APPLIED GENERAL TOPOLOGY

Remarks on fixed point assertions in digital topology, 4

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Abstract

We continue the work of [4, 2, 3], in which we discuss published assertions concerning fixed points in digital topology - assertions that are incorrect or incorrectly proven; that are severely limited or reduce to triviality; or that we improve upon.

1. Introduction

As stated in [2]:

> The topic of fixed points in digital topology has drawn much attention in recent papers. The quality of discussion among these papers is uneven; while some assertions have been correct and interesting, others have been incorrect, incorrectly proven, or reducible to triviality.

Paraphrasing [2] slightly: in [4, 2, 3], we have discussed many shortcomings in earlier papers and have offered corrections and improvements. We continue this work in the current paper.

A common theme among many weak papers concerning fixed points in digital topology is the use of a "digital metric space" (see section 2.2 for its definition). This seems to be a bad idea.

- Nearly all correct nontrivial published assertions concerning digital metric spaces use either the adjacency of the digital image or the metric, but not both. Where our sources do not use adjacencies, we will state our results using the more general framework of a metric space.
- If X is finite (as in a "real world" digital image) or the metric d is a common metric such as any ℓ_p metric, then (X, d) is uniformly discrete, hence not very interesting either as a topological space or as a metric space.
- Many of the published assertions concerning digital metric spaces mimic analogues for connected subsets of Euclidean $\mathbb{R}^n$. Often, the authors neglect important differences between the topological space $\mathbb{R}^n$ and digital images, resulting in assertions that are incorrect, trivial, or trivial when restricted to conditions that many others regard as essential. E.g., in many cases, functions that satisfy fixed point assertions must be constant or fail to be digitally continuous [4, 2, 3].

This paper continues the work of [4, 2, 3] in discussing shortcomings of published assertions concerning fixed points in digital topology.

2. Preliminaries

We use $\mathbb{N}$ to represent the natural numbers, $\mathbb{Z}$ to represent the integers, and $\mathbb{R}$ to represent the reals.

A *digital image* is a pair (X, κ), where $X \subset \mathbb{Z}^n$ for some positive integer n, and κ is an adjacency relation on X. Thus, a digital image is a graph. In order to model the "real world," we usually take X to be finite, although there are several papers that consider infinite digital images. The points of X may be thought of as the "black points" or foreground of a binary, monochrome "digital picture," and the points of $\mathbb{Z}^n \setminus X$ as the "white points" or background of the digital picture.

2.1. **Adjacencies, connectedness, continuity, fixed point.** In a digital image (X, κ), if $x, y \in X$, we use the notation $x \leftrightarrow_\kappa y$ to mean x and y are κ-adjacent; we may write $x \leftrightarrow y$ when κ can be understood. We write $x \leftrightarroweq_\kappa y$, or $x \leftrightarroweq y$ when κ can be understood, to mean $x \leftrightarrow_\kappa y$ or $x = y$.

The most commonly used adjacencies in the study of digital images are the c_u adjacencies. These are defined as follows.

Definition 2.1. Let $X \subset \mathbb{Z}^n$. Let $u \in \mathbb{Z}$, $1 \le u \le n$. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$. Then $x \leftrightarrow_{c_u} y$ if

- $x \ne y$,
- for at most u distinct indices i, $|x_i - y_i| = 1$, and
- for all indices j such that $|x_j - y_j| \ne 1$ we have $x_j = y_j$.

Definition 2.2 ([14]). A digital image (X, κ) is κ-*connected*, or just *connected* when κ is understood, if given $x, y \in X$ there is a set $\{x_i\}_{i=0}^n \subset X$ such that $x = x_0$, $x_i \leftrightarrow_\kappa x_{i+1}$ for $0 \le i < n$, and $x_n = y$.

Definition 2.3 ([14, 1])**.** Let (X, κ) and (Y, λ) be digital images. A function $f : X \to Y$ is (κ, λ)-*continuous*, or κ-*continuous* if $(X, \kappa) = (Y, \lambda)$, or *(digitally) continuous* when κ and λ are understood, if for every κ-connected subset X' of X, $f(X')$ is a λ-connected subset of Y.

Theorem 2.4 ([1])**.** *A function* $f : X \to Y$ *between digital images* (X, κ) *and* (Y, λ) *is* (κ, λ)-*continuous if and only if for every* $x, y \in X$, *if* $x \leftrightarrow_\kappa y$ *then* $f(x) \leftrightarrow_\lambda f(y)$.

Theorem 2.5 ([1])**.** *Let* $f : (X, \kappa) \to (Y, \lambda)$ *and* $g : (Y, \lambda) \to (Z, \mu)$ *be continuous functions between digital images. Then* $g \circ f : (X, \kappa) \to (Z, \mu)$ *is continuous.*

We use 1_X to denote the identity function on X, and $C(X, \kappa)$ for the set of functions $f : X \to X$ that are κ-continuous.

A *fixed point* of a function $f : X \to X$ is a point $x \in X$ such that $f(x) = x$.

Functions $f, g : X \to X$ are *commuting* if $f(g(x)) = g(f(x))$ for all $x \in X$.

2.2. **Digital metric spaces.** A *digital metric space* [8] is a triple (X, d, κ), where (X, κ) is a digital image and d is a metric on X. We are not convinced that this is a notion worth developing; under conditions in which a digital image models a "real world" image, X is finite or d is (usually) an ℓ_p metric, so that (X, d) is *uniformly discrete* as a topological space, i.e., there exists $\varepsilon > 0$ such that for $x, y \in X$, $d(x, y) < \varepsilon$ implies $x = y$. Typically, assertions in the literature do not make use of both d and κ, so that this notion has an artificial feel. E.g., for a discrete topological space X, all functions $f : X \to X$ are continuous, although on digital images, many functions $g : X \to X$ are not digitally continuous.

We say a sequence $\{x_n\}_{n=0}^\infty$ is *eventually constant* if for some $m > 0$, $n > m$ implies $x_n = x_m$. The notions of convergent sequence and complete digital metric space are often trivial, e.g., if the digital image is uniformly discrete, as noted in the following, a minor generalization of results of [9, 4].

Proposition 2.6. *Let* (X, d) *be a metric space. If* (X, d) *is uniformly discrete, then any Cauchy sequence in* X *is eventually constant, and* (X, d) *is a complete metric space.*

Remarks 2.7. If X is finite or $X \subset \mathbb{Z}^n$ and d is an ℓ_p metric, then (X, d) is uniformly discrete.

2.3. **Common conditions, limitations, and trivialities.** In this section, we state results that limit or trivialize several of the assertions discussed later in this paper.

Although there are papers that discuss infinite digital images, a "real world" digital image is a finite set. Further, most authors writing about a digital metric space choose their metric from the Euclidean metric, the Manhattan metric, or some other ℓ_p metric.

Other frequently used conditions:

- The adjacencies most often used in the digital topology literature are the c_u adjacencies.
- Functions that attract the most interest in the digital topology literature are digitally continuous.

Thus, the use of c_u-adjacency and the continuity assumption (as well as the assumption of an ℓ_p metric) in the following Proposition 2.8 should not be viewed as major restrictions. The following is taken from the proof of **Remark 5.2 of** [4].

Proposition 2.8. *Let X be c_u-connected. Let $T \in C(X, c_u)$. Let d be an ℓ_p metric on X, and $0 < \alpha < \frac{1}{u^{1/p}}$. Let $S : X \to X$ such that $d(S(x), S(y)) \leq \alpha d(T(x), T(y))$ for all $x, y \in X$. Then S must be a constant function.*

Similar reasoning leads to the following.

Proposition 2.9. *Let (X, d) be a uniformly discrete metric space. Let $f \in C(X, \kappa)$. Then if $\{x_n\}_{n=1}^\infty \subset X$ and $\lim_{n\to\infty} x_n = x_0 \in X$, then for almost all n, $f(x_n) = f(x_0)$.*

Other choices of (X, d) need not lead to the conclusion of Proposition 2.9, as shown by the following example.

Example 2.10. Let $X = \mathbb{N} \cup \{0\}$,

$$d(x, y) = \begin{cases} 0 & \text{if } x = 0 = y; \\ 1/x & \text{if } x \neq 0 = y; \\ 1/y & \text{if } x = 0 \neq y; \\ |1/x - 1/y| & \text{if } x \neq 0 \neq y. \end{cases}$$

Then d is a metric, and $\lim_{n\to\infty} d(n, 0) = 0$. However, the function $f(n) = n+1$ satisfies $f \in C(X, c_1)$ and

$$\lim_{n\to\infty} d(0, f(n)) = 0 \neq f(0).$$

Proof. Example 2.10 of [4] notes that d has the properties of a metric for values of $X \setminus \{0\}$. Since clearly $d(0, 0) = 0$ and $d(x, 0) = d(0, x)$, we must show that the triangle inequality holds when 0 is one of the points considered. We have the following.

- If $x, y > 0$, then $d(0, y) = 1/y \leq 1/x + |1/y - 1/x| = d(0, x) + d(x, y)$.
- Similarly, if $x, y > 0$ then $d(x, 0) \leq d(x, y) + d(y, 0)$.
- If $x, y > 0$ then $d(x, y) = |1/x - 1/y| < 1/x + 1/y = d(x, 0) + d(0, y)$.

Thus, the triangle inequality is satisfied.

Note $f \in C(X, c_1)$, $\lim_{n\to\infty} d(n, 0) = 0$, and $\lim_{n\to\infty} d(f(n), f(0)) = 1$. $\square$

3. Compatible functions and weakly compatible functions

The papers [11, 13, 7] discuss common fixed points of compatible and weakly compatible functions (the latter also know as "coincidentally commuting") and related notions.

3.1. **Compatibility - definition and basic properties.**

Definition 3.1 ([6])**.** Suppose S and T are self-functions on a metric space (X, d). Consider $\{x_n\}_{n=1}^{\infty} \subset X$ such that

(3.1) $$\lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(x_n) = t \in X.$$

If every sequence satisfying (3.1) also satisfies $\lim_{n \to \infty} d(S(T(x_n)), T(S(x_n))) = 0$, then S and T are *compatible functions*.

Proposition 3.2. *Let $f, g : X \to X$ be commuting functions on a metric space (X, d). Then f and g are compatible.*

Proof. Since f and g are commuting, the assertion is immediate. □

Definition 3.3 ([11])**.** Let S and T be self-functions on a metric space (X, d). The pair (S, T) satisfies the property *E.A.* if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ that satisfies (3.1).

Definition 3.4 ([11])**.** Let S and T be self-functions on a metric space (X, d). The pair (S, T) satisfies the property *common limit in the range of T*, denoted $\mathrm{CLR}(T)$, if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ that satisfies

$$\lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(x_n) = T(x) \text{ for some } x \in X.$$

Proposition 3.5. *Let S and T be self-functions on a metric space (X, d).*

1. *Suppose the pair (S, T) satisfies the property CLR(T). Then the pair (S, T) satisfies the property E.A.*
2. *Suppose S and T have a coincidence point, i.e., there exists $x \in X$ such that $S(x) = T(x)$. Then the pair (S, T) satisfies the property CLR(T).*
3. *If X is finite, then the following are equivalent.*
   (a) *(S, T) satisfies the property E.A.*
   (b) *(S, T) satisfies the property CLR(T).*
   (c) *S and T have a coincidence point, i.e., there exists $x \in X$ such that $S(x) = T(x)$.*

Proof.

1. Suppose the pair (S, T) satisfies the property $\mathrm{CLR}(T)$. It is trivial that the pair (S, T) satisfies the property E.A.
2. Suppose $S(x) = T(x)$. Then the sequence $x_n = x$ satisfies Definition 3.4, so (S, T) has property $\mathrm{CLR}(T)$.
3. Now suppose X is finite.

   $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$: Suppose (S, T) satisfies the property E.A. Let $\{x_n\}_{n=1}^{\infty} \subset X$ satisfy (3.1). Since X is finite, there is a subsequence $\{x_{n_i}\}$ that is eventually constant; say, x_{n_i} is eventually equal to $x \in X$. Hence $T(x_n)$ is eventually $\lim_{n_i \to \infty} T(x_{n_i}) = T(x)$. Thus (S, T) satisfies the property $\mathrm{CLR}(T)$. Similarly, $S(x_n)$ is eventually $\lim_{n_i \to \infty} S(x_{n_i}) = S(x)$. Thus

   $$S(x) = \lim_{n_i \to \infty} S(x_{n_i}) = \lim_{n_i \to \infty} T(x_{n_i}) = T(x).$$

$(b) \Rightarrow (a)$: This is shown in part (1).
$(c) \Rightarrow (b)$: This is shown in part (2).

$\square$

3.2. **Variants on compatibility.** In classical topology and real analysis, there are many papers that study variants of compatible (as defined above) functions. Several authors have studied analogs of these variants in digital topology. Often, the variants turn out to be equivalent.

Definition 3.6 ([5]). Let $S, T : X \to X$. Then S and T are *weakly compatible* or *coincidentally commuting* if, for every $x \in X$ such that $S(x) = T(x)$ we have $S(T(x)) = T(S(x))$.

Theorem 3.7. *Let $S, T : X \to X$. Compatibility implies weak compatibility; and if X is finite, weak compatibility implies compatibility.*

Proof. Suppose S and T are compatible. We show they are weakly compatible as follows. Let $S(x) = T(x)$ for some $x \in X$. Let $x_n = x$ for all $n \in \mathbb{N}$. Then

$$\lim_{n \to \infty} S(x_n) = S(x) = T(x) = \lim_{n \to \infty} T(x_n).$$

By compatibility,

$$0 = \lim_{n \to \infty} d(S(T(x_n)), T(S(x_n))) = d(S(T(x)), T(S(x))).$$

Thus, S and T are weakly compatible.

Suppose S and T are weakly compatible and X is finite. We show S and T are compatible as follows. Let $\{x_n\}_{n=1}^{\infty} \subset X$ such that

$$\lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(x_n) = t \in X.$$

Proposition 2.6 yields that for almost all n, $S(x_n) = T(x_n) = t$. Since X is finite, there is an infinite subsequence $\{x_{n_i}\}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_i} = y \in X$, hence $S(y) = T(y)$. Therefore, for almost all n and almost all n_i, weak compatibility implies

$$S(T(x_n)) = S(T(x_{n_i})) = S(T(y)) = T(S(y)) = T(S(x_{n_i})) = T(S(x_n)).$$

It follows that S and T are compatible. $\square$

We have the following, in which we restate (3.1) for convenience.

Definition 3.8. Suppose S and T are self-functions on a metric space (X, d). Consider $\{x_n\}_{n=1}^{\infty} \subset X$ such that

(3.2)
$$\lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(x_n) = t \in X.$$

- S and T are *compatible of type A* [6] if every sequence satisfying (3.2) also satisfies

$$\lim_{n \to \infty} d(S(T(x_n)), T(T(x_n))) = 0 = \lim_{n \to \infty} d(T(S(x_n)), S(S(x_n))).$$

- S and T are *compatible of type B* [7] if every sequence satisfying (3.2) also satisfies

$$\lim_{n \to \infty} d(S(T(x_n)), T(T(x_n))) \leq$$

(3.3) $\qquad 1/2 \left[\lim_{n \to \infty} d(S(T(x_n)), S(t)) + d(S(t), S(S(x_n))) \right]$

and

$$\lim_{n \to \infty} d(T(S(x_n)), S(S(x_n))) \leq$$

(3.4) $\qquad 1/2 \left[\lim_{n \to \infty} d(T(S(x_n)), T(t)) + d(T(t), T(T(x_n))) \right].$

Note this is a correction of the definition as stated in [7], *where the inequality here given as* (3.4) *uses a left side equivalent to*

$$\lim_{n \to \infty} d(T(S(x_n)), T(T(x_n))) \text{ instead of } \lim_{n \to \infty} d(T(S(x_n)), S(S(x_n))).$$

The version we have stated is the version used in proofs of [7] *and corresponds to the version of* [12] *that inspired the definition of* [7].

- S and T are *compatible of type C* [7] if every sequence satisfying (3.2) also satisfies

$$\lim_{n \to \infty} d(S(T(x_n)), T(T(x_n))) \leq$$

(3.5) $\qquad 1/2 \left[\begin{array}{c} \lim_{n \to \infty} d(S(T(x_n)), S(t)) + \lim_{n \to \infty} d(S(t), S(S(x_n))) + \\ \lim_{n \to \infty} d(S(t), T(T(x_n))) \end{array} \right]$

and

$$\lim_{n \to \infty} d(T(S(x_n)), S(S(x_n))) \leq$$

(3.6) $\qquad 1/2 \left[\begin{array}{c} \lim_{n \to \infty} d(T(S(x_n)), T(t)) + \lim_{n \to \infty} d(T(t), T(T(x_n))) + \\ \lim_{n \to \infty} d(T(t), S(S(x_n))) \end{array} \right].$

- S and T are *compatible of type P* [6] if every sequence satisfying (3.2) also satisfies

$$\lim_{n \to \infty} d(S(S(x_n)), T(T(x_n))) = 0.$$

We augment Theorem 3.7 with the following.

Theorem 3.9. *Let* (X, d) *be a metric space that is uniformly discrete. Let* $S, T : X \to X$. *The following are equivalent.*

- *S and T are compatible.*
- *S and T are compatible of type A.*
- *S and T are compatible of type B.*
- *S and T are compatible of type C.*
- *S and T are compatible of type P.*

Proof. The equivalence of compatible, compatible of type A, and compatible of type P was shown in Theorem 3.3 of [2], where the assumption that X is finite or d is an ℓ_p metric easily generalizes to the assumption that (X, d) is uniformly discrete.

Compatible of type A implies compatible of type B, by Proposition 4.7 of [7].

We show compatible of type B implies compatible, as follows. Let S and T be compatible of type B. Let $\{x_n\}_{n=1}^{\infty} \subset X$ satisfy (3.2). By Proposition 2.6, $S(x_n) = t = T(x_n)$ for almost all n. From (3.3) we have

$$d(S(t), T(t)) \leq 1/2 \left[d(S(t), S(t)) + d(S(t), S(t))\right] = 0,$$

so $S(t) = T(t)$. Thus

$$\lim_{n \to \infty} d(S(T(x_n)), T(S(x_n))) = \lim_{n \to \infty} d(S(t), T(t)) = 0.$$

Therefore, S and T are compatible.

We show compatible implies compatible of type C, as follows. Let S and T be compatible. Let $\{x_n\}_{n=1}^{\infty} \subset X$ satisfy (3.2). By Proposition 2.6, $S(x_n) = t = T(x_n)$ for almost all n, and by compatibility, $S(t) = T(t)$. Therefore,

$$\lim_{n \to \infty} d(S(T(x_n)), T(T(x_n))) = \lim_{n \to \infty} d(S(t), T(t)) = 0,$$

so (3.5) is satisfied, and

$$\lim_{n \to \infty} d(T(S(x_n)), S(S(x_n))) = d(T(t), S(t)) = 0,$$

so (3.6) is satisfied. Thus S and T are compatible of type C.

We show compatible of type C implies compatible, as follows. Let S and T be compatible of type C. Let $\{x_n\}_{n=1}^{\infty} \subset X$ satisfy (3.2). By Proposition 2.6, $S(x_n) = t = T(x_n)$ for almost all n. From (3.5) it follows that

$$d(S(t), T(t)) \leq 1/2 \left[d(S(t), S(t)) + d(S(t), S(t)) + d(S(t), T(t))\right],$$

or

$$d(S(t), T(t)) \leq 1/2 \left[0 + 0 + d(S(t), T(t))\right],$$

which implies

$$0 = d(S(t), T(t)) = \lim_{n \to \infty} d(S(T(x_n)), T(S(x_n))).$$

Therefore, S and T are compatible. $\square$

3.3. **Fixed point assertions of** [11]**.** The following assertion appears as **Theorem 3.1.1 of** [11] and as **Theorem 4.12 of** [7] (there is a minor difference between these: [11] requires $\mu \in (0, 1/2)$ while [7] requires $\mu \in (0, 1)$).

Assertion 3.10. Let (X, d, κ) be a complete digital metric space. Let S and T be compatible self-functions on X. Suppose

(i) $S(X) \subset T(X)$;

(ii) S or T is continuous; and

(iii) for all $x, y \in X$ and some $\mu \in (0, 1/2)$,

$$d(Sx, Sy) \leq \mu \max\{d(Tx, Ty), d(Tx, Sx), d(Tx, Sy), d(Ty, Sx), d(Ty, Sy)\}.$$

Then S and T have a unique common fixed point in X.

Remarks 3.11. The argument given as proof in [11] for this assertion clarifies that the continuity assumed is topological (the classical $\varepsilon - \delta$ continuity), not digital.

Further, Assertion 3.10 and the argument offered for its proof in [11] are flawed as discussed below (this is the first of several assertions with related flaws; we discuss these assertions together), beginning at Remark 3.17. Flaws in the treatment of Assertion 3.10 in [7] are discussed below, beginning at Remark 3.32.

The following assertion appears as **Theorem 3.2.1 of** [11].

Assertion 3.12. Let (X, d, κ) be a complete digital metric space. Let S and T be weakly compatible self-functions on X. Suppose

(i) $S(X) \subset T(X)$;

(ii) $S(X)$ or $T(X)$ is complete; and

(iii) for all $x, y \in X$ and some $\mu \in (0, 1/2)$,

$$d(Sx, Sy) \leq \mu \max\{d(Tx, Ty), d(Tx, Sx), d(Tx, Sy), d(Ty, Sx), d(Ty, Sy)\}.$$

Then S and T have a unique common fixed point in X.

However, this assertion and the argument offered for its proof are flawed as discussed below, beginning at Remark 3.17.

The following is **Theorem 3.3.2 of** [11].

Theorem 3.13. *Let (X, d, κ) be a digital metric space. Let $S, T : X \to X$ be weakly compatible functions satisfying the following.*

(i) For some $\mu \in (0, 1)$ and all $x, y \in X$, $d(Sx, Sy) \leq \mu d(Tx, Ty)$.

(ii) S and T satisfy property E.A.

(iii) $T(X)$ is a closed subspace of X.

Then S and T have a unique common fixed point in X.

However, this result is limited, as discussed below, beginning at Remark 3.17.

The following appears as **Theorem 3.3.3 of** [11].

Assertion 3.14. Let (X, d, κ) be a complete digital metric space. Let S and T be weakly compatible self-functions on X. Suppose

(i) for all $x, y \in X$ and some $\mu \in (0, 1/2)$,

$$d(Sx, Sy) \leq \mu \max\{d(Tx, Ty), d(Tx, Sx), d(Tx, Sy), d(Ty, Sx), d(Ty, Sy)\};$$

(ii) S and T satisfy the property E.A.; and

(iii) $T(X)$ is a closed subspace of X.

Then S and T have a unique common fixed point in X.

However, this assertion and the argument offered for its proof are flawed as discussed below, beginning at Remark 3.17.

The following is **Theorem 3.4.3 of** [11].

Theorem 3.15. *Let S and T be weakly compatible self-functions on a digital metric space (X, d, κ) satisfying*

(i) for some $\mu \in (0, 1)$ and all $x, y \in X$, $d(Sx, Sy) \leq \mu d(Tx, Ty)$; and

(ii) the CLR(T) property.
Then S and T have a unique common fixed point in X.

However, this result is quite limited, as discussed below at Remark 3.19.
The following appears as **Theorem 3.4.3 of** [11].

Assertion 3.16. Let S and T be weakly compatible self-functions on a digital
metric space (X, d, κ) satisfying

(i) for all $x, y \in X$ and some $\mu \in (0, 1/2)$,

$$d(Sx, Sy) \leq \mu \max\{d(Tx, Ty), d(Tx, Sx), d(Tx, Sy), d(Ty, Sx), d(Ty, Sy)\}; \text{ and}$$

(ii) the $\text{CLR}(T)$ property.

Then S and T have a unique common fixed point in X.

However, this assertion and the argument offered for its proof are flawed as
discussed below.

Remarks 3.17. Several times in the arguments offered as proofs for Asser-
tions 3.10, 3.12, 3.14, and 3.16, inequalities appear that seem to confuse "min"
and "max". E.g., in the argument for Assertion 3.10, it is claimed that the
right side of the inequality

$$d(y_n, y_{n+1}) \leq \mu \max \left\{ \begin{array}{c} d(y_{n-1}, y_n), \ d(y_{n-1}, y_n), \ d(y_{n-1}, y_{n+1}), \\ d(y_n, y_n), \ d(y_n, y_{n+1}) \end{array} \right\}$$

is less than or equal to $\mu d(y_{n-1}, y_{n+1})$, which would follow if "max" were re-
placed by "min". Thus, these assertions as given in [11] must be regarded as
unproven.

Remarks 3.18. Further, suppose "min" is substituted for "max" so that (iii) in
each of the Assertions 3.10 and 3.12 and (i) in each of Assertions 3.14 and 3.16
becomes

for all $x, y \in X$ and some $\mu \in (0, 1/2)$,

$$d(Sx, Sy) \leq \mu \min\{d(Tx, Ty), d(Tx, Sx), d(Tx, Sy), d(Ty, Sx), d(Ty, Sy)\}.$$

Then for all $x, y \in X$, $d(Sx, Sy) \leq \mu d(Tx, Ty)$. If $T \in C(X, c_u)$, d is an ℓ_p
metric, and $\mu < 1/u^{1/p}$, then by Proposition 2.8, S is constant. It would then
follow from compatibility (respectively, from weak compatibility) that S and
T have a unique fixed point coinciding with the value of S.

Remarks 3.19. Similarly, in Theorems 3.13 and 3.15, if $T \in C(X, c_u)$, d is an
ℓ_p metric, and $\mu < 1/u^{1/p}$, then by Proposition 2.8, S is constant. It would
then follow from compatibility (respectively, from weak compatibility) that S
and T have a unique fixed point coinciding with the value of S.

3.4. **Fixed point assertions of** [13]**.** The following is stated as **Lemma 3.3.5
of** [13].

Assertion 3.20. Let $S, T : (X, d, \kappa) \to (X, d, \kappa)$ be compatible.
  1) If $S(t) = T(t)$ then $S(T(t)) = T(S(t))$.
  2) Suppose $\lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(x_n) = t \in X$.

(a) If S is continuous at t, $\lim_{n \to \infty} T(S(x_n)) = S(t)$.

(b) If S and T are continuous at t, then $S(t) = T(t)$ and $S(T(t)) = T(S(t))$.

But the continuity used in the proof of this assertion is topological continuity, not digital continuity. We observe that if (X, d) is uniformly discrete, then the assumption of continuity need not be stated, as every self-function on X is continuous in the topological sense.

The argument given as proof of this assertion in [13] depends on the principle that $a_n \to a_0$ implies $S(a_n) \to S(a_0)$ if S is continuous at a_0, a valid principle for topological continuity and also for digital continuity if (X, d) is uniformly discrete, but, as shown in Example 2.10, not generally true for digital continuity. Thus the assertion must be regarded as unproven.

We can modify this assertion as follows. Notice we do not use a continuity hypothesis, but for part 2) we assume (X, d) is uniformly discrete.

Lemma 3.21. *Let* $S, T : (X, d) \to (X, d)$ *be compatible.*

1) If $S(t) = T(t)$ *then* $S(T(t)) = T(S(t))$.

2) Suppose (X, d) *is uniformly discrete. If*

$$\lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(x_n) = t \in X,$$

then $\lim_{n \to \infty} T(S(x_n)) = S(t) = T(t)$ *and* $S(T(t)) = T(S(t))$.

Proof. We modify the argument of [13].

Suppose $S(t) = T(t)$. Let $x_n = t$ for all $n \in \mathbb{N}$. Then $S(x_n) = T(x_n) = S(t) = T(t)$, so $d(S(T(t)), T(S(t))) = d(S(T(x_n)), T(S(x_n))) \to_{n \to \infty} 0$ by compatibility. This establishes 1).

Suppose $\lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(x_n) = t \in X$. Since we assume X is uniformly discrete, we have $S(x_n) = T(x_n) = t$ for almost all n. Therefore, for almost all n, the triangle inequality and compatibility give us

$$d(T(S(x_n)), T(t)) \leq d(T(S(x_n)), S(T(x_n))) + d(S(T(x_n)), T(t))$$

$$\to 0 + \lim_{n \to \infty} d(S(T(x_n)), T(S(x_n))) = 0,$$

so $\lim_{n \to \infty} T(S(x_n)) = T(t)$. Since X is uniformly discrete, by compatibility we have

$$d(S(t), T(t)) = \lim_{n \to \infty} d(S(t), T(S(x_n))) = \lim_{n \to \infty} d(S(T(x_n)), T(S(x_n))) = 0.$$

Therefore, $S(t) = T(t)$ and by part 1), $S(T(t)) = T(S(t))$. □

The following is stated as **Theorem 3.3.6 of** [13].

Assertion 3.22. Let S and T be continuous compatible functions of a complete digital metric space (X, d, κ) to itself. Then S and T have a unique common fixed point in X if for some $\alpha \in (0, 1)$,

(3.7) $S(X) \subset T(X)$ and $d(S(x), S(y)) \leq \alpha d(T(x), T(y))$ for all $x, y \in X$.

Remarks 3.23. The argument given as proof of this assertion in [13] clarifies that the assumed continuity is topological, not digital; the argument is also flawed by its reliance on Assertion 3.20, which we have seen is not generally valid. Thus, the assertion must be regarded as unproven.

As above, we can drop the assumption of continuity from Assertion 3.22 if we assume X is uniformly discrete, as shown in the following.

Theorem 3.24. *Let S and T be compatible functions of a uniformly discrete metric space (X, d) to itself. If S and T satisfy (3.7) for some $\alpha \in (0, 1)$, then they have a unique common fixed point in X.*

Proof. We use ideas from the analogue in [13].

Let $x_0 \in X$. Since $S(X) \subset T(X)$, we can let $x_1 \in X$ such that $T(x_1) = S(x_0)$, and, inductively, $x_n \in X$ such that $T(x_n) = S(x_{n-1})$ for all $n \in \mathbb{N}$. Then for all $n > 0$,

$$d(T(x_{n+1}), T(x_n)) = d(S(x_n), S(x_{n-1})) \le \alpha d(T(x_n), T(x_{n-1})).$$

It follows that $d(T(x_{n+1}), T(x_n)) \le \alpha^n d(T(x_1), T(x_0))$. By Proposition 2.6, there exists $t \in X$ such that $T(x_n) = t$ for almost all n. Our choice of the sequence x_n then implies $S(x_n) = t$ for almost all n. By Lemma 3.21, $S(t) = T(t)$ and $S(T(t)) = T(S(t))$. Then

$$d(S(t), S(S(t))) \le \alpha d(T(t), T(S(t))) = \alpha d(S(t), S(T(t))) = \alpha d(S(t), S(S(t))),$$

so $(1 - \alpha)d(S(t), S(S(t))) \le 0$. Therefore, $d(S(t), S(S(t))) = 0$, so

$$S(t) = S(S(t)) = S(T(t)) = T(S(t)).$$

Thus $S(t)$ is a common fixed point of S and T.

To show the uniqueness of t as a common fixed point, suppose $S(x) = T(x) = x$ and $S(y) = T(y) = y$. Then

$$d(x, y) = d(S(x), S(y)) \le \alpha d(T(x), T(y)) = \alpha d(x, y),$$

so $(1 - \alpha)d(x, y) \le 0$, so $x = y$. $\qquad\qquad\square$

The following is stated as **Theorem 3.4.3 of** [13].

Assertion 3.25. Let S and T be weakly compatible functions of a complete digital metric space (X, d, κ) to itself. Then S and T have a unique common fixed point in X if either of $S(X)$ or $T(X)$ is complete, and for some $\alpha \in (0, 1)$, statement (3.7) is satisfied.

Remarks 3.26. The argument given in [13] as a proof for Assertion 3.25 defines a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(x_n) = t \in X$. From this is claimed that a subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to a limit in X. How this is justified is unclear. Therefore, Assertion 3.25 as stated is unproven. If we additionally assume that X is finite, then the claim, that a subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to a limit in X, is certainly justified.

The following is a version of Assertion 3.25 with the additional hypothesis that X is finite. We have not stated an assumption of completeness, since a finite metric space must be complete.

Theorem 3.27. *Let S and T be weakly compatible functions of a uniformly discrete metric space (X, d) to itself. Then S and T have a unique common fixed point in X if for some $\alpha \in (0, 1)$, (3.7) is satisfied.*

Proof. Since X is finite, it follows from Theorem 3.9 that S and T are compatible. The assertion follows from Theorem 3.24. □

Note also that Assertion 3.22 and Theorems 3.24 and 3.27 are limited by Proposition 2.8.

3.5. **Fixed point assertions of** [7]**.** The following appears as **Proposition 4.10 of** [7]. Apparently, the authors neglected to state a hypothesis that S and T are compatible; they used this hypothesis in their "proof", and with this hypothesis, the desired conclusion is correctly reached.

Assertion 3.28. Let $S, T \in C(X, \kappa)$ for a digital metric space (X, d, κ). If $S(t) = T(t)$ for some $t \in X$, then

$$S(T(t)) = T(S(t)) = S(S(t)) = T(T(t)).$$

As stated, this is incorrect, as shown by the following example.

Example 3.29. Let $S, T : \mathbb{N} \to \mathbb{N}$ be the functions

$$S(x) = 2, \quad T(x) = x + 1.$$

Then $S, T \in C(\mathbb{N}, c_1)$ and $S(1) = T(1) = 2$, but

$$S(T(1)) = S(S(1)) = 2, \quad T(S(1)) = T(T(1)) = 3.$$

Further, the argument of [7] uses neither the hypothesis of continuity nor the adjacency κ. A corrected version of Assertion 3.28 is above at Lemma 3.21.

The following appears as **Proposition 4.11 of** [7].

Assertion 3.30. Let (X, d, κ) be a digital metric space and let $S, T \in C(X, \kappa)$. Suppose $\lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(x_n) = t \in X$. Then
(i) $\lim_{n \to \infty} T(S(x_n)) = S(t)$;
(ii) $\lim_{n \to \infty} S(T(x_n)) = T(t)$; and
(iii) $S(T(t)) = T(S(t))$ and $S(t) = T(t)$.

The "proof" of this assertion in [7] confuses topological and digital continuity. The following shows that the assertion is not generally true.

Example 3.31. Let $S, T : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be the functions $S(x) = 0, T(x) = x + 1$. Let d be the metric of Example 2.10. Clearly, $S, T \in C(\mathbb{N} \cup \{0\}, c_1)$, and with respect to d, we have $\lim_{n \to \infty} S(n) = 0 = \lim_{n \to \infty} T(n)$. However, with respect to d we have
(i) $\lim_{n \to \infty} T(S(n)) = T(0) = 1, S(0) = 0$.
(ii) $\lim_{n \to \infty} S(T(n)) = 0, T(0) = 1$;
(iii) $S(T(0)) = 0, T(S(0)) = 1, S(0) = 0, T(0) = 1$.

Remarks 3.32. We have stated **Theorem 4.12 of** [7] above as Assertion 3.10. The argument for this assertion in [7] is flawed as follows.

The argument considers the case $x_n = x_{n+1}$ and reaches the statement

$$d(T(x_n), T(x_{n+1})) = d(S(x_{n-1}), S(x_n)) \leq$$

$$\alpha \max\{d(T(x_{n-1}), T(x_n)), d(T(x_{n-1}), T(x_{n+1})), d(T(x_n), T(x_{n+1}))\}.$$

This yields three cases, each of which is handled incorrectly:

(1) $d(T(x_n), T(x_{n+1})) \leq \alpha d(T(x_{n-1}), T(x_n))$. Nothing further is stated about this case.

(2) $d(T(x_n), T(x_{n+1})) \leq \alpha d(T(x_{n-1}), T(x_{n+1}))$. The authors misstate this case as $d(T(x_n), T(x_{n+1})) \leq \alpha d(T(x_n), T(x_{n+1}))$ and propagate this error forward.

(3) $d(T(x_n), T(x_{n+1})) \leq \alpha d(T(x_n), T(x_{n+1}))$. This implies $T(x_n) = T(x_{n+1})$, since $0 < \alpha < 1$, but the authors reach a slightly weaker conclusion differently. They reason that $d(T(x_n), T(x_{n+1})) \leq \alpha^n d(T(x_0), T(x_1))$, from an implied induction with the unjustified assumption that this case applies at every level of the induction.

Later in the argument, the error of confusing topological and digital continuity also appears.

Therefore, we must consider Assertion 3.10 unproven.

The following is stated as **Theorem 4.13 of** [7].

Assertion 3.33. Let $S, T : (X, d, \kappa) \to (X, d, \kappa)$ be functions that are compatible of type A on a digital metric space, such that

(i) $S(X) \subset T(X)$;

(ii) S or T is (κ, κ)-continuous; and

(iii) for all $x, y \in X$ and some $\alpha \in (0, 1)$,

$$d(Sx, Sy) \leq \alpha \max\{d(Tx, Ty), d(Tx, Sy), d(Ty, Sx), d(Tx, Sx), d(Ty, Sy)\}.$$

Then S and T have a unique common fixed point in X.

However, the argument given in [7] to prove this assertion relies on Assertion 3.30, which we have shown above is unproven.

Assertion 3.34. Let $S, T : (X, d, \kappa) \to (X, d, \kappa)$ be functions on a digital metric space satisfying (i), (ii), and (iii) of Assertion 3.33. If

(a) (stated as **Theorem 4.14 of** [7]) S and T are compatible of type B, or

(b) (stated as **Theorem 4.15 of** [7]) S and T are compatible of type C, or

(c) (stated as **Theorem 4.16 of** [7]) S and T are compatible of type P,

then S and T have a unique common fixed point in X.

Remarks 3.35. Each part of Assertion 3.34 must be regarded as unproven, as each has a "proof" in [7] that depends on Assertion 3.10, which we have shown above to be unproven. (The arguments in [7] for parts (b) and (c) also depend on the unproven part (a).) Note also that, by Theorem 3.9, a correct proof of any of (a), (b), or (c) for the case that (X, d) is uniformly discrete would prove the other parts correct for this case.

4. COMMUTATIVE AND WEAKLY COMMUTATIVE FUNCTIONS

The paper [13] discusses common fixed points for commutative and weakly commutative functions on digital metric spaces.

Definition 4.1. Let (X, d) be a metric space. Functions $f, g : X \to X$ are *commutative* if $f \circ g(x) = g \circ f(x)$ for all $x \in X$. They are *weakly commutative* if $d(f(g(x)), g(f(x))) \leq d(f(x), g(x))$ for all $x \in X$.

Proposition 4.2 ([13])**.** *Let* (X, d, κ) *be a digital metric space. Let* $T : X \to X$. *Then* T *has a fixed point in* X *if and only if there is a constant function* $S : X \to X$ *such that* S *commutes with* T.

The following is Theorem 3.1.4 of [13].

Theorem 4.3. *Let* T *be a continuous self-function on a complete digital metric space* (X, d, κ) *into itself. Then* T *has a fixed point in* X *if and only if there exists* $\alpha \in (0, 1)$ *and a function* $S : X \to X$ *that commutes with* T *and satisfies (3.7).*
If (3.7) holds then S *and* T *have a unique common fixed point.*

We give a modified version of Theorem 4.3 as follows.

Theorem 4.4. *Let* T *be a function of a metric space* (X, d) *into itself.*
- *If* T *has a fixed point in* X, *then there exists* $\alpha \in (0, 1)$ *and a function* $S : X \to X$ *that commutes with* T *and satisfies (3.7).*
- *Suppose* (X, d) *is uniformly discrete. If there exists* $\alpha \in (0, 1)$ *and a function* $S : X \to X$ *that commutes with* T *and satisfies (3.7), then* T *has a fixed point in* X.

Proof. It follows from Proposition 4.2 that if T has a fixed point, then there is a function $S : X \to X$ that commutes with T and satisfies (3.7).

Suppose X is uniformly discrete. Suppose there exists $\alpha \in (0, 1)$ and a function $S : X \to X$ that commutes with T and satisfies (3.7). Then S and T are compatible by Proposition 3.2. It follows from Theorem 3.24 that T has a fixed point. $\qquad\square$

We will use the following.

Example 4.5 ([4])**.** Let $X = \{p_1, p_2, p_3\} \subset \mathbb{Z}^5$, where

$$p_1 = (0, 0, 0, 0, 0), \quad p_2 = (2, 0, 0, 0, 0), \quad p_3 = (1, 1, 1, 1, 1).$$

Let d be the Manhattan metric and let $T : (X, c_5) \to (X, c_5)$ be defined by $T(p_1) = T(p_2) = p_1$, $T(p_3) = p_2$. Clearly $T(X) \subset 1_X(X)$, $1_X \in C(X, c_5)$, and for all $x, y \in X$ we have $d(T(x), T(y)) \leq 2/5 \, d(1_X(x), 1_X(y))$. However, $T \notin C(X, c_5)$ since $p_2 \leftrightarrow_{c_5} p_3$ and $T(p_2) \not\leftrightarrow_{c_5} T(p_3)$.

In the following, given a function $S : X \to X$ and $k \in \mathbb{N}$, S^k is the k-fold iterate of S, i.e., $S^1 = S$ and $S^{j+1} = S \circ S^j$.

Proposition 4.6 ([13])**.** *Let T and S be commuting functions of a digital metric space (X, d, κ) into itself. Suppose T is continuous and $S(X) \subset T(X)$. If there exists $\alpha \in (0, 1)$ and $k \in \mathbb{N}$ such that $d(S^k(x), S^k(y)) \leq \alpha d(T(x), T(y))$ for all $x, y \in X$, then T and S have a common fixed point.*

Remarks 4.7. The continuity hypothesized for Proposition 4.6 in [13] is topological continuity, not digital continuity. The assumption is used in the proof to argue that (3.7) implies S is (topologically) continuous. Note if X is a discrete topological space, then every self-function on X is topologically continuous. If we were to assume instead that T is digitally continuous, it would not follow from (3.7) that S is digitally continuous, as shown by Example 4.5.

The following is a modified version of Proposition 4.6. In it, there is no continuity assumption.

Corollary 4.8. *Let T and S be commuting functions of a complete metric space (X, d) into itself. Suppose $S(X) \subset T(X)$. If there exists $\alpha \in (0, 1)$ and $k \in \mathbb{N}$ such that $d(S^k(x), S^k(y)) \leq \alpha d(T(x), T(y))$ for all $x, y \in X$, then T and S have a common fixed point.*

Proof. As above, we modify the analogous argument of [13].

We see easily that S^k commutes with T and $S^k(X) \subset S(X) \subset T(X)$. By Theorem 4.3 - whose proof in [13] does not use an adjacency κ, hence is applicable in the more general setting of a metric space - there exists $a \in X$ such that a is the unique common fixed point of S^k and T. Then $a = S^k(a) = T(a)$. Since S and T commute, we can apply S to the above to get

$$S(a) = S(S^k(a)) = S(T(a)) = T(S(a))$$

and, from the first equation in this chain, $S(a) = S^k(S(a))$, so $S(a)$ is a common fixed point of T and S^k. Since a is unique as a common fixed point of T and S^k, we must have $a = S(a) = T(a)$. □

A function $T : X \to X$ on a digital metric space (X, d, κ) is a *digital expansive function* [10] if for some $k > 1$ and all $x, y \in X$, $d(T(x), T(y)) \geq kd(x, y)$. However, this definition is quite limited, as shown by the following, which combines Theorems 4.8 and 4.9 of [4].

Theorem 4.9. *Let (X, d, κ) be a digital metric space. Suppose there are points $x_0, y_0 \in X$ such that*

$$d(x_0, y_0) \in \{\min\{d(x, y) \,|\, x, y \in X, x \neq y\}, \ \max\{d(x, y) \,|\, x, y \in X, x \neq y\}\}.$$

Then there is no $T : X \to X$ that is both onto and a digital expansive function.

Note the hypothesis of Theorem 4.9 is satisfied by every finite digital metric space.

The following appears as Corollary 3.1.6 of [13].

Assertion 4.10. Let $n \in \mathbb{N}$, $K \in \mathbb{R}$, $K > 1$. Let $S : X \to X$ be a κ-continuous onto function of a complete digital metric space (X, d, κ) such that $d(S^n(x), S^n(y)) \geq Kd(x, y)$ for all $x, y \in X$. Then S has a unique fixed point.

Remarks 4.11. Theorem 4.9 shows that Assertion 4.10 is vacuous for finite digital metric spaces, since S being onto implies S^n is onto. Similarly, Assertion 4.10 is vacuous whenever $\kappa = c_1$, since $x \leftrightarrow_{c_1} y$ implies $S^n(x) \leftrightarrow_{c_1} S^n(y)$, hence $d(S^n(x), S^n(y)) \leq d(x, y)$.

We get Corollary 4.12 below by modifying Assertion 4.10 to consider contractive rather than expansive functions, making the following changes.

- We do not require S to be either continuous or onto, nor do we require completeness.
- We use $K \in (0, 1)$ rather than $K > 1$.
- We use $d(S^n(x), S^n(y)) \leq Kd(x, y)$ instead of $d(S^n(x), S^n(y)) \geq Kd(x, y)$.

Corollary 4.12. *Let $n \in \mathbb{N}$ and let $K \in (0, 1)$. Let $S : X \to X$ for a metric space (X, d) such that $d(S^n(x), S^n(y)) \leq Kd(x, y)$ for all $x, y \in X$, then S has a unique fixed point.*

Proof. Take $T = 1_X$. Then this assertion follows from Proposition 4.6, whose proof in [13] does not use an adjacency and therefore is applicable to metric spaces. □

We modify assumptions of the second bullet of Theorem 4.4 to obtain a similar result with a much shorter proof.

Theorem 4.13. *Let (X, d, c_u) be a digital metric space, where d is an ℓ_p metric and X is c_u-connected. Let $T \in C(X, c_u)$. Suppose we have a function $S : X \to X$ such that S commutes with T, $S(X) \subset T(X)$, and for some $\alpha \in (0, 1/u^{1/p})$ and all $x, y \in X$, $d(S(x), S(y)) \leq \alpha d(T(x), T(y))$. Then S is constant, and S and T have a unique common fixed point.*

Proof. It follows from Proposition 2.8 that S is a constant function. Since $S(X) \subset T(X)$, the value x_0 taken by S is a member of $T(X)$, and since S commutes with T,

$$T(x_0) = T(S(x_0)) = S(T(x_0)) = x_0 = S(x_0).$$

Since S is constant, x_0 is a unique common fixed point. □

The following is Theorem 3.2.3 of [13].

Theorem 4.14. *Let T be a function on a complete digital metric space (X, d, κ) into itself. Then T has a fixed point in X if and only if there exists $\alpha \in (0, 1)$ and a function $S : X \to X$ that commutes weakly with T and satisfies (3.7). Indeed T and S have a unique common fixed point if (3.7) holds.*

However, Theorem 4.14 is limited by Proposition 2.8, which gives conditions implying that the function S must be constant.

5. COMMUTING FUNCTIONS

The paper [7] studies common fixed points for commuting functions.
The following appears as **Theorem 3.2 of** [7].

Assertion 5.1. Let $\varnothing \neq X \subset \mathbb{Z}^n$, $n \in \mathbb{N}$, and let S and T be commuting functions of a complete digital metric space (X, d, κ) into itself such that
  (i) $T(X) \subset S(X)$;
  (ii) $S \in C(X, \kappa)$; and
  (iii) for some $\alpha \in (0, 1)$ and all $x, y \in X$, $d(T(x), T(y)) \leq \alpha d(S(x), S(y))$.
  Then S and T have a common fixed point in X.

Remarks 5.2. The argument given as proof for Assertion 5.1 in [7] claims that (ii) and (iii) imply $T \in C(X, \kappa)$, but this is incorrect (another instance of confusing topological and digital continuity), as shown in Example 4.5.

However, if we add to Assertion 5.1 the hypothesis that (X, d) is uniformly discrete, then we can delete the continuity assumption and get the following corrected version of Assertion 5.1.

Theorem 5.3. *Let $\varnothing \neq X \subset \mathbb{Z}^n$, $n \in \mathbb{N}$. Let S and T be commuting functions of a uniformly discrete metric space (X, d) into itself such that*
  (i) $T(X) \subset S(X)$; and
  (ii) for some $\alpha \in (0, 1)$ and all $x, y \in X$, $d(T(x), T(y)) \leq \alpha d(S(x), S(y))$.
  Then S and T have a unique common fixed point in X.

Proof. By Proposition 3.2, S and T are compatible. The result follows from Theorem 3.24. □

The following is presented as **Corollary 3.3 of** [7].

Assertion 5.4. Let S and T be commuting functions of a complete digital metric space (X, d, κ) into itself such that
  (i) $T(X) \subset S(X)$;
  (ii) $S \in C(X, \kappa)$; and
  (iii) for some $\alpha \in (0, 1)$ and $k \in \mathbb{N}$ we have

$$d(T^k(x), T^k(y)) \leq \alpha d(S(x), S(y)) \text{ for all } x, y \in X.$$

Then S and T have a unique common fixed point.

However, the argument given in [7] for this assertion depends on Assertion 5.1, shown above as unproven. Assertion 5.4 can be modified as follows.

Corollary 5.5. *Let S and T be commuting functions of a metric space (X, d) into itself such that*
  (i) $T(X) \subset S(X)$; and
  (ii) for some $\alpha \in (0, 1)$ and $k \in \mathbb{N}$ we have

$$d(T^k(x), T^k(y)) \leq \alpha d(S(x), S(y)) \text{ for all } x, y \in X.$$

If (X, d) is uniformly discrete, then S and T have a unique common fixed point.

Proof. We use the analogous argument of [7].
  Clearly, T^k commutes with S and $T^k(X) \subset T(X) \subset S(X)$. From Theorem 5.3, there is a unique $a \in X$ such that $a = S(a) = T^k(a)$. By applying the function T and the commuting property, we have

$$T(a) = T(S(a)) = S(T(a)) \text{ and } T(a) = T(T^k(a)) = T^k(T(a)),$$

so $T(a)$ is a common fixed point of S and T^k. But a is the unique common fixed point of S and T^k, so we must have $a = T(a)$, and we have already observed that $a = S(a)$, so a is a common fixed point of S and T.

To show the uniqueness of a as a common fixed point, suppose x, y are common fixed points of S and T. From hypothesis (ii),

$$d(x, y) = d(T(x), T(y)) = d(T^k(x), T^k(y)) \leq \alpha d(S(x), S(y)) = \alpha d(x, y).$$

Since $0 < \alpha < 1$, it follows that $x = y$. $\square$

Remarks 5.6. Note that Theorem 5.3 and Corollary 5.5 are limited by Proposition 2.8.

6. Further remarks

We have discussed assertions that appeared in [11, 13, 7]. We have discussed errors or corrections for some, shown some to be limited or trivial, and offered improvements for some.

References

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