w(k)^{1/(p-1)}k^{\alpha p/(p-1)}}\right)^p$$
$$\le \left[\frac{w(n)^{1/p}}{n^{1-\alpha}w(1)^{1/(p-1)}} + \frac{w(n)^{1/p}c}{n^{1-\alpha}w(n)^{1/(p-1)}}(n^{\frac{p(1-\alpha)-1}{p-1}} - 1)\right]^p$$
$$\le \left[\frac{w(n)^{1/p}}{n^{1-\alpha}w(n)^{1/(p-1)}} + \frac{w(n)^{1/p}c}{n^{1-\alpha}w(n)^{1/(p-1)}}(n^{\frac{p(1-\alpha)-1}{p-1}} - 1)\right]^p$$
$$= \left[(1-c)\frac{w(n)^{1/p}}{n^{1-\alpha}w(n)^{1/(p-1)}} + \frac{w(n)^{1/p}c}{n^{1-\alpha}w(n)^{1/(p-1)}}n^{\frac{p(1-\alpha)-1}{p-1}}\right]^p$$
$$= \left(\frac{-\alpha p}{p(1-\alpha)-1}\frac{w(n)^{1/p}}{n^{1-\alpha}w(n)^{1/(p-1)}} + \frac{w(n)^{-1/p(p-1)}c}{n^{1-\alpha}}n^{\frac{p(1-\alpha)-1}{p-1}}\right)^p$$
$$\le \left(\frac{w(n)^{-1/p(p-1)}c}{n^{1-\alpha}}n^{\frac{p(1-\alpha)-1}{p-1}}\right)^p$$
$$= c^p w(n)^{-1/(p-1)}n^{-\alpha p/(p-1)}.$$

Hence, for every $m \in \mathbb{N}$, we have that

$$\sum_{n=1}^{m}\left(\frac{w(n)^{1/p}}{n^{1-\alpha}}\sum_{k=1}^{n}\frac{1}{w(k)^{1/(p-1)}k^{\alpha p/(p-1)}}\right)^p \le c^p\sum_{n=1}^{m}\frac{1}{w(n)^{1/(p-1)}n^{\alpha p/(p-1)}}. \tag{25}$$

The inequalities (23), (24) and (25) imply that $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$; indeed, in each case, suitable choices of a_n and b_k (with $p = q$) allow us to apply Theorem 2(ii) of [5]. This establishes the claim and hence, also Step 5.

Step 6. $\sigma(\mathsf{C}^{(p,w)}) \subseteq \{\lambda \in \mathbb{C}\colon |\lambda| \le \|\mathsf{C}^{(p,w)}\|\}$.

This is well known, [10, Ch.VII Lemma 3.4].

Steps 5 and 6 clearly yield (15). The proof of part (i) is thereby complete.

(ii) Suppose first that $R_w \neq \mathbb{R}$. Fix any $1 < p < \infty$.

Step 7. *Both of the inclusions in (16) are valid.*

The Cesàro operator $\mathsf{C}^{(p,w)}$ is clearly injective. So, $0 \notin \sigma_{pt}(\mathsf{C}^{(p,w)})$. Let $\lambda \in \mathbb{C} \setminus \{0\}$. Consider the equation $(\lambda I - \mathsf{C})x = 0$ with $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \setminus \{0\}$. Then $x_1 = \lambda x_1$ and $(2\lambda - 1)x_2 = x_1$ and $(n\lambda - 1)x_n = \lambda(n-1)x_{n-1}$ for all $n \ge 3$. If $m \in \mathbb{N}$ denotes the smallest positive integer such that $x_m \neq 0$, then it follows that $\lambda = \frac{1}{m}$ and so $x_n = \frac{n-1}{n-m}x_{n-1}$ for all $n > m$. Thus, we deduce that

$$x_n = x_{m+(n-m)} = \frac{(n-1)!}{(m-1)!(n-m)!}x_m, \quad n \ge m. \tag{26}$$

According to (9) we have $\frac{(n-1)!}{(m-1)!(n-m)!} \simeq \frac{1}{(m-1)!} \cdot n^{m-1}$, for each $m \in \mathbb{N}$. So, $x \in \ell_p(w)$ if, and only if, the series $\sum_{n=m+1}^{\infty} n^{(m-1)p}w(n)$ converges. But, the series $\sum_{n=m+1}^{\infty} n^{(m-1)p}w(n)$ converges precisely when $(m-1)p \in R_w$. In this case, $(m-1)p \le t_0$, i.e., $m \le \frac{t_0}{p} + 1$. So, $\sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \{\frac{1}{m}\colon m \in \mathbb{N},\ 1 \le m \le \frac{t_0}{p} + 1\}$.

Conversely, if $m < \frac{t_0}{p} + 1$ for some $m \in \mathbb{N}$, i.e., $(m-1)p < t_0$, then $(m-1)p \in R_w$ as $t_0 = \sup R_w$. Then the vector $x \in \mathbb{C}^{\mathbb{N}}$ defined according to (26), with $x_1 = \ldots = x_{m-1} = 0$ and for any arbitrary $x_m \neq 0$, belongs to $\ell_p(w)$. Therefore, $\frac{1}{m} \in \sigma_{pt}(\mathsf{C}^{(p,w)})$.

Step 8. *Assume now that $R_w = \mathbb{R}$. Then (17) is valid.*

Fix $1 < p < \infty$. As argued in Step 7, the point $\frac{1}{m} \in \sigma_{pt}(\mathsf{C}^{(p,w)})$ if and only if $(m-1)p \in R_w$. But, for $R_w = \mathbb{R}$, this is satisfied for every $m \in \mathbb{N}$ and so $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$. On the other hand, it is also shown in the proof of Step 7 that *every* eigenvalue λ of $\mathsf{C}\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ must have the form $\lambda = \frac{1}{m}$ for some $m \in \mathbb{N}$. Since every eigenvalue of $\mathsf{C}^{(p,w)}$ is also an eigenvalue of C (as $\ell_p(w) \subseteq \mathbb{C}^{\mathbb{N}}$), it follows that $\sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \Sigma$. □

Remark 1. (i) If $s_p \notin S_w(p)$, for some $1 < p < \infty$, then the argument of Step 4 in the proof of Theorem 3.3 implies that (12) reduces to the equality

$$\sigma_{pt}((\mathsf{C}^{(p,w)})') = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| < \frac{p'}{2s_p}\right\} \cup \Sigma.$$

Also, if $t_0 \notin R_w$, then (16) reduces to the equality

$$\sigma_{pt}(\mathsf{C}^{(p,w)}) = \left\{\frac{1}{m}\colon m \in \mathbb{N},\ 1 \le m < \frac{t_0}{p} + 1\right\},\ 1 < p < \infty.$$

(ii) For $w(n) = 1$ for all $n \in \mathbb{N}$, in which case $\ell_p(w) = \ell_p$ and $s_p = 1$, we have that $\mathsf{C}^{(p,w)} = \mathsf{C}^{(p)}$ for all $1 < p < \infty$ with $\|\mathsf{C}^{(p,w)}\| = \|\mathsf{C}^{(p)}\| = p'$. Then

(14) and (15) imply the known fact that

$$\sigma(\mathsf{C}^{(p)}) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2}\right| \le \frac{p'}{2}\right\}. \tag{27}$$

Since $t_0 = -1$, we also recover from (16) the known fact that $\sigma_{pt}(\mathsf{C}^{(p)}) = \emptyset$.

(iii) According to (14), for w positive, decreasing and with $S_w(p) \neq \emptyset$ we have

$$\frac{p'}{s_p} \le \max\{1, \frac{p'}{s_p}\} \le \|\mathsf{C}^{(p,w)}\| \le p'. \tag{28}$$

In particular, whenever $s_p = 1$ (see e.g., Example 3(i) below), the inequalities in (28) imply that necessarily $\|\mathsf{C}^{(p,w)}\| = p'$ is as large as possible.

For the special case when $w(n) = \frac{1}{n^\alpha}$, $n \in \mathbb{N}$, for some $\alpha > 0$, direct calculation yields that $s_p = 1 + \frac{\alpha p'}{p}$ and so $S_w(p) \neq \emptyset$ for all $1 < p < \infty$. It follows that

$$\frac{p'}{s_p} = \frac{p}{\alpha + p - 1} = m_1,$$

where m_1 occurs in the lower bound for $\|\mathsf{C}^{(p,w)}\|$ as given in (5); see Proposition 2.3. Hence, (28) yields that $m_1 \le \|\mathsf{C}^{(p,w)}\|$. Combined with Example 1(iii) we can conclude that

$$\max\{m_1, m_2\} \le \|\mathsf{C}^{(p,w)}\|.$$

This provides an alternate proof, to that in [12], of the same estimate in (5).

(iv) An examination of the argument for Step 2 in the proof of Theorem 3.3(i) shows that the assumption $S_w(p) \neq \emptyset$ is not used there, i.e., we always have

$$\Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(p,w)})')$$

for every $1 < p < \infty$ and every positive, decreasing weight w.

We now present some relevant examples.

EXAMPLE 3. (i) Suppose that $w(n) = \frac{1}{(\log(n+1))^\gamma}$ for $n \in \mathbb{N}$ with $\gamma \ge 0$. Then $\sum_{n=1}^\infty \frac{1}{n^s w(n)^{p'/p}} < \infty$ if and only if $s > 1$ and hence, $s_p = 1$ for every $1 < p < \infty$. In view of Remark 1(iii) we have that $\|\mathsf{C}^{(p,w)}\| = p'$. Moreover, $\sum_{n=1}^\infty n^t w(n) < \infty$ if and only if $t < -1$ or $t \le -1$ in case $\gamma > 1$. Hence, $t_0 = -1$. According to Theorem 3.3 we have, for each $1 < p < \infty$, that

$$\sigma(\mathsf{C}^{(p,w)}) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2}\right| \le \frac{p'}{2}\right\}, \quad \sigma_{pt}(\mathsf{C}^{(p,w)}) = \emptyset.$$

In particular, equality may occur in (15). For the case when $\gamma = 0$ (so that $w(n) = 1$ for $n \in \mathbb{N}$), we recover the known result about the spectrum of $\mathsf{C}^{(p)} \in \mathcal{L}(\ell_p)$, for $1 < p < \infty$, [6], [15].

(ii) More generally, suppose that $w(n) = \frac{1}{n^\beta (\log(n+1))^\gamma}$ for $n \in \mathbb{N}$ with $\beta \geq 0$ and $\gamma \geq 0$. Then $\sum_{n=1}^\infty \frac{1}{n^s w(n)^{p'/p}} < \infty$ if and only if $s > \beta \frac{p'}{p} + 1$ and so $s_p = \beta \frac{p'}{p} + 1$ for every $1 < p < \infty$. Moreover, $\sum_{n=1}^\infty n^t w(n) < \infty$ if and only if $t < (\beta - 1)$ or $t \leq (\beta - 1)$ in case $\gamma > 1$. Hence, $t_0 = \beta - 1$. According to Theorem 3.3 we have, for each $1 < p < \infty$, that

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2((\beta p'/p) + 1)}\right| \leq \frac{p'}{2((\beta p'/p) + 1)}\right\} \cup \Sigma \subseteq \sigma(\mathsf{C}^{(p,w)})$$

and

$$\sigma_{pt}(\mathsf{C}^{(p,w)}) = \left\{\frac{1}{m}\colon m \in \mathbb{N},\ 1 \leq m < \frac{\beta - 1}{p} + 1\right\}.$$

In particular, $\sigma_{pt}(\mathsf{C}^{(p,w)}) = \emptyset$ whenever $\beta \in [0,1]$. We claim that actually

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2((\beta p'/p) + 1)}\right| \leq \frac{p'}{2((\beta p'/p) + 1)}\right\} \cup \Sigma = \sigma(\mathsf{C}^{(p,w)}), \quad (29)$$

which shows that equality may occur in (14).

Keeping in mind the argument for Step 5 in the proof of Theorem 3.3, to verify (29) it suffices to prove that every $\lambda \in \mathbb{C}\backslash\{0\}$ satisfying $\left|\lambda - \frac{p'}{2((\beta p'/p)+1)}\right| > \frac{p'}{2((\beta p'/p)+1)}$ belongs to $\rho(\mathsf{C}^{(p,w)})$, i.e., that the operator $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$. So, fix such a λ and note that $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right) < \left(\beta \frac{p'}{p} + 1\right)/p' = \frac{\beta}{p} + \frac{1}{p'}$. We also observe, for our particular w, that the operator $\tilde{G}_\lambda$ is given by

$$(\tilde{G}_\lambda(x))_n = \frac{1}{n^{1-\alpha+(\beta/p)} \log^{\gamma/p}(n+1)} \sum_{k=1}^n \frac{x_k}{k^{\alpha-(\beta/p)} \log^{-\gamma/p}(k+1)},$$

for $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^\mathbb{N}$. So, $\tilde{G}_\lambda$ is given by the factorable matrix with $a_n := n^{-(1-\alpha+(\beta/p))} \log^{-\gamma/p}(n+1)$ and $b_k := k^{-(\alpha-(\beta/p))} \log^{\gamma/p}(k+1)$, where $\alpha < \frac{\beta}{p} + \frac{1}{p'} = \frac{\beta}{p} + 1 - \frac{1}{p}$ implies that $1 - \alpha + \frac{\beta}{p} > \frac{1}{p}$ and we have that $\left(1 - \alpha + \frac{\beta}{p}\right) + \left(\alpha - \frac{\beta}{p}\right) = 1 = \frac{1}{p} + \frac{1}{p'}$ and also that $\left(\frac{\gamma}{p}\right) + \left(-\frac{\gamma}{p}\right) = 0$. According to Corollary 9(ii) of [5] it follows that $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$ and the claim is proved.

Finally, since $s_p = \frac{\beta+p-1}{p-1}$, it follows from (28) that

$$p' \cdot \frac{p-1}{\beta + p - 1} \leq \|\mathsf{C}^{(p,w)}\| \leq p', \quad 1 < p < \infty,$$

with $\frac{p-1}{\beta+p-1} \uparrow 1$ for $\beta \downarrow 0$. This example also shows that the inequality $t_0 \leq \frac{s_p p'}{p}$ (cf. Proposition 3.1(i)) can be strict. For $\beta \downarrow 0$ it follows from (14) and (15) that

$$\sigma(\mathsf{C}^{(p,w)}) \uparrow \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2}\right| < \frac{p'}{2}\right\},$$

whose closure equals $\sigma(\mathsf{C}^{(p)}) = \sigma(\mathsf{C}^{(p,w)})$ for w as in (i).

It is clear from (16) that $\mathsf{C}^{(p,w)}$ has at most finitely many eigenvalues whenever $t_0 \in \mathbb{R}$. The following result characterizes when $\sigma_{pt}(\mathsf{C}^{(p,w)})$ is an *infinite set*; see also Remark 2(i) below. Recall that a sequence $u = (u_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ is *rapidly decreasing* if $(n^m u_n)_{n\in\mathbb{N}} \in \ell_1$ for every $m \in \mathbb{N}$. The space of all rapidly decreasing, $\mathbb{C}$-valued sequences is usually denoted by s.

PROPOSITION 3.4. *Let $w = (w(n))_{n\in\mathbb{N}}$ be a positive, decreasing sequence.*
(i) *The following assertions are equivalent.*

(1) $R_w = \mathbb{R}$.

(2) $(n^m w(n))_{n\in\mathbb{N}} \in \ell_1$ *for all* $m \in \mathbb{N}$.

(3) $(n^m w(n))_{n\in\mathbb{N}} \in c_0$ *for all* $m \in \mathbb{N}$.

(4) $w \in s$.

(ii) *For each $1 < p < \infty$, the following assertions are equivalent.*

(5) $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$.

(6) $(n^m w(n))_{n\in\mathbb{N}} \in \ell_p$ *for all* $m \in \mathbb{N}$.

(iii) *Any one of the equivalent assertions* (1)-(4) *implies that both* (5) *and* (6) *are valid, for every $1 < p < \infty$.*
(iv) *If* (6) *holds for some $1 < p < \infty$, then each assertion* (1)-(4) *is satisfied.*

PROOF. (i) (1)⇔(2) follows from the definition of R_w.
(2)⇔(3). That (2)⇒(3) is immediate from $\ell_1 \subseteq c_0$.
Assume (3). Fix $t \in \mathbb{N}$ and set $m = t + 2$. Then $(n^m w(n))_{n\in\mathbb{N}} \in c_0$ implies that $\sup_{n\in\mathbb{N}} n^m w(n) < \infty$. Accordingly,

$$\sum_{n=1}^{\infty} n^t w(n) = \sum_{n=1}^{\infty} \frac{1}{n^2} n^m w(n) \leq \frac{\pi^2}{6} \sup_{n\in\mathbb{N}} n^m w(n) < \infty.$$

Since t is arbitrary, we can conclude that (2) holds.
(2)⇔(4). Clear from the definition of the space s.
(ii) Since $\mathsf{C}^{(p,w)}$ is injective, $0 \notin \sigma_{pt}(\mathsf{C}^{(p,w)})$. By (9) and (26), $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $\mathsf{C}^{(p,w)}$ if and only if $\lambda = \frac{1}{m}$ for some $m \in \mathbb{N}$ with the corresponding 1-dimensional eigenspace generated by a vector $x^{[m]} = (x_n^{[m]})_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ satisfying $x_n^{[m]} \simeq n^{m-1}$. So, $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$ if and only if $(n^{m-1})_{n\in\mathbb{N}} \in \ell_p(w)$ for all $m \in \mathbb{N}$, that is, if and only if $(n^m w(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p$ for all $m \in \mathbb{N}$, which is equivalent to (6) via Lemma 3.2(iii).
(iii) Follows immediately from parts (i) and (ii) and the fact that (2)⇒(6) since $\ell_1 \subseteq \ell_p$ for every $1 < p < \infty$.
(iv) Immediate from $\ell_p \subseteq c_0$ for every $1 < p < \infty$. □

Given a decreasing sequence $w = (w(n))_{n\in\mathbb{N}}$ of positive real numbers, set $\alpha_n := -\log w(n)$, for $n \in \mathbb{N}$. Then $w(n) = e^{-\alpha_n}$, for $n \in \mathbb{N}$. Moreover, $\alpha_n \to \infty$ for $n \to \infty$ if and only if $w(n) \to 0$ for $n \to \infty$.

COROLLARY 3.5. *Let $w = (w(n))_{n\in\mathbb{N}}$ be a decreasing, positive sequence.*

(i) *If $w \in s$, then $\lim_{n\to\infty} \frac{\log n}{\alpha_n} = 0$.*

(ii) *If $\lim_{n\to\infty} \frac{\log n}{\alpha_n} = 0$ and $w(N) < 1$ for some N, then $w \in s$.*

PROOF. (i) Since $w \in s$, condition (3) in Proposition 3.4 implies that

$$\forall m \in \mathbb{N}\ \exists n_m \in \mathbb{N}\ \forall n \geq n_m\colon\ n^m w(n) = \frac{n^m}{e^{\alpha_n}} < 1,$$

i.e., that

$$\forall m \in \mathbb{N}\ \exists n_m \in \mathbb{N}\ \forall n \geq n_m\colon\ n^m < e^{\alpha_n}.$$

It follows that

$$\forall m \in \mathbb{N}\ \exists n_m \in \mathbb{N}\ \forall n \geq n_m\colon\ m \log n < \alpha_n.$$

This implies that necessarily $\alpha_n > 0$ for all $n \geq n_m$ and so

$$\forall m \in \mathbb{N}\ \exists n_m \in \mathbb{N}\ \forall n \geq n_m\colon\ \frac{\log n}{\alpha_n} < \frac{1}{m}.$$

This means precisely that $\lim_{n\to\infty} \frac{\log n}{\alpha_n} = 0$.

(ii) Fix $m \in \mathbb{N}$. Then there is $n_0 \in \mathbb{N}$ with $n_0 \geq N$ such that $\frac{\log n}{\alpha_n} < \frac{1}{m+1}$ for all $n \geq n_0$. Since $w(N) < 1$ implies that $\alpha_n = -\log w(n) > 0$ for all $n \geq n_0$, we can conclude that $(m+1)\log n < \alpha_n$, i.e., $n^{m+1}w(n) < 1$ for all $n \geq n_0$. So, $\sup_{n\in\mathbb{N}} n^{m+1}w(n) < \infty$. It follows that

$$n^m w(n) \leq \frac{1}{n} \sup_{r\in\mathbb{N}} r^{m+1} w(r), \quad n \in \mathbb{N},$$

with $\frac{1}{n}\sup_{r\in\mathbb{N}} r^{m+1}w(r) \to 0$ as $n \to \infty$. By (3)$\Leftrightarrow$(4) in Proposition 3.4(i) it follows that $w \in s$. □

REMARK 2. (i) Concerning condition (5) in Proposition 3.4 (for any given $1 < p < \infty$), we claim that the *entire* set $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$ whenever $\sigma_{pt}(C^{(p,w)})$ is an infinite set. To see this, suppose that $\frac{1}{m} \in \sigma_{pt}(C^{(p,w)})$ for some $m \in \mathbb{N}$. According to the argument in Step 7 of the proof of Theorem 3.3, we can conclude that $(n^{m-1})_{n\in\mathbb{N}} \in \ell_p(w)$. So, for all $1 \leq k < m$, it follows that

$$\sum_{n=1}^{\infty} (n^k)^p w(n) \leq \sum_{n=1}^{\infty} (n^{m-1})^p w(n) < \infty$$

and hence, via (9), that the vector $(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ given by (26), with k in place of m, also belongs to $\ell_p(w)$, i.e., it is an eigenvector of $\mathsf{C}^{(p,w)}$ corresponding to $\lambda = \frac{1}{k}$. This shows that $\{\frac{1}{k}\}_{k=1}^{m} \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$ whenever $\frac{1}{m} \in \sigma_{pt}(\mathsf{C}^{(p,w)})$, which clearly implies the stated claim.

(ii) Let $1 < p_0 < \infty$. The constant vector $\mathbf{1} := (1,1,\ldots) \in \mathbb{C}^{\mathbb{N}}$ satisfies $\mathsf{C}\mathbf{1} = \mathbf{1}$ and so $1 \in \sigma_{pt}(C^{(p_0,w)})$ if and only if, $\mathbf{1} \in \ell_{p_0}(w)$, i.e., if, and only if, $w \in \ell_1$. In this case, $1 \in \sigma_{pt}(C^{(p,w)})$ for every $1 < p < \infty$. Then Theorem 3.3(ii) implies that necessarily $t_0 \in (0,\infty]$.

(iii) Let $w(n) = \frac{1}{n^\alpha}$, for all $n \in \mathbb{N}$ and some $\alpha > 0$. Then $\sum_{n=1}^\infty n^t w(n) < \infty$ if, and only if, $t < (\alpha-1)$ and so $t_0 = (\alpha-1)$. In particular, $R_w \neq \mathbb{R}$. Moreover, for any $1 < p < \infty$, we have

$$\left\{\frac{1}{m}\colon m\in\mathbb{N},\ 1\le m<\frac{t_0}{p}+1\right\} = \left\{\frac{1}{m}\colon m\in\mathbb{N},\ 1\le m<\frac{(\alpha-1)}{p}+1\right\}.$$

So, given any $1 < p < \infty$, it is possible to choose an appropriate $\alpha > 0$ such that $\sigma_{pt}(\mathsf{C}^{(p,w)})$ is a *finite* set with any pre-assigned cardinality; see (16).

(iv) Condition (1) of Proposition 3.4, i.e., $R_w = \mathbb{R}$, implies that necessarily $S_w(p) = \emptyset$ for every $1 < p < \infty$; see Proposition 3.1(i).

Let $w = (w(n))_{n\in\mathbb{N}}$ be any decreasing, (strictly) positive sequence and let $1 < p < \infty$. The Cesàro operator $\mathsf{C}^{(p,w)}$ is similar (via an isometry) to an operator $T_w \in \mathcal{L}(\ell_p)$ which is defined by the factorable matrix $A(w) = (a_{nk})_{n,k\in\mathbb{N}}$ with entries $a_{nk} = a_n b_k = \frac{w(n)^{1/p}}{n} \cdot w(k)^{-1/p}$ for $1 \le k \le n$ and $a_{nk} = 0$ for $k > n$ (see the proof of Lemma 2.1). In particular, $\sigma(\mathsf{C}^{(p,w)}) = \sigma(T_w)$. Moreover, the matrix $A(w)$ satisfies the following two conditions:

(i) $\sup_{n\in\mathbb{N}} \sum_{k=1}^\infty |a_{nk}| = \sup_{n\in\mathbb{N}} \frac{w(n)^{1/p}}{n} \sum_{k=1}^n w(k)^{-1/p} \le 1$, because w decreasing implies that $\sum_{k=1}^n w(k)^{-1/p} \le n w(n)^{-1/p}$, $n \in \mathbb{N}$, and

(ii) $f_k := \lim_{n\to\infty} a_{nk} = w(k)^{-1/p} \lim_{n\to\infty} \frac{w(n)^{1/p}}{n} = 0$, $k \in \mathbb{N}$, because $w \in \ell_\infty$.

If, in addition, the matrix $A(w)$ also satisfies the condition

(iii) $\alpha := \lim_{n\to\infty} \sum_{k=1}^\infty a_{nk} = \lim_{n\to\infty} \frac{w(n)^{1/p}}{n} \sum_{k=1}^n w(k)^{-1/p}$ exists,

then the linear operator corresponding to $A(w)$ is a selfmap of c, the space of all convergent sequences, that is, $A(w)$ is conservative, [20, p.112].

Suppose now that the matrix $A(w)$ satisfies condition (iii) with $\alpha = 1$. Then $A(w)$ is *regular* and the linear operator corresponding to $A(w)$ is limit preserving over c, [20, p.114]. Define $\eta := \limsup_{n\to\infty} a_n b_n$. For the operator T_w (which is similar to the Cesàro operator $\mathsf{C}^{(p,w)}$) it turns out that $\eta = 0$ and so a result of Rhoades and Yildirim [20, Theorem 3] yields that

$$\left\{\lambda\in\mathbb{C}\colon \left|\lambda-\frac{1}{2}\right|\le\frac{1}{2}\right\} \subseteq \sigma(\mathsf{C}^{(p,w)}), \tag{30}$$

after noting that $S := \overline{\{a_n b_n : n \in \mathbb{N}\}} = \Sigma_0 \subseteq \{\lambda \in \mathbb{C}\colon |\lambda - \frac{1}{2}| \leq \frac{1}{2}\}$.

It is worthwhile to compare (14) with (30). So, let $1 < p < \infty$ and w be a positive, decreasing sequence such that $S_w(p) \neq \emptyset$. Then

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\} \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\} \subseteq \sigma(\mathsf{C}^{(p,w)})$$

with the first inclusion holding if, and only if, $s_p \leq p'$. Observe that if $\left(\frac{w(n)^{-1/p}}{n}\right)_{n\in\mathbb{N}} \in \ell_{p'}$, then $s_p \leq p'$ is valid and conversely, if $s_p < p'$, then $\left(\frac{w(n)^{-1/p}}{n}\right)_{n\in\mathbb{N}} \in \ell_{p'}$. In this case, (14) is a better inclusion than (30). For instance, if $w(n) := \frac{1}{n^r}$ for all $n \in \mathbb{N}$ and some $r > 0$, then $\left(\frac{w(n)^{-1/p}}{n}\right)_{n\in\mathbb{N}} \in \ell_{p'}$ if, and only if, $r < 1$. On the other hand, the reverse inclusion

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\} \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$$

holds if, and only if, $p' \leq s_p$. Observe that if $\left(\frac{w(n)^{-1/p}}{n}\right)_{n\in\mathbb{N}} \notin \ell_{p'}$, then $p' \leq s_p$ is valid and conversely, if $p' < s_p$, then $\left(\frac{w(n)^{-1/p}}{n}\right)_{n\in\mathbb{N}} \notin \ell_{p'}$. In this case, modulo the additional requirement that $\alpha = 1$ (see condition (iii)), in which case (30) is actually valid, we see that (30) is a better inclusion than (14).

The following example shows that condition (iii) above and the property $S_w(p) \neq \emptyset$ can be compatible.

Example 4. Fix $1 < p < \infty$. For each $n \in \mathbb{N}$ set $w(n) = \frac{1}{(\log(n+1))^p}$, in which case $w(n) \downarrow 0$. Then $S_w(p) = (1, \infty)$ and

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\right\} = \sigma(C^{(p,w)}) \quad \text{with} \quad \sigma_{pt}(C^{(p,w)}) = \emptyset;$$

see Example 3(i) with $\gamma = p$. Moreover, concerning condition (iii) observe that

$$\frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} w(k)^{-1/p} = \frac{1}{n \log(n+1)} \sum_{k=1}^{n} \log(k+1), \quad n \in \mathbb{N}.$$

The inequalities

$$[(n+1)\log(n+1) - n] \leq \sum_{k=1}^{n} \log(k+1) \leq [(n+2)\log(n+2) - n - 2\log 2], \quad n \in \mathbb{N},$$

then imply that

$$\alpha = \lim_{n\to\infty} \frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} w(k)^{-1/p} = 1.$$

Note also that $\left(\frac{w(n)^{-1/p}}{n}\right)_{n\in\mathbb{N}} = \left(\frac{\log(n+1)}{n}\right)_{n\in\mathbb{N}} \in \ell_{p'}$.

We conclude this section with some comments about the mean ergodicity and the linear dynamics of $\mathsf{C}^{(p,w)}$. For X a Banach space, recall that $T \in \mathcal{L}(X)$ is *mean ergodic* if its sequence of Cesàro averages $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m$, for $n \in \mathbb{N}$, converges to some operator $P \in \mathcal{L}(X)$ for the strong operator topology, i.e., $\lim_{n\to\infty} T_{[n]}x = Px$ for each $x \in X$, [10, Ch.VIII]. Since $\frac{1}{n}T^n = T_{[n]} - \frac{n-1}{n}T_{[n-1]}$, for $n \in \mathbb{N}$ (with $T_{[0]} := I$), a necessary condition for T to be mean ergodic is that $\lim_{n\to\infty} \frac{1}{n}T^n = 0$ (in the strong operator topology).

Let w be a positive, decreasing sequence and $1 < p < \infty$ with $S_p(w) \neq \emptyset$. If $s_p < p'$, then it follows from (12) that $\mu := \frac{1}{2}\left(1 + \frac{p'}{s_p}\right) \in \sigma_{pt}((\mathsf{C}^{(p,w)})')$ and so there exists a non-zero vector $x' \in \ell_{p'}(w^{-p'/p})$ such that $(\mathsf{C}^{(p,w)})'x' = \mu x'$. Choose any $x \in \ell_p(w) \setminus \{0\}$ satisfying $\langle x, x'\rangle \neq 0$. Then

$$\langle \frac{1}{n}(\mathsf{C}^{(p,w)})^n x, x'\rangle = \frac{1}{n}\langle x, ((\mathsf{C}^{(p,w)})')^n x'\rangle = \frac{\mu^n}{n}\langle x, x'\rangle, \quad n \in \mathbb{N},$$

with $\mu > 1$ and so the set $\left\{\frac{1}{n}(\mathsf{C}^{(p,w)})^n x \colon n \in \mathbb{N}\right\}$ is unbounded in $\ell_p(w)$. In particular, the sequence $\left\{\frac{1}{n}(\mathsf{C}^{(p,w)})^n\right\}_{n\in\mathbb{N}}$ cannot converge to 0 for the strong operator topology in $\mathcal{L}(\ell_p(w))$. Accordingly, $\mathsf{C}^{(p,w)}$ *fails to be mean ergodic whenever* $s_p < p'$. This is the case when $w(n) = 1$, for all $n \in \mathbb{N}$, in which case $s_p = 1$, and we recover the known fact that the classical Cesàro operator $\mathsf{C}^{(p)}$ fails to be mean ergodic for every $1 < p < \infty$; see [3, Section 4], where it is also shown that the Cesàro operator fails to be mean ergodic in the classical Banach sequence spaces c_0, c, ℓ_p $(1 < p \leq \infty)$, bv_0 and bv but, that it is mean ergodic in bv_p $(1 < p < \infty)$. For w as in Example 3(i) we recall that also $s_p = 1$, for every $1 < p < \infty$, and so $\mathsf{C}^{(p,w)}$ is not mean ergodic.

Concerning the dynamics of a continuous linear operator T defined on a separable Banach space X, recall that T is *hypercyclic* if there exists $x \in X$ such that the orbit $\{T^n x \colon n \in \mathbb{N}_0\}$ is dense in X. If, for some $x \in X$, the projective orbit $\{\lambda T^n x \colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then X is called *supercyclic*. Clearly, hypercyclicity always implies supercyclicity.

Let now w be a positive, decreasing sequence and $1 < p < \infty$. According to Remark 1(iv) the *infinite* set $\Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(p,w)})')$. Then, by a result of Ansari and Bourdon [4, Theorem 3.2], $\mathsf{C}^{(p,w)}$ *is not supercyclic and hence, also not hypercyclic.*

4. Compactness of $\mathsf{C}^{(p,w)}$

According to (27), for each $1 < p < \infty$ the classical Cesàro operator $\mathsf{C}^{(p)} \in \mathcal{L}(\ell_p)$ is surely not compact. However, in the presence of a positive weight $w \downarrow 0$, this may no longer be the case for $\mathsf{C}^{(p,w)}$ acting on $\ell_p(w)$. We begin with the following fact.

PROPOSITION 4.1. *Let w be a positive, decreasing weight.*

(i) *For every* $1 < p < \infty$ *we have* $\Sigma \subseteq \sigma(\mathsf{C}^{(p,w)})$.

(ii) *Suppose that* $\mathsf{C}^{(p,w)}$ *is a compact operator, for some* $1 < p < \infty$. *Then*

$$\sigma(\mathsf{C}^{(p,w)}) = \Sigma_0 \quad \textit{and} \quad \sigma_{pt}(\mathsf{C}^{(p,w)}) = \Sigma. \tag{31}$$

Moreover, $w \in s$ *and* $r(\mathsf{C}^{(p,w)}) < \|\mathsf{C}^{(p,w)}\|$.

PROOF. (i) According to Remark 1(iv) we have $\Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(p,w)})')$. But, always $\sigma_{pt}((\mathsf{C}^{(p,w)})') \subseteq \sigma(\mathsf{C}^{(p,w)})$, [10, p. 581], and so $\Sigma \subseteq \sigma(\mathsf{C}^{(p,w)})$.

(ii) Since $\mathsf{C}^{(p,w)}$ is injective, $0 \notin \sigma_{pt}(\mathsf{C}^{(p,w)})$. The compactness of $\mathsf{C}^{(p,w)}$ then implies that $\sigma_{pt}(\mathsf{C}^{(p,w)}) = \sigma(\mathsf{C}^{(p,w)}) \setminus \{0\}$, [16, Theorem 3.4.23]. According to the proof of Step 8 for Theorem 3.3 we also have that $\sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \Sigma$. In view of part (i), the equalities in (31) follow.

By Theorem 3.3(ii) we must have $R_w = \mathbb{R}$ (if not, then t_0 is finite and so (16) would imply that $\sigma_{pt}(\mathsf{C}^{(p,w)})$ is finite which is a contradiction to (31). Then, via Proposition 3.4(i), we can conclude that $w \in s$.

It follows from (4) and the equality $r(\mathsf{C}^{(p,w)}) = 1$ (see (31)) that $r(\mathsf{C}^{(p,w)}) < \|\mathsf{C}^{(p,w)}\|$. □

To decide when $\mathsf{C}^{(p,w)}$ is compact, first observe that $\mathsf{C}^{(p,w)} = \Phi_w^{-1} T_w \Phi_w$ (see Lemma 2.1 and its proof), where $T_w \in \mathcal{L}(\ell_p)$ is given by (3). Given any $x \in B_p := \{x \in \ell_p \colon \|x\| \leq 1\}$ and $i \in \mathbb{N}$, it follows from Hölder's inequality that

$$\sum_{n=i}^{\infty} |(T_w x)_n|^p = \sum_{n=i}^{\infty} \frac{w(n)}{n^p} \left| \sum_{k=1}^{n} \frac{1}{w(k)^{1/p}} \cdot x_k \right|^p \leq \sum_{n=i}^{\infty} \frac{w(n)}{n^p} \left(\sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \right)^{p/p'}.$$

So, T_w (hence, also $\mathsf{C}^{(p,w)}$) will be compact whenever w satisfies the following

$$\text{Compactness criterion:} \qquad \sum_{n=1}^{\infty} \frac{w(n)}{n^p} \left(\sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \right)^{p/p'} < \infty. \tag{32}$$

Indeed, (32) implies that $\lim_{i\to\infty} \sum_{n=i}^{\infty} |(T_w x)_n|^p = 0$ *uniformly* with respect to $x \in B_p$, from which the relative compactness in ℓ_p of the bounded set $T_w(B_p) \subseteq \ell_p$ follows, [10, pp.338-339].

We introduce some notation. Let w be a positive, decreasing sequence. Define

$$A_n(p,w) := w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}}, \quad n \in \mathbb{N},\ 1 < p < \infty.$$

The compactness criterion (32) then states that $\mathsf{C}^{(p,w)}$ is a compact operator if $\sum_{n=1}^{\infty} (A_n(p,w))^{p/p'}/n^p < \infty$.

THEOREM 4.2. *Suppose, for some $1 < p < \infty$, that there exist constants $M > 0$ and $0 \leq \alpha < 1$ such that*

$$A_n(p, w) \leq M n^{\alpha}, \quad n \in \mathbb{N}. \tag{33}$$

Then $\mathsf{C}^{(q,w)}$ is a compact operator for every $1 < q \leq p$. In particular, $w \in s$.

PROOF. Observe, for fixed $1 < q \leq p$, that

$$\gamma := \frac{q'}{q} - \frac{p'}{p} = \frac{1}{q-1} - \frac{1}{p-1} = \frac{p-q}{(q-1)(p-1)} \geq 0.$$

For each $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} \frac{1}{w(k)^{q'/q}} = \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \cdot w(k)^{-\gamma}.$$

Accordingly, for each $n \in \mathbb{N}$,

$$\begin{aligned} A_n(q, w) &= \frac{w(n)^{q'/q}}{w(n)^{p'/p}} \cdot w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \cdot w(k)^{-\gamma} \\ &= w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \cdot \left(\frac{w(n)}{w(k)}\right)^{\gamma}. \end{aligned}$$

Since w is decreasing, $\frac{w(n)}{w(k)} \leq 1$ for all $1 \leq k \leq n$ and so

$$A_n(q, w) \leq w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} = A_n(p, w) \leq M n^{\alpha}.$$

Accordingly,

$$\sum_{n=1}^{\infty} \frac{(A_n(q, w))^{q/q'}}{n^q} \leq M^{q/q'} \sum_{n=1}^{\infty} \frac{n^{\alpha q/q'}}{n^q} = M^{q/q'} \sum_{n=1}^{\infty} \frac{1}{n^{q-(\alpha q/q')}}.$$

But, $q - \frac{\alpha q}{q'} = q - \alpha(q-1) = q(1-\alpha) + \alpha > (1-\alpha) + \alpha = 1$ and so

$$\sum_{n=1}^{\infty} \frac{(A_n(q, w))^{q/q'}}{n^q} < \infty.$$

Then the compactness criterion yields that $\mathsf{C}^{(q,w)}$ is a compact operator.

That $w \in s$ is a consequence of Proposition 4.1(ii). □

The following consequence of Theorem 4.2 leads to a rich supply of weights w for which $\mathsf{C}^{(p,w)}$ is compact.

Corollary 4.3. *Let w be a positive weight with $w \downarrow 0$. If the limit*

$$l := \lim_{n\to\infty} \frac{w(n)}{w(n-1)} \tag{34}$$

exists in $\mathbb{R} \setminus \{1\}$, then $\mathsf{C}^{(p,w)}$ is compact for every $1 < p < \infty$.

Proof. Fix $1 < p < \infty$. According to Theorem 4.2 (with $\alpha = 0$) it suffices to prove that $\sup_{n\in\mathbb{N}} A_n(p,w) < \infty$. Set $a_n := \sum_{k=1}^{n} w(k)^{-p'/p}$ and $b_n := w(n)^{-p'/p}$ for $n \in \mathbb{N}$. Since $w \downarrow 0$, we have $b_n \uparrow \infty$. Moreover, the limit

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} &= \lim_{n\to\infty} \frac{w(n)^{-p'/p}}{w(n)^{-p'/p} - w(n-1)^{-p'/p}}\\ &= \lim_{n\to\infty} \frac{1}{1-(w(n)/w(n-1))^{p'/p}} = \frac{1}{1-l^{p'/p}}\end{aligned}$$

exists in $\mathbb{R}$ as $l \neq 1$. According to the Stolz-Cesàro criterion, [17, Theorem 1.22], it follows that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1/(1-l^{p'/p}) \in \mathbb{R}$, i.e., $\lim_{n\to\infty} A_n(p,w) = 1/(1-l^{p'/p}) \in \mathbb{R}$. In particular, $\sup_{n\in\mathbb{N}} A_n(p,w) < \infty$ is indeed satisfied. □

Remark 3. (i) Let w be a positive, decreasing weight.

(a) According to (14), if $\mathsf{C}^{(p,w)}$ is a compact operator for some $1 < p < \infty$, then $S_w(p) = \emptyset$.

(b) The condition $w \downarrow 0$ by itself need not imply that $S_w(p) = \emptyset$ (see Examples 3, for instance).

(ii) Suppose $S_w(p) \neq \emptyset$ for some $1 < p < \infty$. Then $\mathsf{C}^{(q,w)}$ fails to be compact for every $q \in [p,\infty)$. This follows from part (i)(a) and Proposition 3.1(iii).

(iii) The following examples (a)-(c) all fall within the scope of Corollary 4.3. So, in each case $w \in s$ and the identities in (31) hold; see Proposition 4.1.

(a) For any fixed $a > 1$ and $r \geq 0$ set $w(n) := n^r/a^n$ for $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \frac{w(n)}{w(n-1)} = a^{-1} \neq 1.$$

(b) For any fixed $a \geq 1$, the weight $w(n) := a^n/n!$ for $n \in \mathbb{N}$ satisfies

$$\lim_{n\to\infty} \frac{w(n)}{w(n-1)} = 0 \neq 1. \tag{35}$$

(c) The weight $w(n) := 1/n^n$ for $n \in \mathbb{N}$ also satisfies (35).

We point out, since w is decreasing, that $\frac{w(n)}{w(n-1)} \in (0,1]$ for all $n \in \mathbb{N}$. Hence, whenever the limit (34) exists, then necessarily $l \in [0,1]$.

As an application, suppose that the positive, decreasing weight w has the property that $l := \lim_{n\to\infty} \frac{w(n)}{w(n-1)}$ exists in $[0,1)$. Then, for each $r > 0$, the positive, decreasing weight $w^r : n \mapsto w(n)^r$, for $n \in \mathbb{N}$, satisfies $\lim_{n\to\infty} \frac{w(n)^r}{w(n-1)^r} = l^r \in [0,1)$. Hence, $\mathsf{C}^{(p,w^r)}$ is a compact operator for every $1 < p < \infty$.

(iv) The following criterion is sufficient to ensure that the limit (34) exists in $\mathbb{R} \setminus \{1\}$. Hence, both Proposition 4.1 and Corollary 4.3 are applicable to such a weight w. In particular, $w \in s$.

Let $\beta = (\beta_n)_{n\in\mathbb{N}}$ be a positive, increasing sequence with $\beta \uparrow \infty$ such that $\lim_{n\to\infty}(\beta_n - \beta_{n-1}) = \infty$. Then the weight $w(n) := e^{-\beta_n}$, for $n \in \mathbb{N}$, satisfies $l := \lim_{n\to\infty} \frac{w(n)}{w(n-1)} = 0 \neq 1$.

It is routine to verify that $\lim_{n\to\infty} \frac{w(n)}{w(n-1)} = 0$.

For the weight $w(n) := a^{-n}$ for $n \in \mathbb{N}$ (with $a > 1$) we have that $\beta_n := -\log w(n) = n\log(a) \uparrow \infty$ but, $(\beta_n - \beta_{n-1})\log(a) \not\to \infty$ for $n \to \infty$. So, the above criterion is *not* applicable to this weight. However, according to part (iii)(a) of this remark (with $r = 0$) the weight w is admissible for Corollary 4.3.

The following examples illustrate that Theorem 4.2 is more general than Corollary 4.3.

Example 5. (i) Fix $0 < \beta < 1$ and set $w_\beta(n) := e^{-n^\beta}$ for $n \in \mathbb{N}$, in which case $w \downarrow 0$, but

$$\lim_{n\to\infty} \frac{w_\beta(n)}{w_\beta(n-1)} = \lim_{n\to\infty} e^{(n-1)^\beta - n^\beta} = \lim_{n\to\infty} e^{-\beta/n^{(1-\beta)}} = 1,$$

as $(n-1)^\beta - n^\beta = n^\beta \left[\left(1 - \frac{1}{n}\right)^\beta - 1\right] = n^\beta \left[1 - \frac{\beta}{n} + o\left(\frac{1}{n}\right) - 1\right] \simeq -\frac{\beta}{n^{1-\beta}}$ for $n \to \infty$. So, Corollary 4.3 is not applicable. We show that Theorem 4.2 does apply.

Fix $1 < p < \infty$ and set $\gamma := \frac{p'}{p}$. Then, for each $n \in \mathbb{N}$, we have that

$$\begin{aligned} A_n(p, w_\beta) &= e^{-\gamma n^\beta} \sum_{k=1}^{n} e^{\gamma k^\beta} \le e^{-\gamma n^\beta} \int_1^{n+1} e^{\gamma x^\beta}\, dx \\ &= \frac{e^{-\gamma n^\beta}}{\beta} \int_1^{(n+1)^\beta} e^{\gamma t} t^{\frac{1}{\beta}-1}\, dt \le \frac{e^{-\gamma n^\beta}}{\beta} \int_1^{(n+1)^\beta} e^{\gamma t} t^m\, dt, \end{aligned}$$

where $m \in \mathbb{N}$ is chosen minimal such that $(m-1) < \frac{1}{\beta} - 1 \le m$. An integration by parts $(m+1)$-times yields that

$$\begin{aligned} \int_1^{(n+1)^\beta} e^{\gamma t} t^m\, dt \le a_0 &+ a_1(n+1)^\beta e^{\gamma(n+1)^\beta} + a_2(n+1)^{2\beta} e^{\gamma(n+1)^\beta} \\ &+ \ldots + a_m(n+1)^{m\beta} e^{\gamma(n+1)^\beta} \end{aligned}$$

for positive constants $a_0,\ a_1,\dots,\ a_m$. It follows that

$$\int_1^{(n+1)^\beta} e^{\gamma t}t^m\,dt \le M(1+n)^{m\beta}e^{\gamma(1+n)^\beta}, \quad n\in\mathbb{N},$$

for some constant $M>0$. Accordingly,

$$A_n(p,w_\beta)\le \frac{M}{\beta}(1+n)^{m\beta}e^{\gamma((1+n)^\beta-n^\beta)}, \quad n\in\mathbb{N}.$$

Since $(n+1)^\beta-n^\beta\simeq\frac{\beta}{n^{1-\beta}}$ and $(1+n)^{m\beta}\simeq n^{m\beta}$ for $n\to\infty$, there exists $K>0$ (independent of n) such that

$$A_n(p,w_\beta)\le Kn^{m\beta}, \quad i\in\mathbb{N}.$$

Since $(m-1)<\frac{1}{\beta}-1$ implies that $\alpha:=m\beta\in(0,1)$, Theorem 4.2 yields that $\mathsf{C}^{(p,w_\beta)}$ is compact.

For $\beta\ge1$ the compactness of $\mathsf{C}^{(p,w_\beta)}$ follows from Corollary 4.3. Indeed, if $\beta=1$, then $w_\beta(n)=e^{-n}$ for $n\in\mathbb{N}$ and so Remark 3(iii)(a) implies the compactness of $\mathsf{C}^{(p,w_\beta)}$. For $\beta>1$, observe from above that

$$\lim_{n\to\infty}\frac{w_\beta(n)}{w_\beta(n-1)}=\lim_{n\to\infty}e^{(n-1)^\beta-n^\beta}=\lim_{n\to\infty}e^{-\beta n^{\beta-1}}=0$$

and so the compactness of $\mathsf{C}^{(p,w_\beta)}$ follows again from Corollary 4.3.

(ii) There also exist positive, decreasing weights $w\in s$ such that the sequence $\{\frac{w(n)}{w(n-1)}\}_{n\in\mathbb{N}}$ *fails to converge* at all, yet $\mathsf{C}^{(p,w)}$ is a compact operator for every $1<p<\infty$.

Define $w(n):=\frac{1}{j^j}$, $n=2j-1$, and $w(n):=\frac{1}{2j^j}$, $n=2j$, for each $j\in\mathbb{N}$. Then w is (strictly) decreasing to 0. For $n_j:=2j$, $j\in\mathbb{N}$, we have $\frac{w(n_j)}{w(n_j-1)}=\frac{1}{2}$ for all $j\in\mathbb{N}$ and so $\lim_{j\to\infty}\frac{w(n_j)}{w(n_j-1)}=\frac{1}{2}$, whereas for $n_r:=2r+1$, $r\in\mathbb{N}$, the subsequence $\{\frac{w(n_r)}{w(n_r-1)}\}_{r\in\mathbb{N}}$ of $\{\frac{w(n)}{w(n-1)}\}_{n\in\mathbb{N}}$ converges to 0. Accordingly, the sequence $\{\frac{w(n)}{w(n-1)}\}_{n\in\mathbb{N}}$ is not convergent and so Corollary 4.3 is not applicable.

Fix $1<p<\infty$ and set $\gamma:=\frac{p'}{p}>0$. To establish the compactness of $\mathsf{C}^{(p,w)}$ observe, for every $j\in\mathbb{N}$, that

$$A_{2j}(p,w)=\frac{1}{(2j^j)^\gamma}\left(\sum_{k=1}^{j}(k^k)^\gamma+\sum_{k=1}^{j}(2k^k)^\gamma\right)=\frac{1+2^\gamma}{2^\gamma}\frac{1}{(j^j)^\gamma}\sum_{k=1}^{j}(k^k)^\gamma, \quad (36)$$

and that

$$A_{2j-1}(p,w)=1+\frac{1}{(j^j)^\gamma}\sum_{k=1}^{2(j-1)}w(k)^{-\gamma}=1+\frac{(j-1)^{(j-1)\gamma}}{(j^j)^\gamma}A_{2(j-1)}(p,w), \quad (37)$$

with $\lim_{j\to\infty} \frac{(j-1)^{(j-1)\gamma}}{(j^j)^\gamma} = 0$. Set $a_j := \sum_{k=1}^j (k^k)^\gamma$ and $b_j := (j^j)^\gamma$ for $j \in \mathbb{N}$. Then $b_j \uparrow \infty$. Moreover,

$$\lim_{j\to\infty} \frac{a_j - a_{j-1}}{b_j - b_{j-1}} = \lim_{j\to\infty} \frac{(j^j)^\gamma}{(j^j)^\gamma - ((j-1)^{j-1})^\gamma} = \lim_{j\to\infty} \frac{1}{1 - \frac{(j-1)^{(j-1)\gamma}}{(j^j)^\gamma}} = 1.$$

According to the Stolz-Cesàro criterion, [17, Theorem 1.22], it follows that also $\lim_{j\to\infty} \frac{a_j}{b_j} = 1$. So, via (36) and (37) we obtain that $\lim_{j\to\infty} A_{2j}(p, w) = \frac{1+2^\gamma}{2^\gamma}$ and $\lim_{j\to\infty} A_{2j-1}(p, w) = 1$. In particular, $\sup_{i\in\mathbb{N}} A_i(p, w) < \infty$ and so Theorem 4.2 applies (with $\alpha = 0$). Hence, $\mathsf{C}^{(p,w)}$ is compact and $w \in s$.

The following result is a comparison type criterion for compactness. One knows something about the compactness of $\mathsf{C}^{(p,w)}$ in $\ell_p(w)$ for a certain weight w and $1 < p < \infty$ and one has a second weight v whose growth relative to w is *controlled*. Then also $\mathsf{C}^{(p,v)} \in \mathcal{L}(\ell_p(v))$ is compact.

PROPOSITION 4.4. *Let w be a positive, decreasing sequence. Suppose, for some $1 < p < \infty$, that there exists $0 \le \alpha < 1$ such that*

$$A_n(p, w) \le M n^\alpha, \quad n \in \mathbb{N}, \tag{38}$$

for some constant $M > 0$.

Let v be any positive, decreasing sequence such that $\{\frac{v(n)}{w(n)}\}_{n\in\mathbb{N}} \in \ell_\infty$ and satisfying

$$w(n) \le K n^\beta v(n), \quad n \in \mathbb{N}, \tag{39}$$

for some $0 \le \beta < (p-1)(1-\alpha)$ and some constant $K > 0$. Then $\mathsf{C}^{(q,v)} \in \mathcal{L}(\ell_q(v))$ is a compact operator for every $1 < q \le p$.

PROOF. Let $L := \sup_{n\in\mathbb{N}} \frac{v(n)}{w(n)}$. Then, for each $n \in \mathbb{N}$, we have via (38) and (39) that

$$A_n(p, v) = v(n)^{p'/p} \sum_{k=1}^n \frac{1}{v(k)^{p'/p}} = \left(\frac{v(n)}{w(n)}\right)^{p'/p} w(n)^{p'/p} \sum_{k=1}^n \frac{1}{w(k)^{p'/p}} \cdot \left(\frac{w(k)}{v(k)}\right)^{p'/p}$$

$$\le L^{p'/p} w(n)^{p'/p} \sum_{k=1}^n \frac{1}{w(k)^{p'/p}} (K k^\beta)^{p'/p} \le (LK)^{p'/p} w(n)^{p'/p} \sum_{k=1}^n \frac{1}{w(k)^{p'/p}} n^{\beta p'/p}$$

$$= (LK)^{p'/p} n^{\beta p'/p} A_n(p, w) \le M (LK)^{p'/p} n^{\alpha + (\beta p'/p)}.$$

Moreover, $\alpha + \frac{\beta p'}{p} = \alpha + \frac{\beta}{(p-1)} < 1$ because $0 \le \beta < (p-1)(1-\alpha)$ implies $\frac{\beta}{(p-1)} < (1-\alpha)$ which implies $\alpha + \frac{\beta}{(p-1)} < 1$. So, Theorem 4.2 applied to v (with $\alpha + \frac{\beta}{(p-1)}$ in place of α) implies that $\mathsf{C}^{(q,v)} \in \mathcal{L}(\ell_q(v))$ is compact for all $1 < q \le p$. □

Example 6. Let $v(n) := \frac{1}{e^{n^\beta} \log^\gamma(n+1)}$ for $n \in \mathbb{N}$, where $0 < \beta < 1$ and $\gamma > 0$. Then $\mathsf{C}^{(p,v)} \in \mathcal{L}(\ell_p(v))$ is compact for every $1 < p < \infty$. Observe that $\lim_{n\to\infty} \frac{v(n)}{v(n-1)} = 1$ and so Corollary 4.3 is not applicable.

So, fix $1 < p < \infty$. Define $w(n) := e^{-n^\beta}$ for $n \in \mathbb{N}$. According to Example 5(i), there exist constants $M > 0$ and $0 < \alpha < 1$ such that

$$A_n(p, w) \leq M n^\alpha, \quad n \in \mathbb{N}.$$

Since $v(n) \leq w(n)$ for $n \in \mathbb{N}$, it is clear that $\left\{\frac{v(n)}{w(n)}\right\}_{n\in\mathbb{N}} \in \ell_\infty$. Choose any $r \in (0, (p-1)(1-\alpha))$. Then

$$\frac{w(n)}{v(n)} = \log^\gamma(n+1) = \frac{\log^\gamma(n+1)}{n^r} \cdot n^r \leq K n^r, \quad n \in \mathbb{N},$$

for some $K > 0$ (as $\lim_{n\to\infty} \frac{\log^\gamma(n+1)}{n^r} = 0$). According to Proposition 4.4, we can conclude that $\mathsf{C}^{(p,v)}$ is compact in $\ell_p(v)$.

Acknowledgements. The research of the first two authors was partially supported by the projects MTM2013-43540-P and GVA Prometeo II/2013/013 (Spain).

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