Study of iterative methods through the Cayley Quadratic Test ☆

D. K. R. Babajee[a], A. Cordero[b,*], J. R. Torregrosa[b]

[a]*Allied Network for Policy Research Actions for Sustainability (ANPRAS), Mauritius*
[b]*Instituto Universitario de Matemática Multidisciplinar Universitat Politècnica de València, Spain*

Abstract

Many iterative methods for solving nonlinear equations have been developed recently. The main advantage claimed by their authors is the improvement of the order of convergence. In this work, we compare their dynamical behavior on quadratic polynomials with the one of Newton's scheme. This comparison is defined in what we call Cayley Quadratic Test (CQT) which can be used as a first test to check the efficiency of such methods. Moreover we make a brief insight in cubic polynomials.

AMS Subject Classification: 65H05, 65D05, 41A25, 65B10

Keywords: Iterative method; order of convergence; Cayley Quadratic Test; stability; dynamical plane

1. Introduction

It is usual to find nonlinear equations in the modelization of many scientific and engineering problems, and a broadly extended tool to solve them are the iterative methods. In the last years, it has become an increasing and fruitful area of research. More recently, complex dynamics has revealed itself as a very useful tool to deep in the understanding of the operator that arise when an iterative scheme is applied to solve the nonlinear equation $f(z) = 0$, with $f : \mathbb{C} \to \mathbb{C}$. The dynamical properties of this operator give us important information about numerical features of the method as its stability and reliability, see for example [1, 2, 3, 4, 5, 6, 7, 8, 9] and the references therein.

The best known method, for being very simple and effective, is Newton's method. In the literature, several modifications have been made on Newton's scheme and other classical ones in order to accelerate the convergence or to reduce the number of operations and functional evaluations in each step of the iterative process.

Let us suppose that $x_{n+1} = \psi(x_n)$ define an Iterative Function (I.F.), where $\psi(x)$ is the fixed point function.

☆This research was supported by Ministerio de Ciencia y Tecnología MTM2011-28636-C02-02

*Corresponding author

Email addresses: `dkrbabajee@gmail.com` (D. K. R. Babajee), `acordero@mat.upv.es` (A. Cordero), `jrtorre@mat.upv.es` (J. R. Torregrosa)

Definition 1.1. *If the sequence $\{x_n\}$ tends to a limit α in such a way that*

$$\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} = C,$$

for $p \geq 1$, then the order of convergence of the sequence is said to be p, and C is known as the asymptotic error constant. If $p = 1$, $p = 2$ or $p = 3$, the convergence is said to be linear, quadratic or cubic, respectively.

Let $e_n = x_n - \alpha$ be the error of the nth iteration, then the relation

$$e_{n+1} = C\ e_n^p + O\left(e_n^{p+1}\right) = O\left(e_n^p\right), \tag{1}$$

is called the error equation.

Let us consider an iterative process in which x_{n+1} is determined by new information at $x_n, \phi_1(x_n), ..., \phi_{\mathrm{i}}(x_n), \mathrm{i} \geq 1$. No old information is reused. Thus,

$$x_{n+1} = \psi(x_n, \phi_1(x_n), ..., \phi_{\mathrm{i}}(x_n)). \tag{2}$$

Then ψ is called an I.F. without memory.

Kung and Traub conjectured in [10] that the upper bound of the order of convergence of an iterative method without memory that uses d functional evaluation per step is 2^{d-1}. If the scheme reaches this bound, it is called optimal.

In order to increase the order of convergence it is necessary to use high-order derivatives in case of schemes point-to-point, or to design multipoint methods. In both cases, the iterative expressions of the resulting methods become more complicated, and this has direct effects in their stability.

Now, we are going to recall some dynamical concepts that we use in this work (see [11]). Given a rational function $R : \mathbb{C} \to \mathbb{C}$, associated to an I.F. acting on a generic polynomial $p(z)$, the *orbit of a point* $z_0 \in \mathbb{C}$ is defined as:

$$\{z_0,\ R\left(z_0\right),\ R^2\left(z_0\right), ..., R^n\left(z_0\right), ...\}.$$

We analyze the phase plane of the map R by classifying the starting points from the asymptotic behavior of their orbits. A $z_0 \in \mathbb{C}$ is called a *fixed point* if $R\left(z_0\right) = z_0$. A *periodic point* z_0 of period $p > 1$ is a point such that $R^p\left(z_0\right) = z_0$ and $R^k\left(z_0\right) \neq z_0$, for $k < p$. A *pre-periodic point* is a point z_0 that is not periodic but there exists a $k > 0$ such that $R^k\left(z_0\right)$ is periodic. Moreover, a fixed point z_0 is called *attractor* if $|R'(z_0)| < 1$, *superattractor* if $|R'(z_0)| = 0$, *repulsor* if $|R'(z_0)| > 1$ and *parabolic* if $|R'(z_0)| = 1$. The fixed points that do not correspond to the roots of $p(z)$ are called *strange fixed points.*

The basin of attraction of an attractor α is defined as:

$$\mathcal{A}\left(\alpha\right) = \{z_0 \in \mathbb{C}\ :\ R^n\left(z_0\right) {\to} \alpha,\ n{\to}\infty\}.$$

The *Fatou set* of the rational function R, $\mathcal{F}\left(R\right)$, is the set of points $z \in \mathbb{C}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\mathbb{C}$ is the *Julia set,* $\mathcal{J}\left(R\right)$. That means that the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

This paper is organized as follows: in Section 2, we introduce the Cayley Quadratic Test (CQT) for classifying several families of iterative schemes of different orders of convergence. Also, some dynamical aspects of these classes of I.F. on quadratic polynomials are studied. In Section 3, a brief introduction about the dynamical behavior of these families on cubic polynomials and the possible effect of the CQT on them is presented. We finish the manuscript with some conclusions and the references used in it.

2. Cayley Quadratic Test

The study of the dynamical behavior of many I.F. in the complex plane can be found in [1, 7, 8, 12, 13, 14, 15, 16]. When a fixed point I.F. associated to a method is applied on a low-degree polynomial, a rational function is obtained. The analysis of this rational function associated to the iterative method gives us important information about its convergence, the set of initial points,.... In order to generalize the qualitative results obtained in the dynamical study, conjugacy classes are defined by using Möbius transformations. All functions that belong to the same conjugacy class have the same (topologically equivalent) Julia set, the same basins of attraction, ... that is, the same dynamical properties.

In the following, we consider the general quadratic polynomial $q(z) = (z - \alpha_1)(z - \alpha_2)$. The Möbius transformation and its inverse are given by

$$M(z) = \frac{z - \alpha_1}{z - \alpha_2}, \quad M^{-1}(z) = \frac{\alpha_2 z - \alpha_1}{z - 1},$$

where $M(\alpha_1) = 0$, $M(\alpha_2) = \infty$ and $M(\infty) = 1$. We are going to design a test in order to classify the different existing methods depending on the analytical expression of the conjugated function to the rational operator associated to the I.F. on $q(z)$, that is, on the resulting Julia set when they are applied on quadratic polynomials. Let us remark that, as the Möbius transformation is applied on the rational function obtained by applying the iterative method on $q(z)$, the resulting rational function is general, as it does not depend on the polynomial used.

We define Newton I.F. for function $f(z)$ as the complex operator

$$\psi_{2^{nd}NR}(z) = z - u(z), \quad u(z) = \frac{f(z)}{f'(z)}.$$

We start with Cayley's result on the dynamical behavior of Newton I.F. for quadratic polynomials.

Theorem 2.1. *[17, 18] Let*

$$\psi_{2^{nd}NR}(z) = \frac{z^2 - \alpha_1\alpha_2}{2z - (\alpha_1 + \alpha_2)},$$

be a rational map obtained from Newton's I.F. applied to the quadratic polynomial $q(z) = (z - \alpha_1)(z - \alpha_2)$, with $\alpha_1 \neq \alpha_2$. Then $\psi_{2^{nd}NR}(z)$ is conjugated to the map $z \to z^2$ by the Möbius transformation $M(z) = \dfrac{z - \alpha_1}{z - \alpha_2}$. Then, the Julia set of Newton's method $J(\psi_{2^{nd}NR}(z))$, that is, the straight line in the complex plane corresponding to the locus of points equidistant from α_1 and α_2, is transformed by the Möbius map into the unit circumference centered at the origin, S^1.

Cayley highlighted the major difficulties in attempting to extend Theorem 2.1 for quadratics to cubics and beyond. It is believed that this circumstance motivated further work of [19] and [1] along these lines (see [20]). Theorem 2.1 also follows from the following more general result.

Theorem 2.2. *[21, page 8] Suppose that $q(z)$ is a polynomial of degree $d \geq 2$. The unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is completely invariant by $q(z)$ if and only if $q(z) = \gamma z^d$, where $|\gamma| = 1$.*

From Theorem 2.1 we have $M \circ \psi_{2^{nd}NR} \circ M^{-1}(z) = z^2$ which means the Julia set is S^1, which can be considered as an ideal fractal. From this result it can be inferred the good behavior of Newton's method on quadratic polynomials: as the associate operator is conjugated to a potence of the variable (that is, it satisfies what we call CQT) there exist no more fixed nor critical points than 0 and ∞. In this way, the dynamics of the method is the simplest one.

Theorem 2.3 (Cayley Quadratic Test (CQT))**.** *Let $\psi(z)$ be the rational map obtained from a general Iteration Function applied to the quadratic polynomial $q(z) = (z-\alpha_1)(z-\alpha_2)$, with $\alpha_1 \neq \alpha_2$. Let us suppose that $\psi(z)$ is conjugated to the map $z \to z^p$, by the Möbius transformation $M(z) = \dfrac{z-\alpha_1}{z-\alpha_2}$. Then, p is the order of the iterative method associated to $\psi(z)$. Moreover, the corresponding Julia set $J(\psi(z))$ is the circumference S^1.*

Proof. The statement "the corresponding Julia set $J(\psi(z))$ is the circumference S^1" follows from Theorems 2.1 and 2.2 and Möbius transformation. It is enough to prove that p is the order of the I.F.; if $p = 1$, the thesis is evident as ψ is a fixed point function. Then, let us consider $p \geq 2$.

$M \circ \psi \circ M^{-1}(z) = z^p$ can be written as $M(\psi(z)) = M(z)^p$. We denote

$$h_p(z) = M(\psi(z)) - M(z)^p \quad = \frac{\psi\,(z)-\alpha_1}{\psi\,(z)-\alpha_2} - \frac{(z-\alpha_1)^p}{(z-\alpha_2)^p} = 0$$

We know that $\psi(\alpha_1) = \alpha_1$ since α_1 is a fixed point of ψ.
Also,

$$M(\alpha_1) = 0,\; M^{(j)}(\alpha_1) = \frac{(-1)^{j+1} j!}{(\alpha_1-\alpha_2)^j},\; j \geq 1$$

Let us consider the case $p = 2$.
By using Computer Algebra Software with $\psi(\alpha_1) = \alpha_1$ and $M(\alpha_1) = 0$, we can obtain

$$h_2(\alpha_1) = 0,$$
$$h_2'(\alpha_1) = M'(\alpha_1)\psi'(\alpha_1) = 0 \Rightarrow \psi'\,(\alpha_1) = 0, \text{ since } M'(\alpha_1) = \frac{1}{\alpha_1-\alpha_2} \neq 0,$$
$$h_2^{(2)}(\alpha_1) = M^{(2)}(\alpha_1)\psi'(\alpha_1)^2 + M'(\alpha_1)\psi^{(2)}(\alpha_1) - 2M'(\alpha_1)^2 = 0 \Rightarrow \psi^{(2)}\,(\alpha_1) = 2M'(\alpha_1) = \frac{2!}{(\alpha_1-\alpha_2)} \neq 0.$$

From Taylor expansion about α_1, we have

$$\psi(z) - \alpha_1 = \psi'(\alpha_1)(z-\alpha_1) + \frac{\psi^{(2)}(\alpha_1)}{2!}(z-\alpha_1)^2 + ... = \frac{1}{\alpha_1-\alpha_2}(z-\alpha_1)^2 + ...$$

which shows that the order of the I.F. is 2.
Let us consider the case $p = 3$.
By using Computer Algebra Software with $\psi(\alpha_1) = \alpha_1$ and $M(\alpha_1) = 0$, we can obtain

$$h_3(\alpha_1) = 0,$$
$$h_3'(\alpha_1) = M'(\alpha_1)\psi'(\alpha_1) = 0 \Rightarrow \psi'\,(\alpha_1) = 0, \text{ since } M'(\alpha_1) = \frac{1}{\alpha_1-\alpha_2} \neq 0,$$
$$h_3^{(2)}(\alpha_1) = M^{(2)}(\alpha_1)\psi'(\alpha_1)^2 + M'(\alpha_1)\psi^{(2)}(\alpha_1) = 0 \Rightarrow \psi^{(2)}\,(\alpha_1) = 0,$$
$$h_3^{(3)}(\alpha_1) = M^{(3)}(\alpha_1)\psi'(\alpha_1)^3 + 3M^{(2)}(\alpha_1)\psi(\alpha_1)\psi^{(2)}(\alpha_1) + M'(\alpha_1)\psi^{(3)}(\alpha_1) - 6M'(\alpha_1)^3 = 0,$$
$$\Rightarrow \psi^{(3)}\,(\alpha_1) = 6M'(\alpha_1)^2 = \frac{3!}{(\alpha_1-\alpha_2)^2} \neq 0.$$

By using again Taylor expansion about α_1, we have

$$\psi(z) - \alpha_1 = \psi'(\alpha_1)(z-\alpha_1) + \frac{\psi^{(2)}(\alpha_1)}{2!}(z-\alpha_1)^2 + \frac{\psi^{(3)}(\alpha_1)}{3!}(z-\alpha_1)^3 + ... = \frac{1}{(\alpha_1-\alpha_2)^2}(z-\alpha_1)^3 + ...$$

which shows that the order of the I.F. is 3.
For the general case p, by using Computer Algebra Software with $\psi(\alpha_1) = \alpha_1$ and $M(\alpha_1) = 0$, we can obtain

$$\begin{aligned}
&h_p(\alpha_1) = 0,\\
&h_p'(\alpha_1) = M'(\alpha_1)\psi'(\alpha_1) = 0 \Rightarrow \psi'(\alpha_1) = 0, \text{ since } M'(\alpha_1) = \frac{1}{\alpha_1 - \alpha_2} \neq 0,\\
&h_p^{(2)}(\alpha_1) = M^{(2)}(\alpha_1)\psi'(\alpha_1)^2 + M'(\alpha_1)\psi^{(2)}(\alpha_1) = 0 \Rightarrow \psi^{(2)}(\alpha_1) = 0,\\
&h_p^{(3)}(\alpha_1) = M^{(3)}(\alpha_1)\psi'(\alpha_1)^3 + 3M^{(2)}(\alpha_1)\psi(\alpha_1)\psi^{(2)}(\alpha_1) + M'(\alpha_1)\psi^{(3)}(\alpha_1) = 0,\\
&\Rightarrow \psi^{(3)}(\alpha_1) = 0.
\end{aligned}$$

Proceeding in a similar way, we can obtain for $j = 2, 3, \ldots, p-1$,

$$\begin{aligned}
&h_p'(\alpha_1) = M'(\alpha_1)\psi'(\alpha_1) = 0,\\
&h_p^{(j)}(\alpha_1) = M^{(j)}(\alpha_1)\psi'(\alpha_1)^j + A_1\psi^{(2)}(\alpha_1) + A_2\psi^{(3)}(\alpha_1) + \ldots + A_{j-2}\psi^{(j-1)}(\alpha_1) + M'(\alpha_1)\psi^{(j)}(\alpha_1) = 0,
\end{aligned}$$

where A_j are functions of α_1.
Thus, we have $\psi^{(j)}(\alpha_1) = 0$, $j = 1, 2, \ldots, p-1$.
Now,

$$\begin{aligned}
&h_p^{(p)}(\alpha_1) = M^{(p)}(\alpha_1)\psi'(\alpha_1)^p + A_1\psi^{(2)}(\alpha_1) + A_2\psi^{(3)}(\alpha_1) + \ldots + A_{p-2}\psi^{(p-1)}(\alpha_1) + M'(\alpha_1)\psi^{(p)}(\alpha_1) - p!M'(\alpha_1)^p = 0\\
&\Rightarrow \psi^{(p)}(\alpha_1) = p!M'(\alpha_1)^{p-1} = \frac{p!}{(\alpha_1 - \alpha_2)^{p-1}} \neq 0.
\end{aligned}$$

Finally, from Taylor expansion about α_1, we obtain

$$\begin{aligned}
&\psi(z) - \alpha_1\\
&= \psi'(\alpha_1)(z - \alpha_1) + \frac{\psi^{(2)}(\alpha_1)}{2!}(z - \alpha_1)^2 + \frac{\psi^{(3)}(\alpha_1)}{3!}(z - \alpha_1)^3 + \ldots + \frac{\psi^{(p-1)}(\alpha_1)}{(p-1)!}(z - \alpha_1)^{p-1} + \frac{\psi^{(p)}(\alpha_1)}{p!}(z - \alpha_1)^p + \ldots\\
&= \frac{1}{(\alpha_1 - \alpha_2)^{p-1}}(z - \alpha_1)^p + \ldots
\end{aligned}$$

which establishes the p-order of the I.F. □

The Iteration Function is said to have passed the Cayley Quadratic Test if it verifies the above Theorem, that is, if it is conjugated to z^p, otherwise it fails CQT.

3. Iterative schemes satisfying CQT

In this section we are going to analyze different known methods in order to establish if they pass, or not, the CQT. Moreover, some aspects of their dynamics will be presented by means of complex dynamical planes. For the representation of the convergence basins of every iterative procedure we have used the software described in [22]. We draw a mesh with eight hundred points per axis; each point of the mesh is a different initial estimation which we introduce in the method. If the scheme reaches one of the attracting fixed points (being or not the roots of the original polynomial) in less than eighty iterations, this point is drawn in different colors, depending on the fixed point that the iterative process converges to. These attracting points are calculated analytically in any case and are marked in the figures by white stars. The color will be more intense when the number of iterations is lower. Otherwise, if the method arrives at the maximum of iterations, the point will be drawn in black.

The following study is made only on iterative schemes that use the derivatives of the nonlinear function f. The reason is that, as the authors prove in [1, 13], Möbius map does not guaranty the existence of conjugacy classes for quadratic polynomials when the scheme is derivative-free.

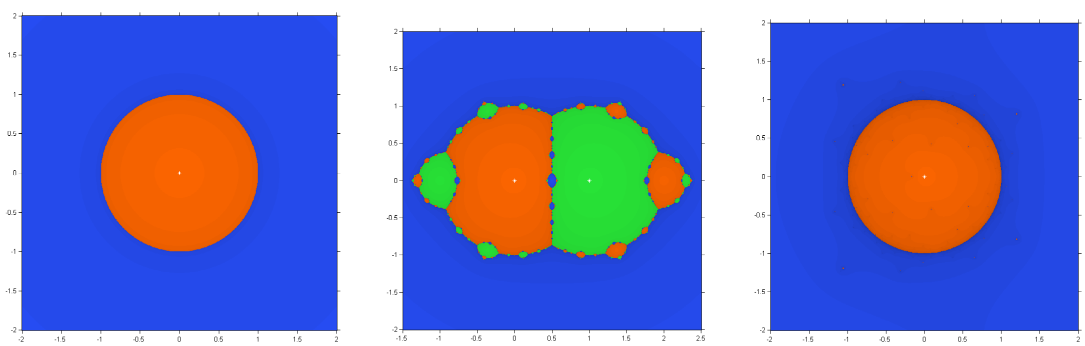
We will start the analysis with an increasing number of functional evaluations. We denote by n_f, $n_{f'}$ and $n_{f''}$ the number or functional evaluations of functions f, f' and f'' per step, respectively.

3.1. Schemes with $n_f = n_{f'} = n_{f''} = 1$

The classical Chebyshev-Halley family involves the functional evaluation of second derivatives. Its associated operator is

$$\psi_{3^{rd}CH}(z) = z - \left(1 + \frac{1}{2}\frac{L_f(z)}{1 - \alpha L_f(z)}\right) u(z), \tag{3}$$

where $u(z) = \frac{f(z)}{f'(z)}$, $L_f(z) = \frac{f(z)f''(z)}{f'(z)^2}$ and $\alpha \in \mathbb{C}$. In [15] it is showed that $M \circ \psi_{3^{rd}CH} \circ M^{-1}(z) = z^3 \frac{z-2(\alpha-1)}{1-2(\alpha-1)z}$, and when $\alpha = 1$, then $M \circ \psi_{3^{rd}CH} \circ M^{-1}(z) = z^4$, which corresponds to the classical super-Halley method. Also, when $\alpha = 0.5$, $M \circ \psi_{3^{rd}CH} \circ M^{-1}(z) = z^3$ and the associated scheme is Halley's one. In Figure 1, the qualitative



(a) $\alpha = 1$ and $\alpha = 0.5$ (b) $\alpha = 2$ (c) $\alpha = -2$

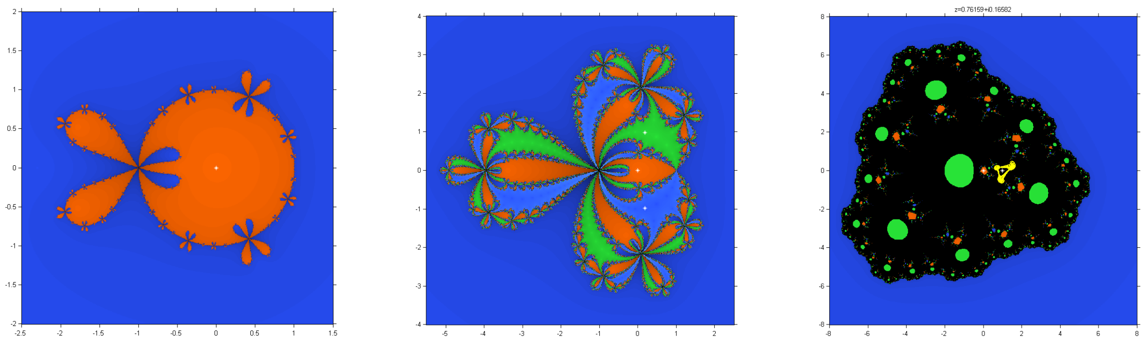
Figure 1: Some dynamical planes from (3) on quadratic polynomials

behavior in the complex plane of some elements of this family of iterative methods on quadratic polynomials is showed. It can be observed (Figure 1a) that the specific schemes verifying CQT have the same dynamical properties as Newton's method. Nevertheless, any other behavior can be found in the vicinity, for values of the parameter that does not satisfy CQT: when $\alpha = 2$, $z = 1$ becomes a strange attractor and our dynamical plane has three different basins of attraction and if $\alpha = -2$ there exist only two basins and the method is quite stable, as we see in Figure 1c. The complete complex dynamical study of this family can be found in [15].

In [1], Amat et al. introduced the third-order parametric family

$$\psi_{3^{rd}c}(z) = z - \left(1 + \frac{1}{2}L_f(z) + cL_f(z)^2\right)\frac{f(z)}{f'(z)}, \tag{4}$$

which is called c-family. It can be proved that $M \circ \psi_{3^{rd}c} \circ M^{-1}(z) = z^3 \frac{-4c+(1+z)^2(2+z)}{1+4z+5z^2+2(1-2c)z^3}$ and Cayley's test cannot be verified for any value of parameter c. A rigorous complex analysis of the c-family on quadratic



(a) $c = \frac{1}{8}$ (b) $c = -2$ (c) $c = 7 - 4\sqrt{3}i$

Figure 2: Some dynamical planes from (4) on quadratic polynomials

polynomials has been made in [23]. Some regions of stable behavior are found, but no element of the family behaves as Newton on quadratic polynomials, in concordance with the theory. In Figure 2 some of these are shown; in particular, case (a) corresponds to a stable behavior with a complicated (but convex) Julia set; case (b) shows four different basins of attraction, two of them not corresponding to the original roots; finally, in case (c) a periodic orbit of period 3 is found.

3.2. *Schemes with* $n_f = 1$ *and* $n_{f'} = 2$

We consider now the two-parameter family of higher order Chebyshev-Halley-like family of ($3^{rd}CHL1$) I.F. due to [24]:

$$\psi_{3^{rd}CHL1}(z) = z - u(z)\left(1 + \frac{1}{2}\frac{f'(z) - f'(z - \beta u(z))}{(\beta - \lambda)f'(z) + \lambda f'(z - \beta u(z))}\right), \quad \beta, \lambda \in \mathbb{C}. \tag{5}$$

Since

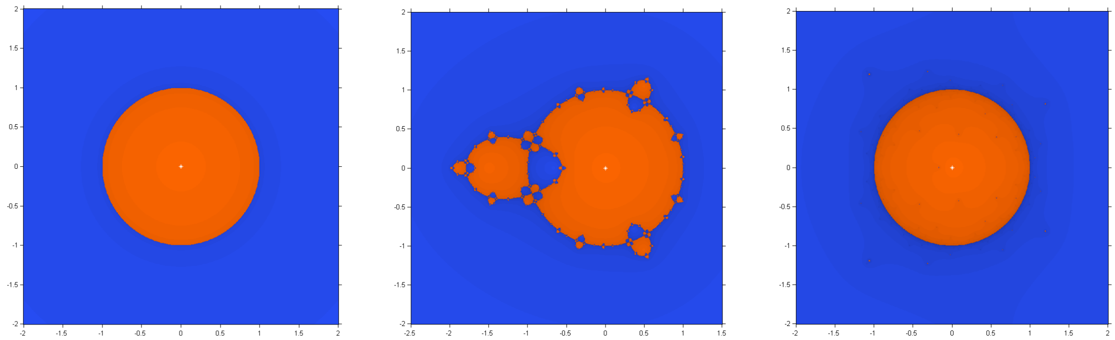
$$M \circ \psi_{3^{rd}CHL1} \circ M^{-1}(z) = \frac{z^3(z + 2(1-\lambda))}{2z(1-\lambda)+1} = \begin{cases} z^4, & \text{for } \lambda{=}1 \\ z^3, & \text{for } \lambda{=}\frac{1}{2}. \end{cases}$$

for any $\beta \neq 0$, we have 3^{rd}*CHL1* family satisfying the CQT for the cases $\lambda = 0.5$ and $\lambda = 1$ which correspond to the third order Halley-like (*HL1*) and third order Super-Halley-like (*SHL1*) families of I.F. respectively [12] (see Figure 3a). Of course, when other values of the parameter are used, unstable behavior can be found, but also very stable behavior, as is shown in Figure 3b (with an apparently connected Julia set) and Figure 3c, where the Julia set is not connected, but is "almost" a circumference: in the immediate basin of attraction associated to one root, there exist little disconnected areas of convergence to the another one.

On the other hand, we know that *SHL1* family of iterative schemes is fourth-order for quadratic polynomials (its asymptotic error constant depends on the third derivative at the root). The special case $\lambda = 1$ and $\beta = \frac{2}{3}$ of the 3^{rd}*CHL1* family corresponds to the well-known fourth order optimal Jarratt I.F.

$$\psi_{4^{th}JM}(z) = z - \frac{3f'(y) + f'(z)}{6f'(y) - 2f'(z)}u(z), \tag{6}$$

where $y = z - \frac{2}{3}u(z)$, which satisfies the CQT.



(a) $\lambda = 1$ and $\lambda = 0.5$ (b) $\lambda = \frac{1}{4}$ (c) $\lambda = -2$

Figure 3: Some dynamical planes from (5) on quadratic polynomials

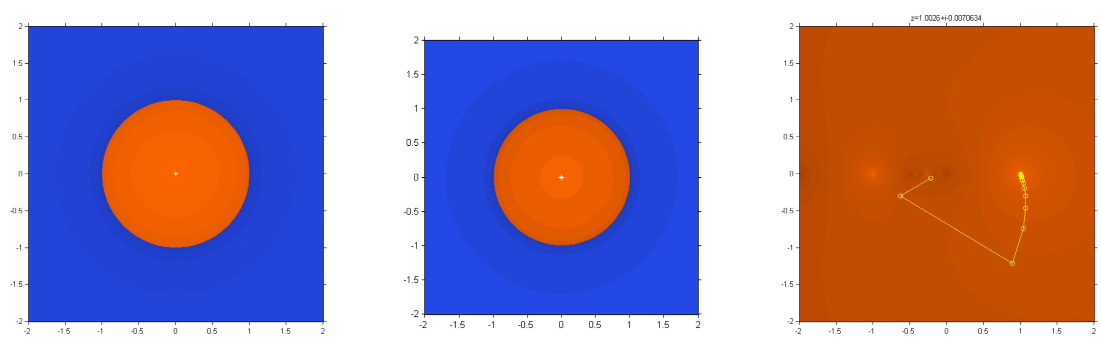
If we analyze the Jarratt-like family, described by the I.F.

$$\psi_{JL}(z) = y - \frac{af'(z) + bf'(y)}{cf'(z) + df'(y)} u(z), \tag{7}$$

where $a, b, c, d \in \mathbb{C}$ and $y = z - \beta u(z)$, $\beta \in \mathbb{C}$, it is easy to show that

- $M \circ \psi_{JL} \circ M^{-1}(z) = z^3$ if and only if $a = d\beta(3 - 2\beta)$, $b = -d\beta$, $c = d(-1 + 2\beta)$ for any $\beta \neq 0$,
- $M \circ \psi_{JL} \circ M^{-1}(z) = z^4$ if and only if $a = \frac{d}{2}(-1 + 4\beta - 2\beta^2)$, $b = \frac{d}{2}(1 - 2\beta)$, $c = d(\beta - 1)$ for any $\beta \neq 0$.

In general, the resulting class of methods have order of convergence one. In Figure 4, some aspects of the



(a) $\beta = a = c = d = 1$ and $b = -1$ (b) $a = -1$, $b = -3$, $c = d = 2$, $\beta = 2$ (c) $a = b = c = d = 1$, $\beta = -2$

Figure 4: Some dynamical planes from (7) on quadratic polynomials

convergence of Jarratt-type methods is presented. The CQT is verified in case of Figures 4a and 4b (the fixed point operator becomes z^3 and z^4 , respectively). However, in Figure 4c, a completely different behavior is showed: in this case, $z = 0$ and $z = \infty$ are repulsive points and the strange fixed point $z = 1$ is the only attractive point. So, no convergence to the roots can be achieved in this case.

3.3. Schemes with $n_f = 2$ and $n_{f'} = 1$

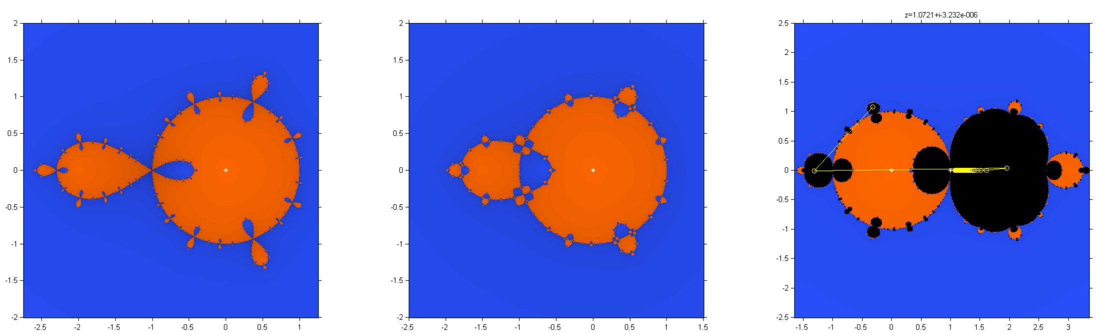
We consider now the two-parameter family of higher order Chebyshev-Halley-like family of (3^{rd}*CHL2*) I.F. described in [12]:

$$\psi_{3^{rd}CHL2}(z) = z - u(z)\frac{\{\beta^2 + (2\lambda - 1)(1-\beta)\}f(z) + (1-2\lambda)f(z-\beta u(z))}{\{\beta^2 + 2\lambda(1-\beta)\}f(z) - 2\lambda f(z - \beta u(z))}, \quad \beta, \lambda \in \mathbb{C}. \tag{8}$$

Since

$$M \circ \psi_{3^{rd}CHL2} \circ M^{-1}(z) = \frac{z^3(2(\lambda-1)-z)}{2(\lambda-1)z-1} = \begin{cases} z^4, & \text{for } \lambda=1 \\ z^3, & \text{for } \lambda=\frac{1}{2}. \end{cases}$$

for any $\beta \neq 0$, we have 3^{rd}*CHL2* family satisfying the CQT for the cases $\lambda = 0.5$ and $\lambda = 1$, which correspond to the third order Halley-like (*HL2*) and third order Super-Halley-like (*SHL2*) families of I.F. respectively [12]. A stable behavior, for two values of λ close to the optimal ones (in the sense of the associated method verifies



(a) $\lambda = 0$ (b) $\lambda = 0.25$ (c) $\lambda = 2.5$

Figure 5: Some dynamical planes from (8) on quadratic polynomials

the CQT) are shown in Figures 5a and 5b. In Figure 5c, a singular circumstance is presented, where $z = 1$ is a parabolic point, and it has its own basin of attraction.

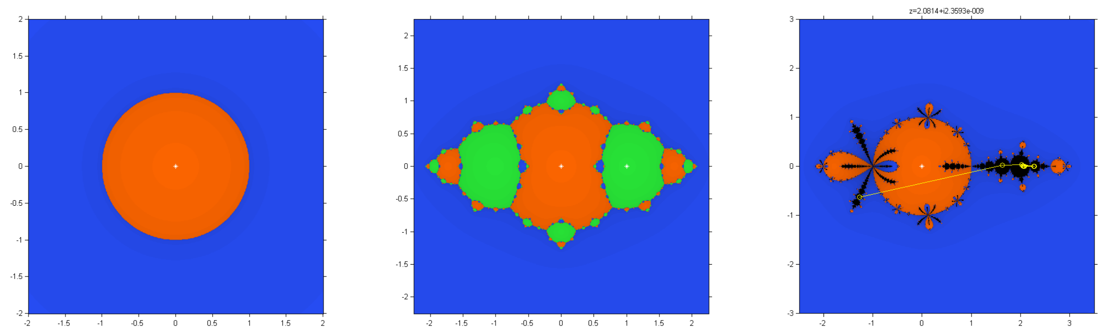
We know that *SHL2* family of iterative schemes is also fourth-order for quadratic polynomials. The special case $\lambda = 1$ and $\beta = 1$ of the 3^{rd}*CHL2* family corresponds to the famous fourth order optimal Ostrowski's method, which satisfies the CQT. Let us consider now the well-known King's family (see [25]), which contains Ostrowski's method as a particular member,

$$\psi_{4^{th}K}(z) = y - \frac{f(z) + (2+\beta)f(y)}{f(z) + \beta f(y)}\frac{f(y)}{f'(z)}, \tag{9}$$

where $\beta \in \mathbb{C}$ and $y = z - u(z)$ is the Newton step.

$$M \circ \psi_{4^{th}K} \circ M^{-1}(z) = z^4 \frac{5 + 4z + z^2 + \beta(2+z)}{1 + (4+\beta)z + (5+2\beta)z^2}.$$

It can be proved that the only case in which the CQT is verified corresponds again with Ostrowski's I.F. The complex dynamics of this family has been studied in [14] and some aspects are shown in Figure 6. Specifically, the ideal fractal for Ostrowski's scheme is shown in Figure 6a meanwhile in Figure 6b a dynamical plane with



(a) $\beta = 0$ (b) $\beta = -2$ (c) $\beta = -3$

Figure 6: Some dynamical planes from (9) on quadratic polynomials

three basins of attraction is shown, two of them corresponding to the original roots of the quadratic polynomial and another one from an attracting strange fixed point, $z = 1$. Finally, in Figure 6c, the proposed scheme has an attracting periodic orbit of period two, whose trajectory is showed in yellow, and the dynamical plane shows three different basins of attraction.

Kim in [26] designs a class of optimal eighth-order methods, whose two first steps yield to a fourth-order scheme

$$\psi_{4^{th}Kim} \quad = \quad y - \frac{1+\beta v+\lambda v^2}{1+(\beta-2)v+\mu v^2}\frac{f(y)}{f'(z)}, \qquad \lambda,\beta,\mu \in \mathbb{C} \tag{10}$$

where y is the step of Newton and $v = \frac{f(y)}{f(z)}$. It is a three-parametric family of iterative schemes whose order of convergence is four, with no conditions on λ, β and μ. The operator associated to the method, after Möbius transformation, can be expressed as

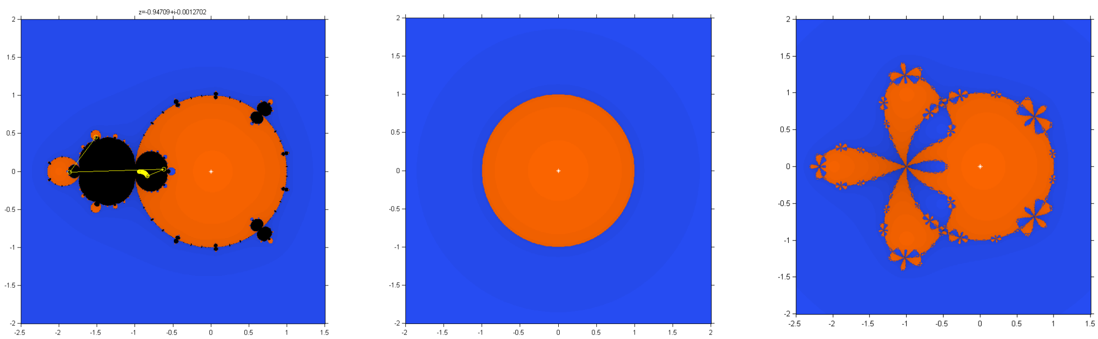
$$M \circ \psi_{4^{th}Kim} \circ M^{-1}(z) = z^4 \frac{1+2\beta-\lambda+\mu+4z+5\beta z+2\mu z+6z^2+4\beta z^2+\mu z^2+4z^3+\beta z^3+z^4}{1+4z+\beta z+6z^2+4\beta z^2+\mu z^2+4z^3+5\beta z^3+2\mu z^3+z^4+2\beta z^4-\lambda z^4+\mu z^4}. \tag{11}$$

Let us note that $M \circ \psi_{4^{th}Kim} \circ M^{-1}(z) = z^4$ if $\lambda = 0$ and $\mu = -2\beta$. Indeed, $M \circ \psi_{4^{th}Kim} \circ M^{-1}(z) = z^5$ when $\lambda = 0$, $\beta = -1$ and $\mu = 1$. A brief glance of the complex dynamics of this family is presented in Figure 7. Different values of the parameters λ, β and μ have been used in order to show different kind of dynamical behavior: in Figure 7a, there are three fixed attracting points that are not repelling, the ones coming from the roots of the polynomial and $z = -1$, which is parabolic and has its own basin of attraction. In the figure, this basin is presented in black and a trajectory of an starting point in this basin is shown in yellow. In Figure 7b, the values of the parameters imply that $M \circ \psi_{4^{th}Kim} \circ M^{-1}(z) = z^5$, and the Julia set is the ideal fractal. A stable behavior is also shown in Figure 7c, where only two different basins of convergence appear, but they have infinite connected components.

In [27], Artidiello et al. designed a class of fourth-order methods, whose iteration function is

$$\psi_{4^{th}A}(z) = y - \frac{(f(z)+4\beta f(y))^2}{f(z)^2+2(4\beta-1)f(z)f(y)+4(4\beta^2-2\beta-1)f(y)^2}\frac{f(y)}{f'(z)}, \tag{12}$$

where y is the step of Newton and $\beta \in \mathbb{C}$. Once the Möbius transformation is made, the rational function



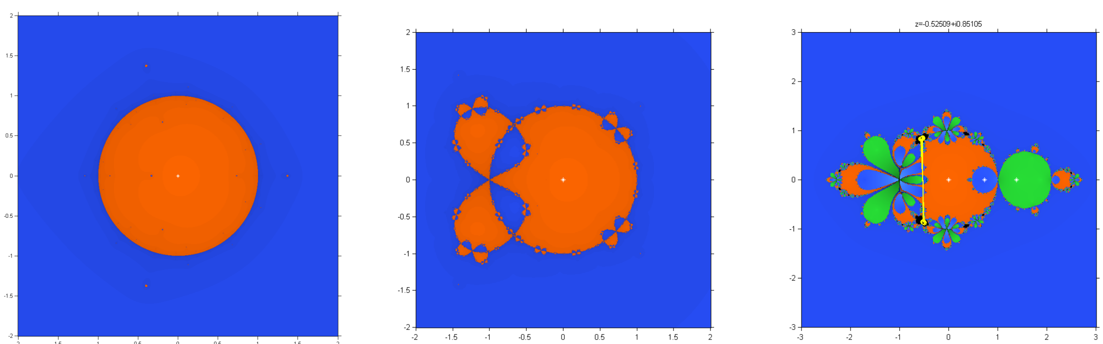
(a) $\lambda = 0, \beta = 1, \mu = -2$ (b) $\lambda = 0, \beta = -1, \mu = 1$ (c) $\lambda = 1, \beta = 0, \mu = 0$

Figure 7: Some dynamical planes from (10) on quadratic polynomials

associated to the iterative method is

$$M \circ \psi_{4^{th}A} \circ M^{-1}(z) = z^4 \frac{-3 + 8\beta - 4z + 24\beta z + 32\beta^2 z + 2z^2 + 24\beta z^2 + 16\beta^2 z^2 + 4z^3 + 8\beta z^3 + z^4}{1 + 4z + 8\beta z + 2z^2 + 24\beta z^2 + 16\beta^2 z^2 - 4z^3 + 24\beta z^3 + 32\beta^2 z^3 - 3z^4 + 8\beta z^4}, \quad (13)$$

and it is easy to show that it fails Cayley's test for any value of β. Although there is no value of β that gives



(a) $\beta = 0$ (b) $\beta = \frac{1}{3}$ (c) $\beta = -\frac{1}{3}$

Figure 8: Some dynamical planes from (12) on quadratic polynomials

the ideal fractal, quite stable behavior can be found, as is shown in Figure 8a. Also in Figure 8b only two basins of attraction are found, although the Julia set is more complicated in this case. However, the opposite value of the parameter gives two strange attracting points and one attracting periodic orbit of period 2, making much lower the probability of converging to one root of the polynomial (see Figure 8c).

3.4. Schemes with $n_f = 2$ and $n_{f'} = 2$

A sixth-order Jarratt-like method ($6^{th}JM$) [28] is given by

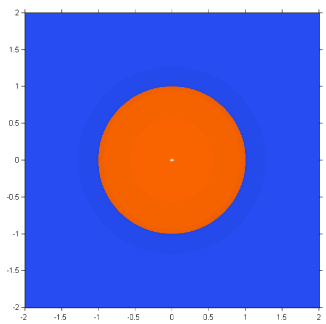
$$\psi_{6^{th}JM}(z) = \psi_{4^{th}JM}(z) - \frac{f(\psi_{4^{th}JM}(z))}{2f[\psi_{4^{th}JM}(z), z] - f'(z) + a_1(\psi_{4^{th}JM}(z) - z)^2}, \quad (14)$$

where $y = z - \frac{2}{3}u(z)$,

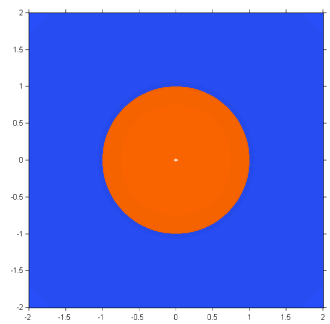
$$a_1 = \frac{2f[\psi_{4^{th}JM}(z), z](-z+y) + (z - 2y + \psi_{4^{th}JM}(z))f'(z) + (z - \psi_{4^{th}JM}(z))f'(y)}{(z-y)(z-\psi_{4^{th}JM}(z))(z - 3y + 2\psi_{4^{th}JM}(z))}$$

and $f[p,q] = \frac{f(p)-f(q)}{p-q}$ is a divided difference of order 1.

Now, it can be checked that $M \circ \psi_{6^{th}JM} \circ M^{-1}(z) = z^8$ which shows that the $6^{th}JM$ I.F. is eighth-order for quadratic polynomials and satisfies the Traub-Kung conjecture. Obviously, the associated dynamical plane is the ideal fractal, see Figure 9a.



(a) $M \circ \psi_{6^{th}JM} \circ M^{-1}(z) = z^8$



(b) $M \circ \psi_{12^{th}JM} \circ M^{-1}(z) = z^{16}$

Figure 9: Ideal fractals from (14) and (15) on quadratic polynomials

3.5. Schemes with $n_f = 3$ and $n_{f'} = 2$

A twelfth order Jarratt-like I.F. ($12^{th}JM$) [28] is given by

$$\psi_{12^{th}JM}(z) = v(z) - \frac{f(v(z))}{2f[z,v(z)] + f[u(z),v(z)] - 2f[z,u(z)] + (u(z)-v(z))f[u(z),z,z]}, \tag{15}$$

where $u(z) = \psi_{4^{th}JM}(z)$, $v(z) = \psi_{6^{th}JM}(z)$ and

$$f[u(z),z,z] = \frac{f[u(z),z] - f'(z)}{u(z) - z}.$$

Now, is easy to prove that $M \circ \psi_{12^{th}JM} \circ M^{-1}(z) = z^{16}$ which shows that the $12^{th}JM$ I.F. is sixteenth order for quadratic polynomials and satisfies the Kung-Traub's conjecture. Again, the corresponding dynamical plane is the ideal fractal as can be seen in Figure 9b.

3.6. Formulas of I.F. satisfying CQT

We can find I.F.'s which satisfy CQT if we consider

$$\psi(z) = M^{-1} \circ z^p \circ M(z) = \frac{\alpha_2(z-\alpha_1)^p - \alpha_1(z-\alpha_2)^p}{(z-\alpha_1)^p - (z-\alpha_2)^p}.$$

For some cases of p, we have:

p = 2, $\psi(z) = \dfrac{z^2 - \alpha_1\alpha_2}{2z - (\alpha_1 + \alpha_2)}$ which corresponds to the Newton I.F.

p = 3, $\psi(z) = \dfrac{z^3 - 3\alpha_1\alpha_2 z + \alpha_1^2\alpha_2 + \alpha_1\alpha_2^2}{3z^2 - 3(\alpha_1 + \alpha_2) + (\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2)}$ which corresponds to the third-order scheme.

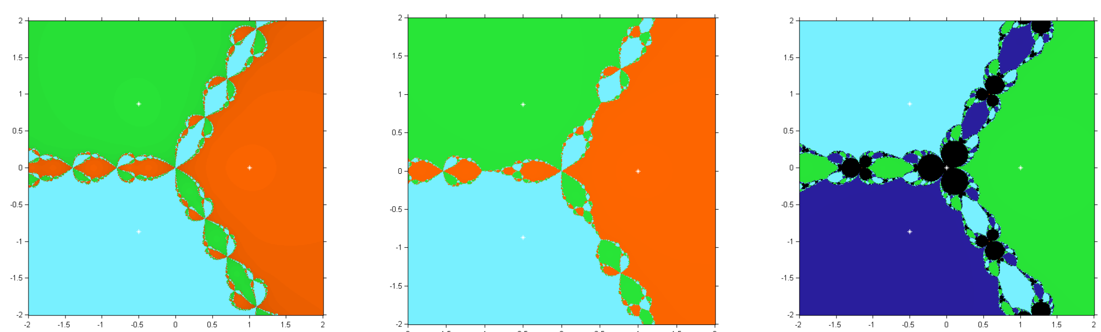
p = 4, $\psi(z) = \dfrac{z^4 - 6\alpha_1\alpha_2 z^2 + 4(\alpha_1^2\alpha_2 + \alpha_1\alpha_2^2)z - (\alpha_1^3\alpha_2 + \alpha_1^2\alpha_2^2 + \alpha_1\alpha_2^3)}{(2z - (\alpha_1 + \alpha_2))(2z^2 - 2(\alpha_1 + \alpha_2)z + (\alpha_1^2 + \alpha_2^2))}$ which corresponds to fourth-order procedure.

4. A brief glance on cubic polynomials

We have analyzed in the previous section different known families of iterative methods in order to establish if there exist any value of the parameters involved that makes the corresponding iterative scheme to verify Cayley Quadratic Test. This aim has been accomplished and also some other behavior have been shown for any other values of the parameters.

Now, the question arisen is the following: when a member of a class of iterative schemes satisfies CQT, is it also stable for cubic polynomials, that is, is the Julia set quite simple? and, are there any basins of attraction of strange fixed points? We will try to conjecture an answer to this question.

In Figure 10, the behavior of the family (5) is presented on the cubic polynomial $z^3 - 1$. In this case, to use a Möbius transformation would force us to add a new parameter; so, the dynamics will not represent the whole behavior on cubic polynomials, but an idea can be obtained on the influence of the CQT on cubic polynomials. Figure 10 corresponds to the values of the parameters that verify CQT. However, the behavior showed by them is not the same; in the case of Figures 10a and 10b, a very stable behavior is shown, very similar to the one of Newton's scheme for the same polynomial; in case of Figures 10c, a new basin of attraction appear, as $z = 0$ becomes an strange attracting point. Different values of the parameters can derive in stable behavior, as can be seen in Figure 10b.

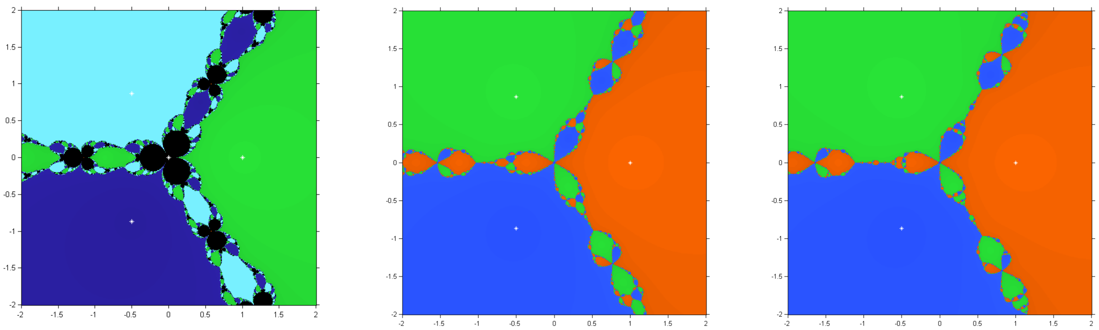


(a) $\lambda = 1$, $\beta = 1$ (b) $\lambda = 1$, $\beta = \frac{2}{3}$ (c) $\lambda = \frac{1}{2}$, $\beta = \frac{2}{3}$

Figure 10: Some dynamical planes from (5) on $x^3 - 1$

Let us notice that in case of Jarratt-type family, (7) (and also in family (8) for $\lambda = \frac{1}{2}$) the behavior is not the same, as for the values that made the CQT be satisfied, strange attractors are found in case of cubic polynomials, as can be seen in Figure 11a. Nevertheless, $\lambda = 1$ in family (8) satisfies CQT on quadratic polynomials and gives a very stable behavior in cubic ones, see Figures 11b and 11c for different values of β.

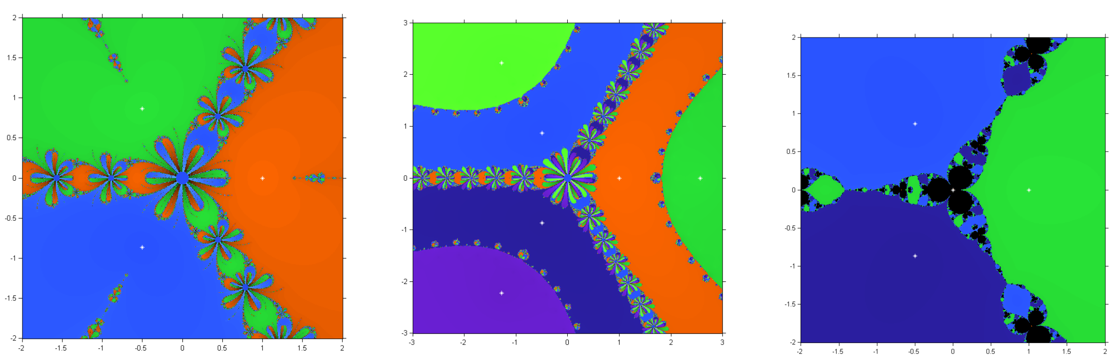
In the case of King's family, (9), it can be observed that $\beta = 0$, that is the value of the parameter which satisfies CQT, gives also a very stable behavior in cubic polynomials, very similar to the one showed in Figure



(a) $\beta = \frac{2}{3}$,$b = -\frac{3a}{5}$, $c = \frac{3a}{10}$, $d = \frac{9a}{10}$ (b) $\lambda = 1$, $\beta = 1$ (c) $\lambda = 1$, $\beta = \frac{2}{3}$

Figure 11: Some dynamical planes from (7) and (8) on $x^3 - 1$

10a. For values that gave unstable behavior on quadratic polynomials, as $\beta = -3$ (periodic orbit) or $\beta = -2$ (strange attractor at $z = 1$), also unstable behavior is found on cubic ones: in the first case, the complexity of the Julia set is increased with the appearance of flower-like structures whose centers are preimages of the infinity (Figure 12a); in the second one, three strange attractors appear, reducing considerably the probability of converging to the roots of the polynomial (Figure 12b). Unstable behavior is showed in general in the case of Kim's family, (10), where even the values of the parameters that made the family to satisfy CQT give strange attractors or other undesirable behaviors on cubic polynomials, see Figure 12c.

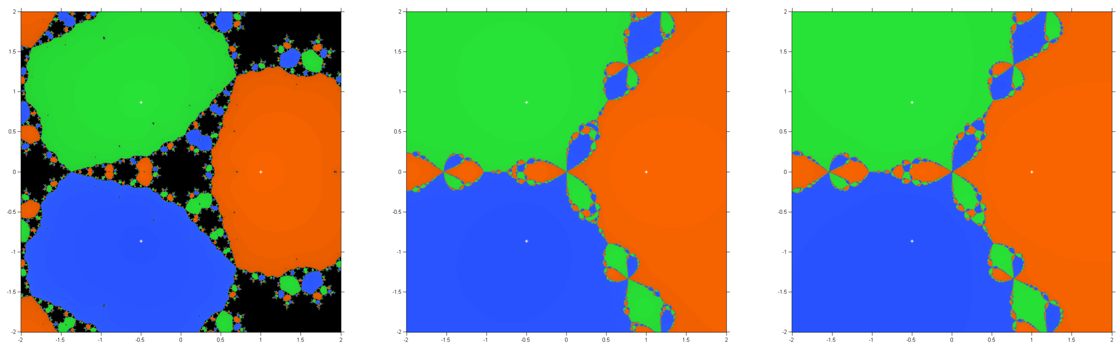


(a) Family (9), $\beta = -3$ (b) Family (9), $\beta = -2$ (c) Family (10), $\lambda = 0$, $\beta = -1$ and $\nu = 1$

Figure 12: Some dynamical planes from (9) and (10) on $x^3 - 1$

Finally, in Figure 13, the behavior of families (12), (14) and (15) is presented on $x^3 - 1$. In the first case, the black region correspond to the basin of the infinity, that is a region of non-convergence for this value of the parameter; a very stable behavior is found in the other two cases, very similar to the one of Newton. Let us remember that these cases are associated to sixth- and twelfth-order of convergence methods verifying CQT, respectively.

From the previous results, we conjecture that new conditions must be satisfied on the I.F. in order to assure stable behavior on cubic polynomials. This can be the aim of future research.



(a) Family (12), $\beta = 0$ (b) (14) (c) (15)

Figure 13: Some dynamical planes from (12), (14) and (15) on $x^3 - 1$

5. Conclusions

From the described results, we can conclude that the CQT is a good indicator about the stability of an iterative method, with independence of its optimality and its order of convergence. Indeed, we have obtained evidences of the need to require some new conditions to ensure the same stability on cubic polynomials. This would imply the definition, in future works, of a Cayley Cubic Test.

Acknowledgments: The authors thank to the anonymous referees for their suggestions to improve the readability of the paper.

6. References

[1] S. Amat, S. Busquier, S. Plaza, Review of some iterative root-finding methods from a dynamical point of view, SCIENTIA, Series A: Math. Sci. 10 (2004) 3–35.

[2] J.L. Hueso, E. Martinez, C. Teruel, Convergence, efficiency and dynamics of new fourth and sixth order families of iterative methods for nonlinear systems, Journal of Computational and Applied Mathematics (2014) doi:10.1016/j.cam.2014.06.010.

[3] Á.A. Magreñán, J. M. Gutiérrez, Real dynamics for damped Newtons method applied to cubic polynomials, Journal of Computational and Applied Mathematics (2014) doi:10.1016/j.cam.2013.11.019.

[4] S. Amat, S. Busquier, Á. A. Magreñán, Reducing Chaos and Bifurcations in Newton-Type Methods. Abstract and Applied Analysis Volume 2013 (2013) Article ID 726701 10 pages http://dx.doi.org/10.1155/2013/726701.

[5] D. K. R. Babajee, A. Cordero, F. Soleymani, J.R. Torregrosa, On improved three-step schemes with high efficiency index and their dynamics, Numerical Algorithms 65(1) (2014) 153–169.

[6] C. Chun, B. Neta, J. Kozdon, M. Scott, Choosing weight functions in iterative methods for simple roots, Applied Mathematics and Computation 227 (2014) 788–800.

[7] Á. A. Magreñán, Estudio de la dinámica del método de Newton amortiguado (PhD Thesis), Servicio de Publicaciones, Universidad de La Rioja, 2013, http://dialnet.unirioja.es/servlet/tesis?codigo=38821.

[8] Á. A. Magreñán, Different anomalies in a Jarratt family of iterative root-finding methods. Applied Mathematics and Computation 233 (2014) 29–38.

[9] B. Neta, C. Chun, M. Scott, Basins of attraction for optimal eighth order methods to find simple roots of nonlinear equations, Applied Mathematics and Computation 227 (2014) 567–592.

[10] H. T. Kung, J. F. Traub, Optimal order of one-point and multi-pointiteration, Journal Assoc. Comput. Math. 21 (1974) 643–651.

[11] P. Blanchard, Complex Analytic Dynamics on the Riemann Sphere, Bulletin of the AMS 11(1) (1984) 85–141.

[12] D. K. R. Babajee, Analysis Of Higher Order Variants Of Newton's Method And Their Applications To Differential And Integral Equations And In Ocean Acidification, PhD thesis, University of Mauritius, 2010.

[13] F. Chicharro, A. Cordero, J.M. Gutiérrez, J.R. Torregrosa, Complex dynamics of derivative-free methods for nonlinear equations, Applied Mathematics and Computation 219 (2013) 7023–7035.

[14] A. Cordero, J. García-Maimó, J. R. Torregrosa, M. P. Vassileva, P. Vindel, Chaos in King's iterative family, Applied Mathematics Letters 26 (2013) 842–848.

[15] A. Cordero, J. R. Torregrosa, P. Vindel, Dynamics of a family of ChebyshevHalley type methods, Applied Mathematics and Computation 219 (2013) 8568–8583.

[16] J. L. Varona, Graphic and numerical comparison between iterative methods, Math. Intelligencer 24 (2002) 37–46.

[17] A. Cayley, The Newton-Fourier Imaginary Problem. Amer. J. Math. 2 (1879) 97.

[18] E. Schroder, Uber unendlich viele Algorithmen zur Auflosung der Gleichungen. Math. Ann. 2 (1870) 317–365.

[19] G. Julia, Sur les èquations fonctionelles, J. Math. Pure Appl. 4 (1918) 47–245.

[20] P. Fatou, Sur les èquations fonctionelles, Bull. Soc. Math. France, 47 (1919) 161–271.

[21] A. F. Beardon, Iteration of Rational Functions, Springer-Verlag, New York, 1991.

[22] F. Chicharro, A. Cordero, J.R. Torregrosa, Drawing dynamical and parameters planes of iterative families and methods, The Scientific World Journal Volume 2013 Article ID: 780153 11 pages.

[23] B. Campos, A. Cordero, Juan R. Torregrosa, P. Vindel, Dynamics of the family of c-iterative methods, International Journal of Computer Mathematics,2014 doi:10.1080/00207160.2014.893608.

[24] G. H. Nedzhibov, V. I. Hasanov, M. G. Petkov, On some families of multi-point iterative methods for solving nonlinear equations, Numer. Alg. 42(2) (2006) 127–136.

[25] R. F. King, A family of fourth-order methods for solving nonlinear equations, SIAM J. Numer. Anal. 10 (1973) 876–879.

[26] Y. I. Kim, A triparametric family of three-step optimal eighth-order methods for solving nonlinear equations, International Journal of Computer Mathematics 89 (2012) 1051–1059.

[27] S. Artidiello, F. Chicharro, A. Cordero, J. R. Torregrosa, Local convergence and dynamical analysis of a new family of optimal fourth-order iterative methods, International Journal of Computer Mathematics 90(10) (2013) 2049–2060.

[28] F. Soleymani, D. K. R. Babajee, M. Sharifi, Modified Jarratt Method without Memory with Twelfth-Order Convergence, Annals of the University of Craiova, Mathematics and Computer Science Series, 39(1) (2012) 21–34.